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Elementary matrices and products of idempotents

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We study the relations between product decomposition of singular matrices into products of idempotent matrices and product decomposition of invertible matrices into elementary ones.

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1. Introduction

J. A. Erdos, in his classical paper [4], showed that singular matrices over fields are product of idempotent matrices. This result was then extended to matrices over division rings and euclidean rings. Fountain [6] studied the problem for commutative Hermite rings using tools from semigroup theory. Inspired by Fountain's paper, Ruitenberg [9] went on considering the question for noncommutative Hermite rings. In the papers by Fountain and Ruitenberg, the connection between product decomposition of singular matrices into idempotents and product decomposition of invertible matrices into elementary ones appeared clearly. The case of a commutative principal ideal domain was analyzed by Bhaskara Rao [3] and Alahmadi and al. [1]. Hannah and O'Meara, in their deep study [7], obtained various (necessary and/or sufficient) conditions for the decomposition of singular elements in regular rings. Various ring conditions where shown to be connected with the decomposition of singular matrices in Alahmadi et al. [2]. Salce and Zanardo studied in [10] relations between the two decompositions mentioned above in the setting of commutative integral domains, and our present paper owes much to their work. Our aim is in fact to see how far we can go in generalizing the results contained in Salce and Zanardo's paper in the noncommutative setting.

We will now briefly describe the content of the article. Section 2 gives necessary preliminaries and examples. We study which elements of a ring can be considered candidates to be products of idempotents. The notion of idempotent complete rings and some of its consequences are studied.

Section 3 gathers more information that will be mainly used in our main results (Theorems 4.1, 4.5, 4.10 and 4.11). The notions of consecutive pairs, regular morphisms and complementary antichains are defined and shown to be strongly related.

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Section 4 contains the main results of the paper. Let us recall that a ring R is IBN if every free R-module has unique rank. In Theorem 4.11 we show that, for IBN rings R which are $2, 3, \dots n$ right regular and such that complements of free direct summands of R^n of rank 1 or $n-1$ are free modules, the following are equivalent:

(i) for any free modules A, B such that $R^n = A \oplus B$ and any map $\beta \in \operatorname{End}_R(R^n)$ with $\ker(\beta) = B$ and $\operatorname{im}(\beta) = A$, there exist consecutive idempotents $\varepsilon_1, \dots, \varepsilon_k$ such that $\beta = \varepsilon_1 \cdots \varepsilon_k$ with $\ker(\varepsilon_i)$ and $\operatorname{im}(\varepsilon_i)$ free for every $i = 1, \dots, k$.

(ii) R is GE_n (i.e., every invertible $n \times n$ matrix is a product of elementary matrices).

2. Idempotent complete rings

Let R be an associative ring with identity. If an element $x \in R$ is a product $x = e_1 \dots e_n$ of finitely many idempotents $e_i \in R$, then x is annihilated by left multiplication by $1 - e_1$ and right multiplication by $1 - e_n$. Let $\ell(x) := \{\, y \in R \mid yx = 0 \,\}$ and $r(x) := \{\, y \in R \mid xy = 0 \,\}$ denote the *left annihilator* and the *right annihilator* of x in R, respectively. Thus, if $x \in R$ is a product of finitely many idempotents, then either $x = 1$, or $\ell(x) \neq 0$ and $r(x) \neq 0$. In this section, we study the rings for which the inverse of this implication holds. We call these rings *idempotent complete rings*. Thus a ring R is *idempotent complete* if, for every $x \in R$, $\ell(x) \neq 0$ and $r(x) \neq 0$ imply that x is a product of finitely many idempotents of R.

Examples 1 Here are some examples of idempotent complete rings.

(a) *Boolean rings*, that is, the rings in which every element is idempotent. They are necessarily commutative rings of characteristic two.

(b) The ring $M_n(k)$ of all $n \times n$ matrices with entries in a commutative field k [4].

(c) Any (not necessarily commutative) integral domain. In fact, if R is an integral domain, $x \in R$, $\ell(x) \neq 0$ and $r(x) \neq 0$, then $x = 0$, so that x is a product of idempotents.

(d) Recall that a right R-module M_R is said to be *Dedekind finite* (or *directly finite*) if, for every right module N_R, $M_R \oplus N_R \cong M_R$ implies $N_R = 0$. A ring R is *Dedekind finite* if the module R_R is Dedekind finite (equivalentily, if ${}_RR$ is Dedekind finite, if and only if every right invertible element of R is invertible, if and only if every left invertible element of R is invertible). Note that if a ring R satisfies the property

$$(*) \qquad \text{“for every } a \in R,\ \ell(a) \neq 0 \text{ if and only if } r(a) \neq 0\text{”},$$

then R must be Dedekind finite. Indeed, if $ab = 1$ and $ba \neq 1$, then $a(1 - ba) = 0$ and $1 - ba \neq 0$. It follows that $r(a) \neq 0$, so that $\ell(a) \neq 0$ by property (*). But $ab = 1$ implies $\ell(a) = 0$, a contradiction. In particular, reversible rings (i.e., the rings in which $ab = 0$ implies $ba = 0$) have this property. Reversible rings are abelian rings. In Proposition 2.1, we will consider idempotent complete abelian rings.

(e) A unit regular ring R satisfies property (*). Indeed, if $a \in R$, where R is a unit regular ring, then there exists a unit u in R such that $a = aua$ and, since ua is an idempotent element, we have $r(a) = r(Ra) = r(Rua) = (1 - ua)R$. Similarly, $\ell(a) = R(1 - au)$. For a unit regular ring R, Hannah and O'Meara proved that an element $a \in R$ is a product of idempotent elements of R if and only if $Rr(a) = R(1 - a)R$ [7, Corollary 1.4]. In particular, for a simple unit regular ring, any zerodivisor is a product of idempotent elements.

(f) Obviously, a nilpotent element a of a ring R always satisfies both the conditions $\ell(a) \neq 0$ and $r(a) \neq 0$. Typically, a strictly upper triangular square matrix is nilpotent. Let us show that, over any ring R, a strictly upper triangular matrix $A \in M_n(R)$ is always a product of idempotent matrices. We prove this by induction on n. If $n = 1$, such a matrix A is zero and there is nothing to prove. Assume that we have proved that any strictly upper square matrix of size $n \times n$ is a product of idempotent matrices. Let $A \in M_{n+1}(R)$ be an upper triangular matrix. The matrix A can be written as $A = \begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix}$, where $B \in M_n(R)$ is upper triangular and $C \in M_{n\times 1}(R)$. The bottom row in A consists entirely of zeros. Then

$$A = \begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_n & C \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}.$$

Now the matrix on the left hand side in the product is an idempotent matrix and $B \in M_n(R)$ is upper triangular, so that it is a product of idempotent matrices. This easily leads to the desired decomposition of A as a product of idempotent matrices.

(g) Any right and left artinian ring R satisfies property (*). In fact, if $a \in R$ and $r(a) = 0$, then left multiplication $\lambda_a \colon R_R \to R_R$ is a monomorphism. But R_R has finite length, so that there exists $b \in R$ with $ab = 1$. This implies that $\ell(a) = 0$.

(h) For any right and left quasi-euclidean domain R and any $n \geq 1$, the matrix ring $M_n(R)$ is idempotent complete [2, Theorem 25].

Recall that a ring is *abelian* if all its idempotents are central. Local rings, reduced rings (i.e., rings with no non-zero nilpotent element) and commutative rings are abelian.

PROPOSITION 2.1 *Let R be an abelian ring. Then R is idempotent complete if and only if either R is an integral domain or R is a boolean ring.*

Proof. We have already seen in Examples 1 ((a) and (c)) that integral domains and boolean rings are idempotent complete. In order to prove the converse, first remark that the product of idempotents is an idempotent, and hence every $x \in R$ with $\ell(x) \neq 0$ and $r(x) \neq 0$ is an idempotent. Let us prove that every $x \in R$ with either $r(x) \neq 0$ or $\ell(x) \neq 0$ is an idempotent. Let $x \in R$ be such that $0 \neq y \in r(x)$. Then $(yx)^2 = 0$, so that both $\ell(yx) \neq 0$ and $r(yx) \neq 0$ and hence yx is an idempotent. This gives $yx = (yx)^2 = 0$. In particular, $\ell(x) \neq 0$, as desired. The same arguments also show that if $\ell(x) \neq 0$, then x is an idempotent. If R is not a domain, there exists non-zerodivisors and hence nontrivial idempotents. Let e be such an idempotent. For any $r \in R$ we have that both er and $(1-e)r$ are zerodivisors and hence idempotents. So, $er = (er)^2 = er^2$ and similarly $(1-e)r = ((1-e)r)^2 = (1-e)r^2$. We thus get that, for any $r \in R$, $r = er + (1-e)r = er^2 + (1-e)r^2 = r^2$. This proves that the ring R is boolean. □

We say that a ring R is *indecomposable* if its central idempotents are only 0 and 1.

LEMMA 2.2 *Let R be an idempotent complete ring that is not indecomposable. Then all elements of R are products of idempotents. In particular, the identity is the unique regular element of R.*

Proof. If R is not an indecomposable ring, there exists a nontrivial central idempotent $e \in R$. Hence the ring decomposes as a ring direct product $R = eRe \times (1-e)R(1-e)$ in a nontrivial way. For every $x \in eRe$, both $\ell(x)$ and $r(x)$ contain $(1-e)R(1-e)$, hence are nonzero. Since R is idempotent complete, x is a product of idempotents in R, hence a product of idempotents in eRe. Similarly, every $y \in (1-e)R(1-e)$ is a product of idempotents in $(1-e)R(1-e)$. It follows that every element of R is a product of finitely many idempotents in R. □

Other families of rings for which right zerodivisors coincide with left zerodivisors are reversible rings (in particular, reduced rings), right and left co-hopfian rings and unit regular rings.

