ARTINIAN PROPERTY OF CONSTANTS OF ALGEBRAIC q-SKEW DERIVATIONS

PIOTR GRZESZCZUK, ANDRE LEROY, AND JERZY MATCZUK

Abstract. Let δ denote a q-skew σ-derivation of an algebra R and $R^{(\delta)} = \{r \in R \mid \delta(r) = 0\}$ stand for the subalgebra of invariants. We prove that $R^{(\delta)}$ is left artinian iff R is left artinian provided R is semiprime and the action of δ on R is algebraic.

The subalgebras of invariants under the action of Hopf algebras have been extensively investigated by many authors (cf. [11]). In particular, the relations between various finiteness conditions of algebras and their subalgebras of invariants have been studied. The action of skew derivations naturally appears in this context, since skew primitive elements of Hopf algebras act as such maps.

In the paper we consider the behaviour of the artinian property under the action of a single q-skew σ-derivation δ on a semiprime algebra R. We prove, in Theorem 2.4, that R has to be left artinian provided the subalgebra of invariants $R^{(\delta)}$ is left artinian and the action of δ on R is algebraic. Theorem 4.5 offers the converse of the above result in case the action of σ on R is also algebraic. We do not know whether the additional assumption on σ is necessary.

When δ is a usual derivation (i.e. $\sigma = \mathrm{id}_R$), then the above theorems are known (cf.[5], [7]). It was also shown in [5] that the analogous theorems hold for the action of algebraic automorphisms under some extra assumptions on the characteristic of R. Notice that $\mathrm{id}_R - \sigma$ is 1-skew σ-derivation of R for any automorphism σ of R. Thus Theorems 2.4 and 4.5 also apply to invariants of the action of algebraic automorphisms and no assumptions on the characteristic of R are necessary.

1. Preliminaries

Let R be an associative algebra over a field K and let σ be a K-linear automorphism of R. Recall that a K-linear map $\delta\colon R \to R$ is a

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1991 *Mathematics Subject Classification.* 16W20, 16W25, 16W55.

This research was supported by the grant 190/R97/R98 in the frame of French-Polish joint projects and by Polish KBN grant no. 2 PO3A 039 14. We would like to thank all three universities for their hospitality.

σ-derivation if

$$\delta(rs) = \delta(r)s + \sigma(r)\delta(s),$$

for all $r, s \in R$. Furthermore, we say that δ is a q-skew derivation if there exists a nonzero $q \in K$ such that $\delta\sigma = q\sigma\delta$.

Subsets A of R such that $\sigma(A) = A$ and $\delta(A) \subseteq A$ are known as σ-stable and δ-stable, respectively. Subsets satisfying both properties are called (σ, δ)-stable.

If A is a (σ, δ)-stable subset of R, we let

$$A^{(\delta)} = \{a \in A \mid \delta(a) = 0\}$$

denote the invariants of A.

Henceforth we will assume that R is a semiprime algebra and δ is a q-skew derivation of R which is algebraic over K. By this we mean that there are $k, n \geq 0$ and elements $a_n, \dots, a_1, a_0 \in K$ such that

$$\text{(1.I)} \qquad a_n\delta^{n+k}(r) + a_{n-1}\delta^{n+k-1}(r) + \cdots + a_1\delta^{k+1}(r) + a_0\delta^k(r) = 0,$$

for all $r \in R$, where $a_0 \neq 0$. Clearly we may assume $a_0 = 1$. We let $t\colon R \to R$ be defined as

$$t = a_n\delta^n + \cdots + a_1\delta + id_R.$$

Sometimes, to emphasize that t is defined with respect to δ, we will write t_δ instead of t.

It is clear that t is a homomorphism of right $R^{(\delta)}$-modules and

$$t(R) = \{r \in R \mid \delta^m(r) = 0 \text{ for some } m \geq 1\} = \{r \in R \mid \delta^k(r) = 0\}.$$

If $k = 1$, then we say that δ is separable and in this special case, t maps R onto $R^{(\delta)}$. It is known (cf.[2]) that $t(R)$ is a (σ, δ)-semiprime subalgebra of R, provided δ is q-skew. For general skew derivations, $t(R)$ does not have to be a subring.

We begin with the following easy observation:

Lemma 1.1. *Let I be a nonzero σ-stable ideal of R. Then I contains a nonzero (σ, δ)-stable ideal and $I \cap R^{(\delta)} \neq 0$.*

Proof. Notice that since I is a σ-ideal of R, $\delta^i(I^{(n+k)}) \subseteq I$ for any $i \geq 0$, where $n+k$ is as in (1.I). Since δ is q-skew, the above inclusion together with semiprimeness of R imply that $\bigcap_{i=0}^{\infty} \delta^{-i}(I) \neq 0$ is a nonzero (σ, δ)-stable ideal of R contained in I. Now, the second part of the lemma follows from [2, Theorem 2]. □

In the sequel we will often make use of the following:

Lemma 1.2. *Suppose that either q is a primitive m-th root of unity or $q = 1$ and char$K = m$. Then:*

(1) *δ^m is a 1-skew σ^m-derivation.*
(2) $t_\delta(R) = t_{\delta^m}(R)$.

Proof. The first statement is a part of [3, Lemma 3.8].

$t_\delta(R)$ is the zero eigenspace of δ, thus it is also the zero eigenspace of δ^m which, by definition, is equal to $t_{\delta^m}(R)$. □

The following lemma is a generalization of [7, Lemma 2.1]; it collects basic properties of the map t.

Lemma 1.3. *Suppose that δ is 1-skew derivation, i.e. $\delta\sigma = \sigma\delta$. Let $T = t(R)$ and $A = \ker t$. Then:*

(1) $t(T) = T$.
(2) $AT \cap T = 0$
(3) *$R = T \oplus A$ as right $R^{(\delta)}$-modules.*
(4) *If I is a left ideal of T then $RI \cap T = I$.*
(5) *If L is a nonzero one-sided ideal of R, then $t(L) \neq 0$.*

Proof. The proof is similar to the one of [7, Lemma 2.1]. As an example we present the proof of (2) and (5).

(2) The restriction of δ to T is a nilpotent 1-skew derivation of T of index k. Let $T_i = \{r \in R \mid \delta^i(r) = 0\}$ for $i = 0, 1, \ldots, k$. Then $T_0 = 0$, $T_k = T$ and $\delta(T_i) \subseteq T_{i-1}$ for $0 < i \leq k$.

Since $\delta\sigma = \sigma\delta$, $t\sigma = \sigma t$ and consequently $A = \ker t$ is (σ, δ)-stable.

Let $1 \leq i \leq k$, $r \in T_i$ and $a \in A$. Using the definition of t it is easy to see that

$$\begin{aligned} t(ar) &\in t(a)r + \operatorname{span}_K \left\{\sigma^j\delta^s(a)\delta^l(r) \mid j, s \geq 0, \quad l \geq 1\right\} \\ &\subseteq t(a)r + AT_{i-1} = AT_{i-1} \end{aligned}$$

as $a \in A = \ker t$. Hence $t^k(AT) \subseteq AT_0 = 0$. Thus for $r \in AT \cap T_1$ $r = t(r) = t^k(r) = 0$. Therefore $AT \cap T = 0$ follows, as $AT \cap T$ is a δ-stable subspace of T.