The commutative domains R with the property that the ring $M_n(R)$ of all $n \times n$ matrices with entries in R is idempotent complete for every $n \geq 1$ have received considerable attention [10]. It is very interesting to note that a Bézout domain R has the property that $M_n(R)$ is idempotent complete for every $n \geq 1$ if and only if every invertible matrix with entries in R is a product of elementary matrices [9].

Proposition 2.3 *If R is a ring that is not indecomposable and $n \geq 2$ is an integer, then the ring $M_n(R)$ is not idempotent complete.*

Proof. If R is a ring that is not indecomposable, R has a nontrivial central idempotent $e \in R$. The scalar matrix $\begin{pmatrix} e & & 0 \\ & \ddots & \\ 0 & & e \end{pmatrix}$ is a nontrivial central idempotent of $M_n(R)$. If $M_n(R)$ is idempotent complete, we can apply Lemma 2.2, getting that the identity is the unique regular element of $M_n(R)$. But the matrix $\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \cdots & \\ & & & & 1 \end{pmatrix}$ is invertible, hence it is a regular element. This contradiction proves the proposition. □

3. Regular maps, consecutive pairs and antichains of direct summands

Since our main objective is to work with products of idempotents, it is natural to analyze when the kernel and the image of a right module morphism are direct summands. This is the content of the first half of this section.

Lemma 3.1 *Let A, B, C be three right R-modules and $\alpha\colon A \to B$, $\beta\colon B \to C$ be morphisms. The following conditions are equivalent:*

(i) $\ker(\beta\alpha) = \ker(\alpha)$ *and* $\operatorname{im}(\beta\alpha) = \operatorname{im}(\beta)$.
(ii) $\operatorname{im}(\alpha) \oplus \ker(\beta) = B$.

Proof. (i) ⇒ (ii) Suppose that (i) holds. We must show that $B = \operatorname{im}(\alpha) \oplus \ker(\beta)$. Suppose $b \in \operatorname{im}(\alpha) \cap \ker(\beta)$. Then $b = \alpha(a)$ for some $a \in A$ and $\beta(b) = 0$. Thus $\beta\alpha(a) = 0$, so that $a \in \ker(\beta\alpha) = \ker(\alpha)$. Thus $\alpha(a) = 0$, that is, $b = 0$. This shows that $\operatorname{im}(\alpha) \cap \ker(\beta) = 0$. Now let b' be an arbitrary element of B. Then $\beta(b') \in \operatorname{im}(\beta) = \operatorname{im}(\beta\alpha)$, so that there exists $a' \in A$ with $\beta(b') = \beta\alpha(a')$. Thus $b' - \alpha(a') \in \ker(\beta)$, so that $b' \in \operatorname{im}(\alpha) + \ker(\beta)$. This proves that $B = \operatorname{im}(\alpha) \oplus \ker(\beta)$.

(ii) $\Rightarrow$ (i) Suppose that $\mathrm{im}(\alpha) \oplus \ker(\beta) = B$. Let $a \in \ker(\beta\alpha)$. Hence we have $\alpha(a) \in \mathrm{im}(\alpha) \cap \ker(\beta) = 0$. We thus get that $a \in \ker(\alpha)$. This shows that $\ker(\beta\alpha) \subseteq \ker(\alpha)$. The other inclusion is trivial.

The inclusion $\mathrm{im}(\beta\alpha) \subseteq \mathrm{im}(\beta)$ is trivial. Conversely, if $c \in \mathrm{im}(\beta)$, then $c = \beta(b)$ for some $b \in B$. Write $b \in \mathrm{im}(\alpha) \oplus \ker(\beta)$ as $b = \alpha(a) + k$, where $k \in \ker(\beta)$. Then $c = \beta(b) = \beta(\alpha(a) + k) = \beta\alpha(a) \in \mathrm{im}(\beta\alpha)$. □

The previous lemma motivates the following definition, that of consecutive pair of morphisms. Let A, B, C be three right R-modules and $\alpha\colon A \to B$, $\beta\colon B \to C$ be morphisms. We say that the pair (α, β) is a *consecutive pair* if the equivalent conditions of Lemma 3.1 hold.

Examples 2 Let us examine some particular cases. As above, let $\alpha\colon A \to B$ and $\beta\colon B \to C$ be morphisms of right R-modules.

(a) If $A = B$ and α is the identity 1_A of A, then the pair $(1_A, \beta)$ is a consecutive pair if and only if β is a monomorphism.
(b) If $B = C$ and β is the identity 1_B of B, then the pair $(\alpha, 1_B)$ is a consecutive pair if and only if α is an epimorphism.
(c) If $A = B = C$ and $\alpha = \beta$ is idempotent, then the pair (α, α) is a consecutive pair.

A notion related to that of consecutive pair is the notion of regular map. Let M and N be right R-modules. A morphism $f\colon A \to B$ is a *regular* morphism (or a *regular map*) if there exists a morphism $g\colon B \to A$ such that $f = fgf$. Any morphism g with this property is called a *quasi inverse* of f.

In the next lemma, we collect some characterizations of regular maps. The notation $A \subseteq^{\oplus} B$ indicates that A is a direct summand of B.

Lemma 3.2 *Let M and N be right R-modules. The following conditions are equivalent for a morphism $f\colon M \to N$.*

(i) *f is a regular map.*
(ii) $\ker(f) \subseteq^{\oplus} M$ *and* $\mathrm{im}(f) \subseteq^{\oplus} N$.
(iii) *There exists a morphism $g\colon N \to M$ such that (f, fg) and (gf, f) are consecutive pairs.*

Proof. (i) $\Rightarrow$ (ii) Suppose $f = fgf$. It is then easy to check that $\ker(f) = \ker(gf)$ and $\mathrm{im}(f) = \mathrm{im}(fg)$. Since fg and gf are idempotent endomorphisms, we immediately get that (ii) holds.

(ii) $\Rightarrow$ (i) Suppose that $M = \ker(f) \oplus M'$ and $N = \mathrm{im}(f) \oplus N'$ for suitable submodules M' and N'. Then $f|_{M'}$ gives an isomorphism between M' and $\mathrm{im}(f)$. Hence, for any $n \in N$, there exist a unique $m' \in M'$ and a unique $n' \in N'$ such that $n = f(m') + n'$. Define $g\colon N \to M$ by $g(n) = m'$. We get $fgf(m) = f(g(f(m) + 0)) = f(m)$ for every $m \in M$, as desired.

(i) $\Rightarrow$ (iii). If (i) holds, we have seen that $\ker(f) = \ker(gf)$ and $\mathrm{im}(f) = \mathrm{im}(fg)$. Since obviously we also have $\ker((fg)f) = \ker(f)$ and $\mathrm{im}(f(gf)) = \mathrm{im}(f)$, we conclude that both the pairs (f, fg) and (gf, f) are consecutive.

(iii) $\Rightarrow$ (ii) From (f, fg) and (gf, f) consecutive pairs, we get that $\mathrm{im}(f) \oplus \ker(fg) = N$ and $\mathrm{im}(gf) \oplus \ker(f) = M$. Thus (ii) holds. □

We say that a morphism $h \in \mathrm{Hom}_R(N, M)$ is a *reflexive inverse* of $f\colon M \to N$ if both $f = fhf$ and $h = hfh$. The next lemma shows that, in this case, there is a remarkable symmetry between the kernels and the images of the maps f and h.

Lemma 3.3 *Let $f\colon M \to N$ be a regular morphism. Then there exists a reflexive inverse $h\colon N \to M$ of f. Moreover, $M = \ker(f) \oplus \operatorname{im}(h)$ and $N = \ker(h) \oplus \operatorname{im}(f)$.*

Proof. Since f is regular, there exists $g\colon N \to M$ such that $f = fgf$. Set $h := gfg$. It is easily checked that $f = fhf$ and $h = hfh$. Now $hf \in \operatorname{End}_R(M)$ and $fh \in \operatorname{End}_R(N)$ are idempotents, so that $M = \ker(hf) \oplus \operatorname{im}(hf) = \ker(f) \oplus \operatorname{im}(h)$ and $N = \ker(fh) \oplus \operatorname{im}(fh) = \ker(h) \oplus \operatorname{im}(f)$. $\square$

Remarks 1 (a) If we assume N projective, then f is regular if and only if $\operatorname{im} f \subseteq^{\oplus} N$. Similarly, if we assume M injective, then f is regular if and only if $\ker f \subseteq^{\oplus} M$.

(b) If $f = fgf$ is a regular element as in Lemma 3.2(iii), the consecutive pair $(\alpha = f, \beta = fg)$ also satisfies that $\operatorname{im}(\beta) \subseteq^{\oplus} N$ and $\ker(\alpha) \subseteq^{\oplus} M$. Conversely, for a consecutive pair (α, β) such that $M \overset{\alpha}{\to} N \overset{\beta}{\to} N$, it is easy to check that $\operatorname{im}(\beta) \subseteq^{\oplus} N$ and $\ker(\alpha) \subseteq^{\oplus} M$ if and only if $\beta\alpha$ is regular, if and only if α and $\beta\alpha$ are both regular.

In the next proposition, we analyze consecutive pairs and their relation with regular maps in more details.