(5) Let L be a one-sided ideal of R. Assume $t(L) = 0$. Since t, σ, δ commute, eventually enlarging L, we may assume that L is (σ, δ)-stable. Thus $\delta_{|L}$ is a 1-skew derivation of L satisfying $a_n(\delta_{|L})^n + \ldots + a_1\delta_{|L} + \mathrm{id}_L = 0$. This implies $L^{(\delta)} = 0$, so $L = 0$ by [2, Theorem 2]. □

As an immediate application of the above lemma we get the following:

Proposition 1.4. *Suppose that q is a root of unity. If R is left artinian then $t(R)$ is left artinian.*

Proof. By Lemma 1.2 we may assume that $q = 1$. Now the thesis is a consequence of Lemma 1.3(4). □

2. $R^{(\delta)}$ IS ARTINIAN IMPLIES R IS ARTINIAN

Suppose $\delta^k = 0$ and set $R_i = \{x \in R \mid \delta^i(x) = 0\}$ for $0 \leq i \leq k$. Notice that $R_0 = 0$, $R_1 = R^{(\delta)}$ and $R_k = R$. It is clear that $\delta(R_i) \subseteq R_{i-1}$ and, since δ is q-skew, $\sigma(R_i) = R_i$ for $1 \leq i \leq k$. We also have

$$\delta(R^{(\delta)}R_i) = \sigma(R^{(\delta)})\delta(R_i) \subseteq R^{(\delta)}R_{i-1}$$

for any $i \in \{1, 2, \ldots, k\}$. Using the above formula and an inductive argument, it is easy to check that $R^{(\delta)}R_i \subseteq R_i$ for any i. This shows that R_i's are left $R^{(\delta)}$-modules. In fact, one can see that R_i's are $(R^{(\delta)}, R^{(\delta)})$-bimodules.

The image of the restriction d_i of $\sigma^{-1}\delta$ to R_i is contained in R_{i-1} and $d_i \colon R_i \to R_{i-1}$ is a homomorphism of left $R^{(\delta)}$-modules whose kernel is contained in $R^{(\delta)}$. Similarly, the restriction of δ to R_i is a homomorphism of left $R^{(\delta)}$-modules into R_{i-1}. Therefore if ${}_{R^{(\delta)}}R^{(\delta)}$ $(R^{(\delta)}_{R^{(\delta)}})$ satisfies a module property which is closed with respect to taking submodules and extensions (for example: DCC, ACC, Goldie, Krull dimensions) then ${}_{R^{(\delta)}}R$ $(R_{R^{(\delta)}})$ satisfies the same property. Thus, in particular, we have:

Proposition 2.1. *Suppose that δ is algebraic. If $R^{(\delta)}$ is left (right) artinian, then $t(R)$ is left (right) artinian.*

Lemma 2.2. *Suppose that $R^{(\delta)}$ is left artinian. Then $t(R)$ is semiprime artinian.*

Proof. By [2, Theorem 6] and Proposition 2.1 we know that $t(R)$ is (σ, δ)-semiprime and left artinian. Moreover, when q is a root of unity, we may use Lemma 1.2 and assume that $q = 1$.

Let $\mathcal{B}$ denote the prime radical of $t(R)$. Notice that $\mathcal{B}$ is nilpotent since $t(R)$ is left artinian.

If either q is not a root of unity or $\mathrm{char}K = 0$ and $q = 1$ then, by [9, Lemma 2.5], $\mathcal{B}$ is (σ, δ)-stable. Hence $\mathcal{B} = 0$ and $t(R)$ is semiprime in this case.

Suppose $\mathrm{char}K = p \neq 0$ and $q = 1$. Then δ^{p^n} is a 1-skew σ^{p^n}-derivation for any $n \in \mathbb{N}$. Thus, eventually replacing δ by its suitable p-power, we may assume that δ is separable, i.e. $t(R) = R^{(\delta)}$. Then, by [2, Corollary 7], $t(R)$ is σ-semiprime and in fact semiprime, since $\mathcal{B}$ is nilpotent and σ-stable. □

Lemma 2.3. *Suppose q is not a root of unity. Then every algebraic q-skew derivation is nilpotent.*

Proof. Let $\bar{K}$ denote the algebraic closure of the field K. Replacing R by $R \otimes_K \bar{K}$, σ by $\sigma \otimes \mathrm{id}_{\bar{K}}$ and δ by $\delta \otimes \mathrm{id}_{\bar{K}}$ we may assume that δ is a

q-skew algebraic derivation of an algebra R over an algebraically closed field K. Let $0 \neq v \in R$ be an eigenvector of δ with eigenvalue λ. Then for any $i \in \mathbb{N}$ we have

$$\delta\sigma^i(v) = q^i\sigma^i\delta(v) = q^i\lambda\sigma^i(v).$$

Thus $q^i\lambda$ is an eigenvalue of δ for any $i \in \mathbb{N}$. Since δ is algebraic, it has only a finite number of different eigenvalues. This implies that $\lambda = 0$ is the only eigenvalue of δ since, by assumption, q is not a root of unity. Therefore δ is nilpotent. □

Now we are ready to prove the main result of this section.

Theorem 2.4. *Suppose that δ is an algebraic q-skew derivation of R. If $R^{(\delta)}$ is left artinian, then R is left artinian.*

Proof. Suppose $R^{(\delta)}$ is left artinian. Let $T = t(R)$. Then, by Lemma 2.2, T is semiprime artinian.

If q is not a root of unity then, by Lemma 2.3, δ is nilpotent. Thus $R = T$ and the thesis follows.

Suppose that q is a root of unity. Then, by Lemma 1.2, we may assume that $q = 1$.

Note that R is left non-singular. Indeed, if the left singular ideal Z of R would be nonzero then, by Lemmas 1.1 and 1.3(5), $Z \cap T$ would be a nonzero ideal of a semiprime artinian algebra. Thus Z would contain a nonzero idempotent, which is impossible.

We claim that if L is an essential left ideal of R, then L contains an essential left ideal L_* which is δ-stable. To this end it is enough to show that $L \cap \delta^{-1}(L)$ is essential. Let $a \in L$ be be such that $(L \cap \delta^{-1}(L)) \cap Ra = 0$ and $A = \{r \in R \mid r\delta(a) \in L\}$. Then A is an essential left ideal of R and for any $r \in A$ we have

$$\sigma^{-1}(r)a \in L \ \text{ and } \ \delta(\sigma^{-1}(r)a) = \delta(\sigma^{-1}(r))a + r\delta(a) \in L.$$

This shows that $\sigma^{-1}(r)(a) \in (L \cap \delta^{-1}(L)) \cap Ra = 0$. Thus $\sigma^{-1}(A)a = 0$ and $a = 0$ follows, since R is left non-singular. This implies that $L \cap \delta^{-1}(L)$ is essential.

Now we will show that R does not contain proper essential left ideals. Suppose that L is a proper essential left ideal of R. By the above we may assume that L is δ-stable. Then $L_0 = t(L) = L \cap T$ and $L_0 \neq T$ as $1 \notin L_0$. Since T is semiprime artinian we can pick $0 \neq b \in T$ such that $L_0 \cap Tb = 0$. First note that $b \notin R^{(\delta)}$. Let $\hat{L} = \{r \in L \mid rb \in L\}$. Then $\hat{L}$ and $\hat{L}_*$ are essential left ideals of R. Moreover $\hat{L}_*b \neq 0$, because R is left non-singular. If $b \in R^{(\delta)}$ then, by Lemma 1.3(5), we would get

$$0 \neq t(\hat{L}_*b) = t(\hat{L}_*)b \subseteq L \cap Tb = L_0 \cap Tb = 0$$

which is impossible.