Proposition 3.4 *Let A, B, C be right R-modules and $\alpha\colon A \to B$, $\beta\colon B \to C$ be morphisms. Suppose $\operatorname{im}(\beta) \subseteq^{\oplus} C$ and $\ker(\alpha) \subseteq^{\oplus} A$. Then the following conditions are equivalent:*

(i) *The pair (α, β) is a consecutive pair.*
(ii) *There exist morphisms $\delta\colon B \to A$ and $\varepsilon\colon C \to B$ such that $\alpha = \varepsilon\beta\alpha$ and $\beta = \beta\alpha\delta$.*
(iii) *There exists an idempotent $e \in \operatorname{End}_R(B)$ such that*

$$\alpha \in e\operatorname{Hom}(A, B),\ e \in \alpha\operatorname{Hom}(B, A),\ \beta \in \operatorname{Hom}(B, C)e \text{ and } e \in \operatorname{Hom}(C, B)\beta.$$

(iv) *There exists an idempotent $e \in \operatorname{End}_R(B)$ such that $\operatorname{im}(e) = \operatorname{im}(\alpha)$ and $\ker(e) = \ker(\beta)$.*

Proof. (i) $\Rightarrow$ (ii) From the hypotheses (i), $\operatorname{im}(\beta) \subseteq^{\oplus} C$ and $\ker(\alpha) \subseteq^{\oplus} A$, it immediately follows that $\beta\alpha$ is regular. In particular, there exists $\gamma \in \operatorname{Hom}_R(C, A)$ such that $\beta\alpha\gamma\beta\alpha = \beta\alpha$, and hence $\beta\alpha\gamma$ is an idempotent in $\operatorname{End}_R(C)$. Since (i) holds, we know that $\operatorname{im}(\beta) = \operatorname{im}(\beta\alpha)$. Then, for any $b \in B$, there exists an element $a \in A$ such that $\beta(b) = \beta\alpha(a)$, so that $((1_C - \beta\alpha\gamma)\beta)(b) = ((1_C - \beta\alpha\gamma)\beta\alpha)(a) = 0$. Thus $(1_C - \beta\alpha\gamma)\beta = 0$, so that $\beta = \beta\alpha\gamma\beta$. Setting $\delta := \gamma\beta \in \operatorname{Hom}(B, A)$, we get $\beta = \beta\alpha\delta$, as desired. Similarly, $\beta\alpha\gamma\beta\alpha = \beta\alpha$ implies $\beta\alpha(1_A - \gamma\beta\alpha) = 0$. From $\ker(\beta\alpha) = \ker(\alpha)$, we obtain that $\alpha(1_A - \gamma\beta\alpha) = 0$, i.e., $\alpha = \alpha\gamma\beta\alpha$. Setting $\varepsilon := \alpha\gamma$, we get $\alpha = \varepsilon\beta\alpha$, as desired.

(ii) $\Rightarrow$ (iii) Suppose $\alpha = \varepsilon\beta\alpha$ and $\beta = \beta\alpha\delta$, where $\varepsilon \in \operatorname{Hom}_R(C, B)$ and $\delta \in \operatorname{Hom}_R(B, A)$. Set $e := \alpha\delta = \varepsilon\beta\alpha\delta = \varepsilon\beta$. We obtain $e\alpha = \alpha\delta\alpha = \varepsilon\beta\alpha = \alpha$ and also $(\alpha\delta)^2 = \alpha\delta$, that is, $e^2 = e$. Moreover, $\beta = \beta\alpha\delta = \beta e$.

(iii) $\Rightarrow$ (iv) It is easy to see that the conditions given in (iii) imply that $\operatorname{im}(\alpha) = \operatorname{im}(e) \subseteq^{\oplus} B$ and $\ker(\beta) = \ker(e) \subseteq^{\oplus} B$.

(iv) $\Rightarrow$ (i) This is obvious. $\square$

Now, let R be a ring and P_R be a finitely generated projective module. Let $\mathcal{S}(P)$

be the set of all direct summands of P_R and $\mathcal{X}$ be a subset of $\mathcal{S}(P)$. Clearly, for every $X, X' \in \mathcal{X}$, every epimorphism $X \to X'$ is an isomorphism if and only if, for every $X, X' \in \mathcal{X}$, every splitting monomorphism $X \to X'$ is an isomorphism, or, equivalently, if and only if, for every $X, X' \in \mathcal{X}$ and every pair of homomorphisms $\varphi\colon X \to X'$, $\psi\colon X' \to X$, $\psi\varphi = 1_X$ implies $\varphi\psi = 1_{X'}$. We will say that the subset $\mathcal{X}$ of $\mathcal{S}(P)$ is *Dedekind finite* if it satisfies any of these three equivalent conditions.

The following easy lemma will be useful.

LEMMA 3.5 *Let P_R be a finitely generated projective module and $\mathcal{X}$ be a Dedekind finite subset of $\mathcal{S}(P)$. If $A, B \in \mathcal{X}$ and $A \subseteq^{\oplus} B$, then $A = B$.*

Proof. Let $C \subseteq P$ be such that $A \oplus C = B$. Consider the projection π of B onto A with kernel C. Since $\mathcal{X}$ is Dedekind finite, we get that $C = 0$, and hence $A = B$. □

For a finitely generated projective module P, we say that a pair $(\mathcal{A}, \mathcal{B})$ of Dedekind finite subsets $\mathcal{A}, \mathcal{B}$ of $\mathcal{S}(P)$ is a pair of *complementary antichains* in $\mathcal{S}(P)$ if the following two conditions hold:

(1) For every $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $A \oplus B = P_R$.

(2) For every $B \in \mathcal{B}$ there exists $A \in \mathcal{A}$ such that $A \oplus B = P_R$.

Notice that if $(\mathcal{A}, \mathcal{B})$ is a pair of complementary antichains, then also $(\mathcal{B}, \mathcal{A})$ is a pair of complementary antichains.

If $(\mathcal{A}, \mathcal{B})$ is a pair of complementary antichains in $\mathcal{S}(P)$, an endomorphism α of P_R is an $(\mathcal{A}, \mathcal{B})$*-endomorphism* if $\operatorname{im}(\alpha) \in \mathcal{A}$ and $\ker(\alpha) \in \mathcal{B}$.

There is a link between $(\mathcal{A}, \mathcal{B})$-endomorphisms and consecutive pairs of maps, as the following lemma shows.

LEMMA 3.6 *Let $(\mathcal{A}, \mathcal{B})$ be a pair of complementary antichains in $\mathcal{S}(P_R)$ and let $f, g \in \operatorname{End}_R(P)$ be two $(\mathcal{A}, \mathcal{B})$-endomorphisms. Then the composite mapping fg is an $(\mathcal{A}, \mathcal{B})$-endomorphism if and only if (f, g) is a consecutive pair.*

Proof. If fg is an $(\mathcal{A}, \mathcal{B})$-endomorphism, then both $\ker(g)$ and $\ker(fg)$ are in $\mathcal{B}$. Since $\ker(g) \subseteq^{\oplus} \ker(fg)$, Lemma 3.5 shows that $\ker(fg) = \ker(g)$. Similarly, we also obtain that $\operatorname{im}(fg) = \operatorname{im}(f)$. This shows that (f, g) is a consecutive pair.

Conversely, if (f, g) is a consecutive pair, then $\ker(fg) = \ker(g) \in \mathcal{B}$ and $\operatorname{im}(fg) = \operatorname{im}(f) \in \mathcal{A}$, so that fg is an $(\mathcal{A}, \mathcal{B})$-endomorphism. □

Remark 1 Let us briefly consider the notion of Dedekind finite module. For any ring R, let $\operatorname{Df}(R)$ be the greatest non-negative integer n such that R^n_R is a Dedekind finite R-module, if such a non-negative integer n exists, and set $\operatorname{Df}(R) := \infty$ if no such a non-negative integer exists. Notice that $R^0_R = 0$ is always Dedekind finite, so that $\operatorname{Df}(R) = \infty$ if and only if R^t_R is Dedekind finite for every integer $t \geq 0$. Also, $\operatorname{Df}(R)$ is an integer n if and only if R^t_R is Dedekind finite for every integer $t \leq n$ and R^t_R is not Dedekind finite for every integer $t > n$.

There exist rings R with $\operatorname{Df}(R) = n$ for every $n = 0, 1, 2, \dots, \infty$. To see this, it suffices to take $R = \mathbb{Z}$, the ring of integers, for $n = \infty$. For $n < \infty$, consider the cyclic monoid $\mathbb{N}_0/\!\sim$, where $\sim$ is the principal congruence on the additive monoid $\mathbb{N}_0$ generated by the relation $n+1 \sim n+2$. Thus $\mathbb{N}_0/\!\sim\ = \{\, [t] \mid t \in \mathbb{N}_0 \,\}$ is the monoid with $n+2$ elements $[0], [1], [2], \dots, [n+1]$ in which $[t] = [n+1]$ for every integer $t \geq n+1$. By [5, Theorem 2.1], there exists a hereditary ring R with $\{0 = R^0_R, R_R = R^1_R, R^2_R, \dots, R^{n+1}_R\}$ as a set of representatives of the finitely generated projective right modules up to isomorphism, and $R^t_R \cong R^{n+1}_R$ for every $t \geq n+1$. Thus $\operatorname{Df}(R) = n$.

Examples 3 Here are some examples of pairs of complementary antichains in $\mathcal{S}(R^n_R)$.

(1) $\mathcal{F}_{n,r}$ *free* R*-modules of rank* r *with free complement of rank* $n-r$. Let r be an integer, $0 \le r \le n$. Let R be any ring with $\mathrm{Df}(R) \ge \max\{r, n-r\}$, so that both R^r_R and R^{n-r}_R are Dedekind finite R-modules (Remark 1). Set $\mathcal{F}_{n,r} := \{\, A \in \mathcal{S}(R^n_R) \mid A \cong R^r_R$ and $R^n_R/A \cong R^{n-r}_R \,\}$ and $\mathcal{F}_{n,n-r} := \{\, B \in \mathcal{S}(R^n_R) \mid B \cong R^{n-r}_R$ and $R^n_R/B \cong R^r_R \,\}$. Then $(\mathcal{F}_{n,r}, \mathcal{F}_{n,n-r})$ is a pair of complementary antichains in $\mathcal{S}(R^n_R)$.

(2) $\mathcal{F}'_{n,r}$ *free* R*-modules of rank* r *with arbitrary complement.* Let r be an integer, $0 \le r \le n$. Let R be a ring for which the module R_R cancels from direct sums, that is, $M_R \oplus R_R \cong N_R \oplus R_R$ implies $M_R \cong N_R$ for every pair M_R, N_R of right R-modules. For instance, R can be any semilocal ring or, more generally, any ring of stable range one. For any such ring R, one has that $\mathrm{Df}(R) = \infty$. Set $\mathcal{F}'_{n,r} := \{\, A \in \mathcal{S}(R^n_R) \mid A \cong R^r_R \,\}$ and $\mathcal{F}''_{n,r} := \{\, B \in \mathcal{S}(R^n_R) \mid R^n_R/B \cong R^r_R \,\}$. Then $(\mathcal{F}'_{n,r}, \mathcal{F}''_{n,r})$ is a pair of complementary antichains in $\mathcal{S}(R^n_R)$.

(3) $\mathcal{A}$ *projective* R*-modules of rank* r *and* $\mathcal{B}$ *projective* R*-modules of rank* $n-r$. Let R be a ring with a faithful projective rank function ρ [12, p. 5–6]. For instance, R can be a commutative integral domain, like in the case of [10], or more generally R can be any ring that can be embedded in a division ring, or a ring that can be embedded in a simple artinian ring, or R can be any IBN ring over which every finitely generated projective module is free (for instance, any local ring). Let $1 \le r \le n$ be any real number. Set $\mathcal{A} := \{\, A \in \mathcal{S}(R^n_R) \mid \rho(A) = r \,\}$ and $\mathcal{B} := \{\, B \in \mathcal{S}(R^n_R) \mid \rho(B) = n-r \,\}$. Then $(\mathcal{A}, \mathcal{B})$ is a pair of complementary antichains in $\mathcal{S}(R^n_R)$.

(4) Let R be a ring with a faithful projective rank function ρ like in Example (3). Let r be an integer, $0 \le r \le n$, $\mathcal{A}' := \{\, A \in \mathcal{S}(R^n_R) \mid A \cong R^r_R \,\}$ and $\mathcal{B}' := \{\, B \in \mathcal{S}(R^n_R) \mid R^n_R/B \cong R^r_R \,\}$ like in Example (2). Then $(\mathcal{A}', \mathcal{B}')$ is a pair of complementary antichains in $\mathcal{S}(R^n_R)$.

For a given decomposition $B \oplus C = M$ of a module M, the following lemma offers a description of all the submodules C_1 of M with $M = B \oplus C_1$.

Lemma 3.7 *Let* $M = B \oplus C$ *be a decomposition of* M *with projections* $\beta \colon M \to B$, $\gamma : M \to C$. *Then* $M = B \oplus C_1$ *if and only if there exists* $\theta \in \mathrm{End}(M)$ *such that* $C_1 = (\gamma - \beta\theta\gamma)(M)$

Proof. Suppose that $M = B \oplus C_1$ with projections β_1 onto B and γ_1 onto C_1. We will show that $\beta_1 = \beta + \beta\theta\gamma$ and $\gamma_1 = \gamma - \beta\theta\gamma$ with $\theta = \gamma - \gamma_1$. We have $B \subseteq \ker(\theta)$, so $\theta = \theta\beta + \theta\gamma = \theta\gamma$.

If $m = b + c = b_1 + c_1$, where $b, b_1 \in B, c \in C, c_1 \in C_1$, then $\theta(m) = \gamma(m) - \gamma_1(m) = c - c_1 = b_1 - b \in B$. Thus $\beta\theta = \theta$. Hence $\gamma_1 = \gamma - \theta = \gamma - \beta\theta\gamma$. Also $\beta_1 = 1_M - \gamma_1 = \beta + \gamma - \gamma_1 = \beta + \beta\theta\gamma$.

Conversely, if β_1, γ_1 are defined as above, that is $\beta_1 = \beta + \beta\theta\gamma$ and $\gamma_1 = \gamma - \beta\theta\gamma$ for any $\theta \in \mathrm{End}(M)$, then $\beta_1 + \gamma_1 = 1_M$, $\beta_1^2 = \beta_1$, $\gamma_1^2 = \gamma_1$, $\beta_1\gamma_1 = \gamma_1\beta_1 = 0$. Therefore, $M = \beta_1 M \oplus \gamma_1 M$. Since $\beta_1(M) \subseteq B$ and $\beta_1(b) = \beta(b) = b$ for $b \in B$, we have $M = B \oplus (\gamma - \beta\theta\gamma)(M)$, as required. □

Lemma 3.7 used repeatedly shows how we can build a sequence of decompositions of a module M changing one direct summand at a time.

4. Products of idempotents versus products of elementary matrices

Let P be a finitely generated projective right module over a ring R. We say that an endomorphism f of P is a *strongly regular* map if there exists $g \in \mathrm{End}_R(P)$ such that $f = fgf$ and $fg = gf$. Notice that, for a strongly regular map $f \in \mathrm{End}(P)$, we have that $\ker f = \ker(gf) = \ker(fg)$ and $\mathrm{im}(f) = \mathrm{im}(gf) = \mathrm{im}(fg)$, so that $P = \ker(gf) \oplus \mathrm{im}(gf) = \ker(f) \oplus \mathrm{im}(f)$ (cf. [8, Proposition 2.22]). Clearly, every idempotent endomorphism is strongly regular.

If $\alpha \in \mathrm{End}(A)$, we say that α is a *product of consecutive strongly regular endomorphisms* if there exist finitely many strongly regular $\varepsilon_1, \dots, \varepsilon_k \in \mathrm{End}_R(A)$ such that $\alpha = \varepsilon_1 \dots \varepsilon_k$ and $(\varepsilon_i, \varepsilon_{i+1})$ is a consecutive pair of morphisms for every $1 \le i \le k-1$.

THEOREM 4.1 *Let P_R be a finitely generated projective right module over a ring R and $(\mathcal{A}, \mathcal{B})$ be a pair of complementary antichains in $\mathcal{S}(P_R)$. The following three conditions are equivalent:*

(H($\mathcal{A}, \mathcal{B}$)) *For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, there exists an endomorphism β of P with $\mathrm{im}(\beta) = A$ and $\ker(\beta) = B$, which is a product $\beta = \varepsilon_1 \dots \varepsilon_k$ of consecutive strongly regular $(\mathcal{A}, \mathcal{B})$-endomorphisms.*

(S($\mathcal{A}, \mathcal{B}$)) *For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, there exist direct-sum decompositions*

$$P = A_1 \oplus B_1 = A_2 \oplus B_1 = A_2 \oplus B_2 = \cdots = A_k \oplus B_{k-1} = A_k \oplus B_k$$

with $A = A_1$ and $B = B_k$.

(HI($\mathcal{A}, \mathcal{B}$)) *For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, there exists an endomorphism β of P with $\mathrm{im}(\beta) = A$ and $\ker(\beta) = B$, which is a product $\beta = \varepsilon_1 \dots \varepsilon_k$ of consecutive idempotent $(\mathcal{A}, \mathcal{B})$-endomorphisms.*

Proof. (H($\mathcal{A}, \mathcal{B}$)) $\Rightarrow$ (S($\mathcal{A}, \mathcal{B}$)) Assume that (H($\mathcal{A}, \mathcal{B}$)) holds. Suppose $A \in \mathcal{A}$ and $B \in \mathcal{B}$. By hypothesis, there exists an endomorphism β of P_R with $\mathrm{im}(\beta) = A$ and $\ker(\beta) = B$, which is a product of consecutive strongly regular $(\mathcal{A}, \mathcal{B})$-endomorphisms ε_i: $\beta = \varepsilon_1 \cdots \varepsilon_k$. For every $i = 1, \dots, k$, set $A_i := \mathrm{im}(\varepsilon_i)$ and $B_i := \ker(\varepsilon_i)$, so that $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$. But ε_i is strongly regular, hence $P = A_i \oplus B_i$.

Let us show that $A = A_1$. The image $\mathrm{im}(\beta) = \mathrm{im}(\varepsilon_1 \dots \varepsilon_k)$ is a direct summand of P contained in the image of ε_1, so that $A \oplus C = A_1$. Lemma 3.5 implies that $A = A_1$.

Let us prove that $B = B_k$. The module $B_k = \ker(\varepsilon_k)$ is a direct summand of P contained in $\ker(\varepsilon_1 \dots \varepsilon_k) = \ker(\beta) = B_k$. Thus $B = B_k \oplus C$. Lemma 3.5 shows that $B = B_k$.

Finally, since $(\varepsilon_i, \varepsilon_{i+1})$ is a consecutive pair for $i = 1, \dots, k-1$, we have that $P = \mathrm{im}(\varepsilon_{i+1}) \oplus \ker(\varepsilon_i)$, that is, $P = A_{i+1} \oplus B_i$. This concludes the proof of the implication.

(S($\mathcal{A}, \mathcal{B}$)) $\Rightarrow$ (HI($\mathcal{A}, \mathcal{B}$)). Suppose that $A \in \mathcal{A}$ and $B \in \mathcal{B}$. By (S($\mathcal{A}, \mathcal{B}$)), there is a sequence of direct-sum decompositions

$$P = A_1 \oplus B_1 = A_2 \oplus B_1 = A_2 \oplus B_2 = \cdots = A_k \oplus B_{k-1} = A_k \oplus B_k.$$

of P with $A = A_1$ and $B = B_k$.