Set $T_i = \ker \delta^i$, $0 \leq i \leq k$ where k is as in (1.I). Let s be the smallest integer such that there exist $0 \neq b \in T_s$ and essential δ-stable left ideal L of R such that $L_0 \cap Tb = 0$. The above implies that $s > 1$. Moreover, since $\delta(b) \in T_{s-1}$, $J_0 \cap T\delta(b) \neq 0$ for any essential δ-stable left ideal J of R. Let $\mathcal{S}$ denote the socle of R. Notice that $\mathcal{S} = \bigcap \{J \mid J \text{ is essential in } R\} = \bigcap \{J_* \mid J \text{ is essential in } R\}$. Hence, since $\mathcal{S}$ is σ-stable, both $\mathcal{S}$ and $\mathcal{S}_0 = t(\mathcal{S}) = \mathcal{S} \cap T$ are (σ, δ)-stable. Because T is left artinian and the intersection of finite number of essential left ideals is essential, there exist an essential δ-stable left ideal J of R such that $J \subseteq L$ and

$$\mathcal{S}_0 \cap T\delta(b) = J_0 \cap T\delta(b) \neq 0. \tag{2.I}$$

On the other hand, $\mathcal{S}_0 b \subseteq \mathcal{S}_0 \cap T \subseteq L \cap T = 0$. Since $\mathcal{S}_0$ is (σ, δ)-stable, this yields also that $\mathcal{S}_0\delta(b) = 0$. Therefore $(\mathcal{S}_0 \cap T\delta(b))^2 \subseteq \mathcal{S}_0 T\delta(b) = \mathcal{S}_0\delta(b) = 0$ and $\mathcal{S}_0 \cap T\delta(b) = 0$ follows, as $\mathcal{S}_0 \cap T\delta(b)$ is a left ideal of a semiprime algebra T. This contradicts (2.I), and shows that R does not contain proper essential left ideals. Therefore every left ideal of R is a direct summand of R and, consequently, R is left artinian. □

3. Going down in case when δ is nilpotent

Throughout this section we additionally assume that R is σ-simple (i.e. R does not contain σ-stable ideals), δ is a nilpotent q-skew σ-derivation.

$R[x; \sigma, \delta]$ denote the skew polynomial ring with coefficients written on the left. It is known (cf. [3])) that σ has an extension to an automorphism of $R[x; \sigma, \delta]$ such that $\sigma(x) = q^{-1}x$.

Consider the natural number $n = n(R) = \min\{k \mid \mathrm{r.ann}_R(\delta^k(R)) \neq 0\}$. Let (x^n) be a two-sided ideal of $R[x; \sigma, \delta]$ generated by x^n. Notice that any element from (x^n) is of the form $\sum_{i \geq 0} r_i x^i$, where r_0 belongs to $R\delta^n(R)$. Therefore $R \cap (x^n) \subseteq R\delta^n(R)$, so $R \cap (x^n)$ is σ-stable ideal of R with nonzero right annihilator. The assumption imposed on R implies that $R \cap (x^n) = 0$. Let M denote a σ-stable ideal of $R[x; \sigma, \delta]$ containing x^n and maximal with respect to the property that $R \cap M = 0$. Next we let $\widetilde{R}$ denote the factor ring $R[x; \sigma, \delta]/M$ and we let y be the image of x in $\widetilde{R}$. Then it is clear that R embeds in $\widetilde{R}$. Note that σ can be extended to an automorphism of $\widetilde{R}$ such that $\sigma(y) = q^{-1}y$ and δ can be viewed as an inner q-skew σ-derivation of $\widetilde{R}$ induced by a nilpotent element y. The construction of $\widetilde{R}$ and σ-simplicity of R yield that $\widetilde{R}$ is also σ-simple.