For every $i = 1, 2, \dots, k$, let $\varepsilon_i \colon P \to P$ be the idempotent endomorphism with $\mathrm{im}(\varepsilon_i) = A_i$ and $\ker(\varepsilon_i) = B_i$. Since $P = A_{i+1} \oplus B_i$, the map $\varepsilon_1 \dots \varepsilon_k$ is a product of consecutive idempotents. Let us prove, by induction on k, that $\operatorname{im}(\varepsilon_1 \dots \varepsilon_k) = \operatorname{im}(\varepsilon_1)$ and $\ker(\varepsilon_1 \dots \varepsilon_k) = \ker(\varepsilon_k)$. The case $k = 1$ is trivial. Notice that $A_2 \cong A_1 = A$ is in $\mathcal{A}$. By the inductive hypothesis, $\operatorname{im}(\varepsilon_2 \dots \varepsilon_k) = \operatorname{im}(\varepsilon_2) = A_2$ and $\ker(\varepsilon_2 \dots \varepsilon_k) = \ker(\varepsilon_k)$. Then

$$\begin{aligned} A = A_1 = \operatorname{im}(\varepsilon_1) = \varepsilon_1(P) = \varepsilon_1(A_2 \oplus B_1) = \\ = \varepsilon_1(A_2) = \varepsilon_1(\operatorname{im}(\varepsilon_2 \dots \varepsilon_k)) = \operatorname{im}(\varepsilon_1 \dots \varepsilon_k). \end{aligned}$$

Moreover, $B_k = \ker(\varepsilon_k)$ is a direct summand of P contained in $\ker(\varepsilon_1 \dots \varepsilon_k) = B$. Thus B_k is a direct summand of B, and these modules B_k and B are in $\mathcal{B}$, so $B_k = B$. This completes the induction and the proof of the implication.

The implication $(\mathrm{HI}(\mathcal{A}, \mathcal{B})) \Rightarrow (\mathrm{H}(\mathcal{A}, \mathcal{B}))$ is trivial, because every idempotent in $\operatorname{End}_R(P)$ is strongly regular. □

Definition 1 We say that a ring R is *right n-regular* if for every $n \times n$ invertible matrix $M = (b_{ij}) \in M_n(R)$ there exists $i, j = 1, 2, \dots, n$ such that $r(b_{ij}) = 0$.

Clearly, every ring is right 1-regular. Every integral domain is right n-regular for every n.

A ring R is not right n-regular if there exists an $n \times n$ invertible matrix $M = (b_{ij}) \in M_n(R)$ with $r(b_{ij}) \neq 0$ for every i and j. It easily follows that if R is not right n-regular and R is not right m-regular, then R is not right $(n+m)$-regular. Thus the set of all integers $n \geq 1$ for which R is not right n-regular is a subsemigroup, possibly empty, of the additive semigroup $\mathbb{N}$ of all positive integers. The subsemigroups of $\mathbb{N}$ have a well known structure. They are either empty, or they are ultimately d-segments, where $d \geq 1$ is an integer and a subset S of $\mathbb{N}$ is said to be *ultimately a d-segment* if there exists an $N \in \mathbb{N}$ such that, for $x \geq N$, we have $x \in S$ if and only if d divides x [11]. In particular, for a ring R, either R is n-regular for every $n \geq 1$, or there exists a smallest $n \geq 2$ for which R is not n-regular.

To give an example that shows that there exist commutative rings that are not right 2-regular, it is enough to consider the ring $R = k \times k$ and the matrix $\begin{pmatrix} (1,0) & (0,-1) \\ (0,1) & (1,0) \end{pmatrix}$, k being a field. The condition of n-regularity might seem harmless but in fact it is quite restrictive as is shown by the following lemma.

Lemma 4.2 *If a ring R is right n-regular for some $n > 1$, then the only idempotents of R are* 0 *and* 1*. In particular, R is Dedekind finite.*

Proof. If e is a nontrivial idempotent of R, then it is easy to check that the following matrices are invertible although all their entries are zerodivisors:

$$\begin{pmatrix} e & 1-e \\ -(1-e) & e \end{pmatrix} \qquad \begin{pmatrix} e & 1-e & 0 \\ 0 & e & 1-e \\ 1-e & 0 & e \end{pmatrix}$$

This shows that R is neither 2 or 3-regular. The subsemigroup of $\mathbb{N}$ generated by 2 and 3 is $\mathbb{N} \setminus \{1\}$. This concludes the proof of the first part of the statement.

It is clear that a ring without nontrivial idempotents must be Dedekind finite. □

Let us give an example showing that the absence of nontrivial idempotents is not sufficient to guarantee 2-regularity.

Examples 4 (a) Consider the commutative ring $R = \mathbb{Z}[x, y]/I$, where I is the ideal generated by the polynomials $2x, 3y$. It is easy to check that the matrix $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ is invertible, but all of its coefficients are zerodivisors. So R is not 2-regular. Let us check that the only idempotent elements in R are 0 and 1. In R, one has that $xy = 0$. In particular, any idempotent e of R can be written in the form $e = a+xp(x)+yq(y)$, where $a \in \mathbb{Z}$, $p(x) \in \mathbb{Z}[x]$ and $q(y) \in \mathbb{Z}[y]$. Applying the ring morphism $R \to \mathbb{Z}$ that sends both x and y to 0, one sees that $a = 0$ or $a = 1$. Possibly replacing $1 - e$ for e, we can suppose $a = 0$, and we must prove that $e = 0$ in R. Applying the ring morphism $R \to \mathbb{Z}_2[x]$ that sends y to 0, one sees that the element $e = xp(x) + yq(y)$ must be sent to an idempotent element $xp(x)$ of $\mathbb{Z}_2[x]$, i.e., it must be sent to the zero of $\mathbb{Z}_2[x]$. Thus all the coefficients of $p(x) \in \mathbb{Z}[x]$ are even. Similarly, applying the ring morphism $R \to \mathbb{Z}_3[y]$ that sends x to 0, one sees that the idempotent $e = xp(x) + yq(y)$ of R must be sent to the zero of $\mathbb{Z}_3[y]$, so that all the coefficients of $q(y) \in \mathbb{Z}[y]$ are divisible by 3. It follows that $e = a + xp(x) + yq(y) = 0$ in R.

(b) Any local ring is n-regular, for any $n \in \mathbb{N}$. Indeed if R is local with maximal ideal J and $A = (a_{ij}) \in M_n(R)$ is invertible, then the image of A in $M_n(R/J)$ is also invertible and hence there exist indices $k, l \in \{1, \dots n\}$ such that $a_{kl} + J$ is nonzero in the division ring R/J. Since J is the Jacobson radical of R we quickly conclude that a_{kl} is also invertible in R. This shows that R is n-regular.

Since we will now deal with $n \times n$ matrices over a (non-necessarily commutative) ring R, we prefer to fix the notation and recall some elementary facts. For a ring R and an integer $n \geq 1$, the free right R-module R^n_R will be viewed as the set of all $n \times 1$ matrices, that is, as a set of columns. Thus the ring $M_n(R)$ can be identified with the endomorphism ring $\mathrm{End}(R^n_R)$, identifying any matrix $A \in M_n(R)$ with the left multiplication $\lambda_A \colon R^n_R \to R^n_R$. For any matrix $A \in M_n(R)$, when we write $A = (a_1, a_2, \dots, a_n)$, we mean that a_j denotes the j-th column of A, which is an $n \times 1$ matrix, that is, an element of R^n_R.

Proposition 4.3 *Let R be a ring, $n \geq 1$ an integer, $A \in M_n(R)$ a matrix, and suppose that $A = (a_1, a_2, \dots, a_n)$. Then:*

(i) *The set $\{a_1, a_2, \dots, a_n\}$ is a set of generators for the right R-module R^n_R if and only if the matrix A is right invertible.*

(ii) *The set $\{a_1, a_2, \dots, a_n\}$ is an R-linearly independent subset of R^n_R if and only if the matrix A is left cancellable (that is, if $AX = AY$ with $X, Y \in M_n(R)$ implies $X = Y$; equivalently, if $AX = 0$ with $X \in M_n(R)$ implies $X = 0$).*

(iii) *The set $\{a_1, a_2, \dots, a_n\}$ is a free set of generators for the right R-module R^n_R if and only if the matrix A is invertible.*

Proof. (i) The set $\{a_1, a_2, \dots, a_n\}$ is a set of generators for R^n_R if and only if the endomorphism $\lambda_A \colon R^n_R \to R^n_R$ is an epimorphism. But the free module R^n_R is projective, so that λ_A is an epimorphism if and only if it is a splitting epimorphism, that is, if and only if the endomorphism λ_A is right invertible. In view of the isomorphism $M_n(R) \cong \mathrm{End}(R^n_R)$, this is equivalent to "the matrix A is right invertible".

(ii) The set $\{a_1, a_2, \dots, a_n\}$ is R-linearly independent if and only if the endomorphism $R^n_R \to R^n_R$, $e_j \mapsto a_j$ is a monomorphism, that is, if and only if $\lambda_A \colon R^n_R \to R^n_R$ is a monomorphism. This happens if and only if $AZ = 0$ implies $Z = 0$ for every $n \times 1$ matrix Z, that is, if and only if $AX = 0$ implies $X = 0$ for every $n \times n$ matrix X.

(iii) In view of (i) and (ii), it suffices to show that if a matrix A is right invertible and left cancellable, then A is invertible. Now A right invertible implies that $AB = 1$ for some $n \times n$ matrix B, so that $ABA = A$. Thus $A(BA - 1) = 0$. But A is left cancellable, so that B is a two-sided inverse of A. $\square$

Proposition 4.4 *The following conditions are equivalent for a ring R and an integer $n \geq 1$:*

(i) *For any free set $\{a_1, \dots, a_n\}$ of generators of R^n_R, there exist indices $i, j = 1, 2, \dots, n$ such that $\{a_1, \dots, a_{i-1}, \widehat{a_i}, a_{i+1}, \dots a_n, e_j\}$ is an R-linearly independent subset of R^n.*

(ii) *The ring R is right n-regular.*

Proof. (i) $\Rightarrow$ (ii). Suppose that (i) holds. Let $M = (b_{ij})$ be an invertible matrix. Let $M^{-1} = (a_1, \dots a_n)$ be its inverse, where a_j is an $n \times 1$ matrix. Since M^{-1} is invertible, the set $\{a_1, \dots, a_n\}$ is a free set of generators of $M_{n\times 1}(R) \cong R^n_R$. By (i), there exist $i, j = 1, 2, \dots, n$ such that $\{a_1, \dots \widehat{a_i}, \dots, a_n, e_j\}$ is an R-linearly independent subset of $M_{n\times 1}(R)$. Thus the matrix $M' = (a_1, \dots, a_{i-1}, e_j, a_{i+1}, \dots, a_n)$ has right annihilator $r(M') = 0$ in $M_n(R)$ (equivalently, the mapping $\lambda_{M'} \colon R^n \to R^n$ given by left multiplication by M' is injective). Set

$$M'' := (e_1, \dots, e_{i-1}, Me_j, e_{i+1}, \dots, e_n),$$

so that $M' = M^{-1}M''$. From $r(M') = 0$, it follows that $r(M'') = 0$. Thus, for every $(x_1, \dots, x_n)^t \in M_{n\times 1}(R)$, we have that $M''(x_1, \dots, x_n)^t = (0, \dots, 0)^t$ implies $(x_1, \dots, x_n) = (0, \dots, 0)$. Now, $M''(x_1, \dots, x_n)^t = (0, \dots, 0)^t$ is the equality

$$\begin{pmatrix} 1 & 0 & \dots & b_{1j} & 0 & \dots & 0 \\ & \ddots & & \vdots & & & \\ & & 1 & \vdots & 0 & & \\ & & & b_{ij} & & & \\ & & & \vdots & 1 & & \\ & & & \vdots & & \ddots & \\ & & & b_{nj} & & & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \\ \vdots \\ \\ \vdots \\ \\ x_n \end{pmatrix} = 0,$$

which is equivalent to the system

$$\begin{cases} x_1 + b_{1j}x_i = 0 \\ x_2 + b_{2j}x_i = 0 \\ \vdots \\ x_i + \widehat{b_{ij}x_i} = 0 \\ \vdots \\ x_n + b_{nj}x_i = 0 \\ b_{ij}x_i = 0 \end{cases} \tag{1}$$

The System (1) has only the zero solution if and only if $r(b_{ij}) = 0$, as we wanted to prove.

(ii) $\Rightarrow$ (i). Suppose R right n-regular. Let $\{a_1, \dots, a_n\}$ be a free set of generators of R^n and let M be the inverse of the matrix $(a_1, \dots, a_n)$. Since $M = (b_{ij})$ is invertible, there exist indices i, j such that $r(b_{ij}) = 0$. In the notation above, the System (1) has only the trivial solution. Equivalently, $r(M'') = 0$, so that $r(M') = r(M^{-1}M'') = 0$. Thus the subset $\{a_1, \dots \widehat{a_i}, \dots, a_n, e_j\}$ is an R-linearly independent subset of R^n_R. $\square$

For brevity, in the statement of the next Theorem and from now on, we will indicate by $(\mathrm{S}'_{n,1})$ the condition that until now we had denoted by $(\mathrm{S}(\mathcal{F}'_{n,1}, \mathcal{F}''_{n,1}))$, that is, the condition $(\mathrm{S}(\mathcal{A}, \mathcal{B}))$ relative to the pair $(\mathcal{F}'_{n,1}, \mathcal{F}''_{n,1})$ of Example 3(2). Recall that, after Definition 1, we have seen that, for a ring R, either R is n-regular for every $n \geq 1$, or there exists a smallest $n \geq 2$ for which R is not n-regular.

THEOREM 4.5 *Let $n \geq 1$ be an integer and R be an m-right regular ring for every $m = 2, 3, \dots, n$. Suppose that R satisfies the following condition:*

$(\mathrm{S}'_{n,1})$ *For any free direct summand A of rank $n-1$ of the free right R-module R^n_R and any free direct summand B of rank 1 of R^n_R, there exist direct-sum decompositions $R^n_R = A_1 \oplus B_1 = A_2 \oplus B_1 = A_2 \oplus B_2 = \cdots = A_k \oplus B_{k-1} = A_k \oplus B_k$ with $A = A_1$ and $B = B_k$.*

Then R also satisfies the following condition:

(GE_n) *Every invertible $n \times n$ matrix is a product of elementary matrices.*

Proof. The proof is by induction on $n \geq 1$. We must show that if R is right m-regular for every $m = 2, 3, \dots, n$ and $(\mathrm{S}'_{n,1})$ holds, then any invertible $n \times n$ matrix $M = (a_1, \dots, a_n) \in M_n(R)$ is a product of elementary matrices. The case $n = 1$ is trivial, because any invertible matrix is an elementary matrix. So assume $n \geq 2$, R m-right regular for every $m = 2, 3, \dots, n$ and that $(\mathrm{S}'_{n,1})$ holds. Let $M = (a_1, \dots, a_n) \in M_n(R)$ be a $n \times n$ invertible matrix. By Proposition 4.4, there exist $i, j = 1, 2, \dots, n$ for which $\{a_1, \dots, a_{i-1}, \widehat{a_i}, a_{i+1}, \dots, a_n, e_j\}$ is an R-linearly independent subset of R^n. Without loss of generality, we can assume that $j = n$, because the matrix M is a product of elementary matrices if and only if the matrix $P_{jn}M$ is a product of elementary matrices, where P_{jn} is the permutation matrix that exchanges row j and row n, and M is invertible if and only if $P_{jn}M$ is invertible. Set $A = \oplus_{i=1}^{n-1} a_i R$ and $B = e_n R$. The condition $(\mathrm{S}'_{n,1})$ is then satisfied, and we obtain a sequence of direct-sum decompositions with free summands

$$R^n_R = A_1 \oplus B_1 = A_2 \oplus B_1 = A_2 \oplus B_2 = \cdots = A_k \oplus B_{k-1} = A_k \oplus B_k,$$

where $A = A_1 \cong A_j$ and $B = B_k \cong B_j$ are free of rank $n-1$ and 1, respectively, for every $j = 1, 2, \dots, k$.

For $j = 1, 2, \dots, k$, let $\{a_{1j}, \dots, a_{n-1,j}\}$ be a free set of generators for the free module A_j and let $\{b_j\}$ be a free set of generators for B_j. In particular, for the free module $A = A_1$, take $a_{i1} = a_i$ for every $i = 1, \dots, n-1$ and, for the free module $B = B_k$, take $b_k = e_n$ We thus have $A_j = \oplus_{i=1}^{n-1} a_{ij} R$ and $B_j = b_j R$. Consider the following invertible matrices:

$$\begin{aligned} &E_j = (a_{1j}, a_{2j}, \dots, a_{n-1j}, b_j), && j = 1, 2, \dots, k \\ &F_j = (a_{1,j+1}, a_{2,j+1}, \dots, a_{n-1,j+1}, b_j), && j = 1, 2, \dots, k-1. \end{aligned}$$

Also, set $F_0 = M$, so $b_0 = a_n$.

Now consider the matrices

$$P_j = F_j^{-1}E_j, \text{ for } 1 \le j \le k, \quad \text{and} \quad Q_j = E_{j+1}^{-1}F_j, \text{ for } 0 \le j \le k-1.$$

Since the last column of E_j and the last column of F_j are equal, the last column of P_j is the unit column vector e_n. Similarly, the first $n-1$ columns of Q_j are the unit column vectors $e_1, \dots, e_{n-1}$. Now write the matrices P_j and Q_j as block matrices as follows:

$$P_j = \begin{pmatrix} S_j & 0 \\ s_j & 1 \end{pmatrix} \quad \text{and} \quad Q_j = \begin{pmatrix} I_{n-1} & t_j \\ 0 & u_j \end{pmatrix},$$

where S_j is an invertible matrix of size $(n-1) \times (n-1)$, $s_j \in M_{1\times(n-1)}(R)$, $t_j \in M_{(n-1)\times 1}(R)$ and u_j is an invertible element of R. The induction hypothesis applied to S_j shows that S_j is a product of elementary matrices. Now 1 is in the bottom right corner of P_j, so that, right multiplying P_j by elementary matrices, we can assume that the row s_j is the zero row $(0, \dots, 0) \in M_{1\times(n-1)}(R)$. From this and the fact that S_j is a product of elementary matrices, we get that P_j is also a product of elementary matrices. Since the element u is invertible, left multiplying by elementary matrices allows us to transform Q_j into a product of elementary matrices. This shows that the matrices P_j and Q_j are product of elementary matrices.

Now, we have that $M = F_0 = E_1Q_0$, so that

$$M = (E_kE_k^{-1})(F_{k-1}F_{k-1}^{-1}) \cdots (E_2E_2^{-1})(F_1F_1^{-1})E_1Q_0.$$

It follows that $M = E_k(E_k^{-1}F_{k-1})(F_{k-1}^{-1}E_{k-1}) \cdots (E_2^{-1}F_1)(F_1^{-1}E_1)Q_0$, that is, $M = E_kQ_{k-1}P_{k-1} \cdots Q_1P_1Q_0$.

The last column of E_k is the vector e_n, hence, as we have seen above, the inductive argument shows that E_k is a product of elementary matrices. Therefore the matrix M itself is a product of elementary matrices. □

For the proof of Theorem 4.10, we need two lemmas. The first one appeared in the master's thesis of Polidoro under the supervision of L. Salce. Recall that, if $a \in R$ and $i \neq j$, the *elementary transvection* $T_{ij}(a)$ is the $n \times n$ matrix $T_{ij}(a) = I_n + aE_{ij}$. If $u \in R$ is an invertible element, the *elementary dilation* $D_i(u)$ is the $n \times n$ matrix $D_i(u) = I_n - E_{ii} + uE_{ii}$.