We use the following rule (cf. [3, 2.5])

$$y^k a = \sum_{i=0}^{k} \binom{k}{i}_q \sigma^{k-i}\delta^i(a) y^{k-i} \tag{3.I}$$

for any $a \in R$ and any nonnegative integer k, where $\binom{k}{i}_q$ is the evaluation at $t = q$ of the polynomial function

$$\binom{k}{i}_t = \frac{(t^k-1)(t^{k-1}-1)\cdots(t^{k-i+1}-1)}{(t^i-1)(t^{i-1}-1)\cdots(t-1)}.$$

Following [3] recall that if q is a primitive m-th root of unity, then

$$\binom{km}{jm}_q = \binom{k}{j} \tag{3.II}$$

for $j = 0, 1, \ldots, k$ and

$$\binom{km}{i}_q = 0 \tag{3.III}$$

for any i which is not divisible by m. Moreover, if $0 \le s \le m-1$ and $0 \le i \le s$, then

$$\binom{mk+s}{i}_q = \binom{s}{i}_q. \tag{3.IV}$$

In particular $\binom{k}{1}_q$ is nonzero for any positive integer k which is not divisible by m.

Lemma 3.1. *There exists a nontrivial right $R^{(\delta)}$-module homomorphism $\theta\colon R \to R^{(\delta)}$ such that $\theta(A) \neq 0$ for every nonzero left (σ,δ)-stable ideal A of R.*

Proof. First notice that if A is a nonzero left (σ,δ)-ideal of R, then $\delta^{n-1}(A) \neq 0$. To this end, assume $\delta^{n-1}(A) = 0$ and take $0 \neq a \in A$, such that $\delta(a) = 0$. Then $0 = \delta^{n-1}(Ra) = \delta^{n-1}(R)a$. Hence $a \in \mathrm{r.ann}_R(\delta^{n-1}(R))$, a contradiction.

Since $y^n = 0$ in $\widetilde{R}$, using the formula (3.I) we get

$$\binom{n}{1}_q \sigma^{n-1}\delta(r)y^{n-1} + \cdots + \binom{n}{n-1}_q \sigma\delta^{n-1}(r)y + \delta^n(r) = 0 \tag{3.V}$$

for any $r \in R$

Consider the case when $\binom{n}{n-1}_q \neq 0$. It holds in particular if either q is not a root of unity or $q = 1$ and $\mathrm{char} K = 0$ or if q is a primitive m-th root of unity and n is not divisible by m. Using the above equality and the fact that δ^{n-1} is nonzero on any nonzero (σ,δ)-stable left ideal of

R one gets, by easy induction, that in this case $L = \mathrm{l.ann}_R(y) \neq 0$. Notice that L is a left (σ, δ)-stable ideal of R. Hence $LR = R$ and $Ry = LRy \subseteq LyR + L\delta(R) = L\delta(R)$. Thus $Ry \subseteq R$.

Now assume that q is a primitive m-th root of unity. If $n = mk$ and k is not divisible by characteristic of K, then by (3.II) $\binom{n}{m(k-1)}_q = \binom{k}{k-1} = k \neq 0$. Applying the same argument one obtains that $L = \mathrm{l.ann}_R(y^m) \neq 0$. Since δ^m is an ordinary derivation, we obtain that $Ry^m \subseteq R$.

Finally if $n = mp^s l$ (where $p = \mathrm{char} K > 0$ and $p \nmid l$), then $\binom{n}{mp^s(l-1)}_q = \binom{p^s l}{p^s(l-1)} = \binom{l}{l-1} \neq 0$. In this case one gets that $L = \mathrm{l.ann}_R(y^{mp^s}) \neq 0$ and since δ^{mp^s} is a derivation, $Ry^{mp^s} \subseteq R$.

Consider the map $\theta\colon R \to \widetilde{R}$ given by

$$\theta(r) = \sum_{k=0}^{n-1} q^{-(n-1)k} y^{n-k-1} \sigma^k(r) y^k \tag{3.VI}$$

Since $y^n = 0$ and $\sigma(\theta(r)) = \sum_{k=0}^{n-1} q^{-(n-1)(k+1)} y^{n-k-1} \sigma^{k+1}(r) y^k$, it follows that $\delta(\theta(r)) = y\theta(r) - \sigma(\theta(r))y = 0$. Therefore θ maps R into $\widetilde{R}^{(\delta)}$. We will show that $\theta(R) \subseteq R^{(\delta)}$. This is clear when n is not divisible by m, since $Ry \subseteq R$.