LEMMA 4.6 *Every permutation matrix is a product of elementary dilations and elementary transvections.*

Proof. Since every permutation is a product of transpositions, it is enough to consider the transposition P_{ij} of two columns i and j. Now

$$P_{ij} = T_{ij}(1)T_{ji}(-1)T_{ij}(1)D_i(-1).$$

□

LEMMA 4.7 *Let R be a ring, $0 < r < n$ be integers, $R^n = R^r \oplus R^{n-r}$ be the canonical direct-sum decomposition of R^n, and $\varphi\colon R^n \to R^n$ be an automorphism. Let $\tau\colon R^n \to R^n$ be another automorphism given by left multiplication by a transvection or a dilation. Suppose that the image of the direct-sum decomposition $R^r \oplus R^{n-r}$*

via φ is $X \oplus Y$, i.e., suppose that $X = \varphi(R^r \oplus 0)$ and $Y = \varphi(0 \oplus R^{n-r})$. Then the image of the direct-sum decomposition $R^r \oplus R^{n-r}$ via $\varphi\tau$ is $X' \oplus Y'$, where either $X = X'$, or $Y = Y'$, or both.

Proof. If τ is left multiplication by a dilation $D_j(u)$ or a transvection $T_{ij}(a)$ with $i, j \leq r$ or $i, j \geq r$, then $\tau(R^r \oplus 0) = R^r \oplus 0$ and $\tau(0 \oplus R^{n-r}) = 0 \oplus R^{n-r}$, so that $\varphi\tau(R^r \oplus 0) = \varphi(R^r \oplus 0) = X$ and $\varphi\tau(0 \oplus R^{n-r}) = Y$.

If τ is left multiplication by a transvection $T_{ij}(a)$ with $i \leq r \leq j$, then $\tau(R^r \oplus 0) = R^r \oplus 0$, so that $\varphi\tau(R^r \oplus 0) = \varphi(R^r \oplus 0) = X$.

If τ is left multiplication by a transvection $T_{ij}(a)$ with $j \leq r \leq i$, then $\tau(0 \oplus R^{n-r}) = 0 \oplus R^{n-r}$, so that $\varphi\tau(0 \oplus R^{n-r}) = Y$. $\square$

Recall that a ring R satisfies $\mathrm{S}(\mathcal{F}_{n,n-r}, \mathcal{F}_{n,r})$ if, for any two direct-sum decompositions $R^n = A \oplus X = Y \oplus B$ with A, X, Y, B free right R-modules of ranks $n-r, r, n-r, r$, respectively, there exist direct-sum decompositions $R^n_R = A_1 \oplus B_1 = A_2 \oplus B_1 = A_2 \oplus B_2 = \cdots = A_k \oplus B_{k-1} = A_k \oplus B_k$ with $A = A_1$, $X = B_1$, $Y = A_k$ and $B = B_k$. From now on, for brevity, we will indicate the condition $\mathrm{S}(\mathcal{F}_{n,n-r}, \mathcal{F}_{n,r})$ by $(\mathrm{S}_{n,r})$. For instance, in the statement of the next Theorem, $(\mathrm{S}_{n,1})$ stands for $\mathrm{S}(\mathcal{F}_{n,n-1}, \mathcal{F}_{n,1})$.

THEOREM 4.8 *The following conditions are equivalent for a ring R and for an integer $n \geq 1$:*

(i) *R satisfies $(\mathrm{S}'_{n,1})$.*
(ii) (a) *R satisfies $(\mathrm{S}_{n,1})$ and*
 (b) *for any two direct-sum decompositions $R^n = A \oplus X = Y \oplus B$ with A, B free right R-modules of ranks $n-1, 1$, respectively, the submodules X, Y are free right R-modules of ranks $1, n-1$, respectively.*

Proof. (i) $\Rightarrow$ (ii) Suppose that R satisfies $(\mathrm{S}'_{n,1})$. Let $R^n = A \oplus X = Y \oplus B$ be two direct-sum decompositions with A, X, Y, B free right R-modules of ranks $n-1, 1, n-1, 1$, respectively. By $(\mathrm{S}'_{n,1})$, applied to the free direct summand Y of rank $n-1$ and X of rank 1, there exist direct-sum decompositions $R^n_R = Y_1 \oplus X_1 = Y_2 \oplus X_1 = Y_2 \oplus X_2 = \cdots = Y_k \oplus X_{k-1} = Y_k \oplus X_k$ with $Y = Y_1$ and $X = X_k$. Relabel the modules X_i and the modules Y_i, so that $X_1, X_2, \ldots, X_k$ become orderly $B_k, B_{k-1}, \ldots, B_1$ and $Y_1, \ldots, Y_k$ become orderly $A_{k+1}, A_k, \ldots, A_2$. Thus we have that there are direct-sum decompositions $R^n_R = A_2 \oplus B_1 = A_2 \oplus B_2 = \cdots = A_k \oplus B_k = A_{k+1} \oplus B_k$ with $Y = A_{k+1}$ and $X = B_1$. Now set $A_1 := A$ and $B_{k+1} := B$, so that the equalities $R^n = A \oplus X = Y \oplus B$ can be rewritten as $R^n = A_1 \oplus B_1 = A_{k+1} \oplus B_{k+1}$. Then we have that $R^n = A_1 \oplus B_1 = A_2 \oplus B_1 = A_2 \oplus B_2 = \cdots = A_k \oplus B_k = A_{k+1} \oplus B_k = A_{k+1} \oplus B_{k+1}$ with $A = A_1$, $X = B_1$, $Y = A_{k+1}$ and $B = B_{k+1}$, so that R satisfies $(\mathrm{S}_{n,1})$.

For (b), suppose that $R^n = A \oplus X = Y \oplus B$, with A, B free right R-modules of ranks $n-1, 1$, respectively. By (i), there exist direct-sum decompositions $R^n_R = A_1 \oplus B_1 = A_2 \oplus B_1 = A_2 \oplus B_2 = \cdots = A_k \oplus B_{k-1} = A_k \oplus B_k$ with $A = A_1$ and $B = B_k$. Then $A_1 \cong A_2, B_1 \cong B_2, \ldots, B_{k-1} \cong B_k$, so that $A_i \cong A$ and $B_i \cong B$ for every $i = 1, 2, \ldots, k$. Also, $X \cong B_1$ and $Y \cong A_k$. Thus $X \cong B$ and $Y \cong A$ are free modules of rank $1, n-1$, respectively.

(ii) $\Rightarrow$ (i) Suppose that (ii) holds. Let A, B be free direct summands of R^n_R of ranks $n-1, 1$, respectively. Thus there exist modules X, Y with $R^n = A \oplus X = Y \oplus B$. By (b), X and Y are free right R-modules of ranks $1, n-1$, respectively. By (a), there exist direct-sum decompositions $R^n_R = A_1 \oplus B_1 = A_2 \oplus B_1 = A_2 \oplus B_2 = \cdots = A_k \oplus B_{k-1} = A_k \oplus B_k$ with $A = A_1$, $X = B_1$, $Y = A_k$ and $B = B_k$. Thus (i) holds. $\square$

We will now briefly analyze the condition in Proposition 4.8(ii)(b). Recall that a *right unimodular row* is a $1 \times n$ matrix $(c_1, \dots, c_n)$ with entries in R such that $c_1R + \dots + c_nR = R$. Right unimodular rows are usually introduced in connection with the definition of right stable range of a ring R [13]. Moreover, "every right unimodular row can be completed to a square invertible matrix" is equivalent to "if $R_R^n \cong P \oplus R_R$, then $P \cong R_R^{n-1}$". Let us generalize this situation a little.

Let R be a ring and $0 < t < n$ be integers. We will say that a $t \times n$ matrix C with entries in R is *right unimodular* if left multiplication by C is a surjective mapping $\lambda_C \colon R_R^n \to R_R^t$. Here we are using the standard identification between matrices and module morphisms between free modules. Since R_R^t is a projective R-module, the mapping λ_C is surjective if and only if it is a split epimorphism, that is, if and only if λ_C is right invertible. It follows that C is right unimodular if and only if there exists a $n \times t$ matrix D with $CD = 1_t$. We will say that a $t \times n$ matrix C *can be completed to a square invertible matrix* if there exists a $(n-t) \times n$ matrix C' for which the $n \times n$ matrix $\binom{C}{C'}$ is an invertible matrix.

Proposition 4.9 *The following conditions are equivalent for a ring R and integers $0 < t < n$:*

(i) *For any direct-sum decomposition $R^n = A \oplus X$ with A free right R-module of rank t, the submodule X is a free right R-module of rank $n - t$.*
(ii) *Every right unimodular $t \times n$ matrix C with entries in R can be completed to a square invertible matrix.*

Proof. (i) $\Rightarrow$ (ii) Suppose that condition (i) holds. Let C be a right unimodular $t \times n$ matrix. The corresponding morphism $\lambda_C \colon R_R^n \to R_R^t$ is a split epimorphism, so that $R_R^n = K \oplus \ker(\lambda_C)$, the restriction of λ_C to K is an isomorphism $K \to R_R^t$ and the restriction of λ_C to $\ker(\lambda_C)$ is the zero morphism $\ker(\lambda_C) \to R_R^t$. Thus K is a free direct summand of R_R^n of rank t, so that, by (i), the right R-module $\ker(\lambda_C)$ is free of rank $n-t$. Thus there is an isomorphism $f \colon \ker(\lambda_C) \to R_R^{n-t}$, which can be be extended to a homomorphism $f' \colon R_R^n = K \oplus \ker(\lambda_C) \to R_R^{n-t}$, whose restriction to K is zero. Thus $\binom{\lambda_C}{f'} \colon R_R^n = \ker(\lambda_C) \oplus K \to R_R^t \oplus R_R^{n-t} = R_R^n$ is an isomorphism, whose matrix is a square invertible matrix that completes C, because the composite mapping of $\binom{\lambda_C}{f'}$ and the canonical projection $R_R^n = R_R^t \oplus R_R^{n-t} \to R_R^t$ has as its matrix the matrix of $(1\ 0)\binom{\lambda_C}{f'} = \lambda_C$, which is exactly C.

(ii) $\Rightarrow$ (i) Assume that (ii) holds and fix a direct-sum decomposition $R^n = A \oplus X$ with A free right R-module of rank t. Thus there is an epimorphism $f \colon R^n \to R^t$ whose restriction to A is an isomorphism $A \to R^t$ and whose restriction to X is the zero morphism. Then f is a split epimorphism, so that its matrix C with respect to the canonical bases is a right unimodular $t \times n$ matrix. By (ii), C can be completed to a square invertible matrix $\binom{C}{D}$, which corresponds to an automorphism g of R_R^n. In "matricial form", g can be written as a 2×2 matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \colon A \oplus X \to R_R^t \oplus R_R^{n-t}$, where α is an isomorphism, $\beta = 0$, the matrix of $(\alpha\ \beta) \colon R_R^n \to R_R^t$ with respect to the canonical bases is C, and the matrix of $(\gamma\ \delta) \colon R_R^n \to R_R^{n-t}$ with respect to the canonical bases is D. Now $\begin{pmatrix} 1 & 0 \\ -\gamma\alpha^{-1} & 1 \end{pmatrix}$ is an automorphism, because

$\begin{pmatrix} 1 & 0 \\ \gamma\alpha^{-1} & 1 \end{pmatrix}$ is its inverse. Thus $\begin{pmatrix} 1 & 0 \\ -\gamma\alpha^{-1} & 1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ -\gamma\alpha^{-1} & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ is an automorphism. It follows that $\delta \colon X \to R_R^{n-t}$ is an isomorphism. □

THEOREM 4.10 *Let R be a ring with the IBN property. If R satisfies* GE_n*, then R also satisfies* $\mathrm{S}(\mathcal{F}_{n,n-r}, \mathcal{F}_{n,r})$ *for every $r = 1, 2, \dots, n-1$.*

Proof. Suppose that GE_n holds. Let $R^n = A \oplus X = Y \oplus B$ be any two direct-sum decompositions of R^n with A, X, Y, B free right R-modules of rank $n-r, r, n-r, r$, respectively. Let φ and ψ be two automorphisms of R^n such that the images of the direct-sum decomposition $R^{n-r} \oplus R^r$ via ψ and φ are $A \oplus X$ and $Y \oplus B$, respectively. Let φ and ψ be given by left multiplication by the invertible matrices M and N respectively. The invertible matrix $X = N^{-1}M$ is a product of elementary matrices by GE_n. By Lemma 4.6, $M = NX = NE_1E_2 \cdots E_l$, where each matrix E_i is either a dilation or a transvection. Let τ_i be the automorphism of R^n given by left multiplication by E_i. Let the image of the direct-sum decomposition $R^r \oplus R^{n-r}$ via $\psi\tau_1\tau_2 \dots \tau_i$ be $Y_i \oplus X_i$. By Lemma 4.7, either $X_{i+1} = X_i$ or $Y_{i+1} = Y_i$. Thus we get a sequence of decompositions $R^n = Y_0 \oplus X_0 = Y_1 \oplus X_1 = \cdots = Y_l \oplus X_l$ with $Y_0 = Y, X_0 = B$, $X_{i+1} = X_i$ or $Y_{i+1} = Y_i$ for every i, $Y_l = A$ and $X_l = X$. Eliminating the useless repetitions that occur when $X_i = X_{i+1} = X_{i+2}$ or $Y_i = Y_{i+1} = Y_{i+2}$, we get the chain of direct-sum decompositions required in $\mathrm{S}(\mathcal{F}_{n,n-r}, \mathcal{F}_{n,r})$. □

THEOREM 4.11 *Let R be a ring with the IBN property, $n \geq 1$ an integer. Suppose that R be m-right regular for every $m = 2, 3, \dots, n$ and that for any two direct-sum decompositions $R^n = A \oplus X = Y \oplus B$ with A, B free right R-modules of ranks $n-1, 1$, respectively, the submodules X, Y are free right R-modules. Then the following conditions are equivalent.*

(H_n) *For every $r = 1, 2, \dots, n$ and every free direct summands $A \subseteq^{\oplus} R_R^n$ and $B \subseteq^{\oplus} R_R^n$, with A, B free R-modules of rank $r, n-r$ respectively, there exists an endomorphism β of R_R^n with $\operatorname{im}(\beta) = A$ and $\ker(\beta) = B$, which is a product $\beta = \varepsilon_1 \dots \varepsilon_k$ of consecutive strongly regular $(\mathcal{F}_{n,n-1}, \mathcal{F}_{n,1})$-endomorphisms.*

(S_n) *For every $r = 1, 2, \dots, n$ and every free direct summands $A \subseteq^{\oplus} R_R^n$ and $B \subseteq^{\oplus} R_R^n$, with A, B free R-modules of rank $r, n-r$ respectively, there exist direct-sum decompositions*

$$R_R^n = A_1 \oplus B_1 = A_2 \oplus B_1 = A_2 \oplus B_2 = \cdots = A_k \oplus B_{k-1} = A_k \oplus B_k$$

with $A = A_1$ and $B = B_k$.

(HI_n) *For every $r = 1, 2, \dots, n$ and every free direct summands $A \subseteq^{\oplus} R_R^n$ and $B \subseteq^{\oplus} R_R^n$, with A, B free R-modules of rank $r, n-r$ respectively, there exists an endomorphism β of R_R^n with $\operatorname{im}(\beta) = A$ and $\ker(\beta) = B$, which is a product $\beta = \varepsilon_1 \dots \varepsilon_k$ of consecutive idempotent $(\mathcal{F}_{n,n-1}, \mathcal{F}_{n,1})$-endomorphisms.*

($\mathrm{H}_{n,1}$) *For every free direct summands $A \subseteq^{\oplus} R_R^n$ and $B \subseteq^{\oplus} R_R^n$, with A, B free R-modules of rank $n-1, 1$ respectively, there exists an endomorphism β of R_R^n with $\operatorname{im}(\beta) = A$ and $\ker(\beta) = B$, which is a product $\beta = \varepsilon_1 \dots \varepsilon_k$ of consecutive strongly regular $(\mathcal{F}_{n,n-1}, \mathcal{F}_{n,1})$-endomorphisms.*

($\mathrm{S}_{n,1}$) *For every free direct summands $A \subseteq^{\oplus} R_R^n$ and $B \subseteq^{\oplus} R_R^n$, with A, B free*

R-modules of rank $n-1, 1$ respectively, there exist direct-sum decompositions

$$R_R^n = A_1 \oplus B_1 = A_2 \oplus B_1 = A_2 \oplus B_2 = \cdots = A_k \oplus B_{k-1} = A_k \oplus B_k$$

with $A = A_1$ and $B = B_k$.

($\mathrm{HI}_{n,1}$) *For every free direct summands $A \subseteq^{\oplus} R_R^n$ and $B \subseteq^{\oplus} R_R^n$, with A, B free R-modules of rank $n-1, 1$ respectively, there exists an endomorphism β of R_R^n with $\mathrm{im}(\beta) = A$ and $\ker(\beta) = B$, which is a product $\beta = \varepsilon_1 \dots \varepsilon_k$ of consecutive idempotent $(\mathcal{F}_{n,n-1}, \mathcal{F}_{n,1})$-endomorphisms.*

(GE_n) *Every invertible $n \times n$ matrix is a product of elementary matrices.*

Proof. First of all notice that the hypotheses of the Theorem imply that all the pairs $(\mathcal{F}_{n,r}, \mathcal{F}_{n,n-r})$, $0 \le r \le n$, are pairs of complementary antichains in $\mathcal{S}(R_R^n)$. To see this, it suffices to check that $\mathrm{Df}(R) \ge n$ (Example 3(1)), that is, it suffices to prove that R_R^n is a Dedekind finite R-module. Assume that $R_R^n \cong R_R^n \oplus C$. From the hypotheses of the Theorem, it follows that $R_R \cong R_R \oplus C$. If $n = 1$, from the same hypotheses, we get that C must be a free module of rank 0, so that $C = 0$. If $n > 1$, then R is Dedekind finite by Lemma 4.2, so that $C = 0$ also in this case. This proves that all the pairs $(\mathcal{F}_{n,r}, \mathcal{F}_{n,n-r})$ are pairs of complementary antichains, and therefore we can apply to them the results of Section 3.

The equivalence of (H_n), (S_n) and (HI_n) was proved in Theorem 4.1. It suffices to take, as $(\mathcal{A}, \mathcal{B}) := (\mathcal{F}_{n,r}, \mathcal{F}_{n,n-r})$, the pair of complementary antichains in $\mathcal{S}(R_R^n)$ where $\mathcal{F}_{n,i}$ consists of all free submodules of R_R^n rank i with free complement of rank $n-i$ (Example 3(1)). Similarly for the equivalence of ($\mathrm{H}_{n,1}$), ($\mathrm{S}_{n,1}$) and ($\mathrm{HI}_{n,1}$). Moreover, clearly, the first three equivalent conditions imply the second three equivalent conditions. Notice that the hypothesis "for any two direct-sum decompositions $R^n = A \oplus X = Y \oplus B$ with A, B free right R-modules of ranks $n-1, 1$, respectively, the submodules X, Y are free right R-modules" implies that X and Y have rank $1, n-1$ respectively, because R is IBN. Thus $(\mathcal{F}_{n,r}, \mathcal{F}_{n,n-r}) = (\mathcal{F}'_{n,r}, \mathcal{F}''_{n,r})$ (Examples 3(1) and (2)), and R satisfies ($\mathrm{S}'_{n,1}$) if and only if R satisfies ($\mathrm{S}_{n,1}$) (Proposition 4.8). Hence ($\mathrm{S}_{n,1}$) implies (GE_n) by Theorem 4.5.

Finally, (GE_n) implies (S_n) by Theorem 4.10. □

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