Assume that n is divisible by m, that is $n = mk$. Making use of the formula (3.I) we may write

$$\theta(r) = \sum_{i=0}^{n-1} S_i \sigma^{n-1-i} \delta^k(r) y^{n-1-i} \quad \text{where } S_i = \sum_{j=i}^{n-1} \binom{j}{i}_q q^{-(n-1-j)(n-1-i)}$$

Moreover an easy computation shows that the above formula reduces to the following

$$\theta(r) = \sum_{j=0}^{k} \binom{k}{j} \sigma^{(k-j)m} \delta^{jm-1}(r) y^{(k-j)m}.$$

This gives $\theta(R) \subseteq R^{(\delta)}$, since $Ry^m \subseteq R$ in this case.

If in addition $\mathrm{char} K = p > 0$ and $n = mp^s l$ where $p \nmid l$, then it is easy to see that

$$\theta(r) = \sum_{j=0}^{l} \binom{l}{j} \sigma^{(l-j)mp^s} \delta^{jmp^s-1}(r) y^{(l-j)mp^s}$$

But, in this case $Ry^{mp^s} \subseteq R$, so $\theta(R) \subseteq R$.

Now let A be a nonzero left (σ,δ)-stable ideal of R. Assume that $\theta(A)=0$. The above formulas for θ imply that depending on characteristic of K we have either

$$\delta^{n-1}(A)y^{n-1}=\theta(A)y^{m-1}=0$$

or

$$\delta^{n-1}(A)y^{m(k-1)}=\theta(A)y^{m(k-1)}=0$$

or

$$\delta^{n-1}(A)y^{mp^s(l-1)}=\theta(A)y^{mp^s(l-1)}=0$$

Therefore $A'=\mathrm{l.ann}_A(y^{n-1})$ (respectively, $A'=\mathrm{l.ann}_A(y^{m(k-1)})$ or $A'=\mathrm{l.ann}_A(y^{mp^s(l-1)})$) is a nonzero (σ,δ)-stable left ideal of R. Repeating this procedure we can construct a nonzero (σ,δ)-stable left ideal B of R such that $\theta(B)=0$ and $By=0$ ($By^m=0$ or $By^{mp^s}=0$). Hence again the above formulas for θ imply that $\delta^{n-1}(B)=0$, a contradiction with non-triviality of δ^{n-1} on nonzero (σ,δ)-stable left ideals of R. □

As a side effect of the proof of the above lemma we obtain the following generalization of a classical result of Herstein:

Corollary 3.2. *Suppose that $q^i+q^{i-1}+\ldots+1\neq 0$ for all $i\in\mathbb{N}$ and δ is a nilpotent q-skew derivation of a σ-simple algebra R. Then there exists a nilpotent element $a\in R$ such that $\delta(r)=ar-r^\sigma a$ for all $r\in R$, i.e. δ is an inner σ-derivation adjoint to a.*

Proof. We know that δ can be view as an inner q-skew σ-derivation of $\widetilde{R}$ adjoint to a nilpotent element y. We have seen in the proof of Lemma 3.1 that the formula (3.V) and the assumption imposed on q imply that $Ry\subseteq R$. This means that $y\in R$. Thus $R=\widetilde{R}$ and the thesis follows. □

Henceforth θ will denote the homomorphism defined in Lemma 3.1 and T will stand for $\theta(R)$.

Corollary 3.3. *If $f\in R^{(\delta)}$ is such that $\sigma(f)=f$ and $Ty^{n-1}f=0$, then $y^{n-1}f=0$.*

Proof. Consider the extension of θ to $\widetilde{R}$. Recall that $\widetilde{R}$ is σ-simple. Lemma 3.1 implies, in particular, that $\theta(\widetilde{A})\neq 0$ for any nonzero (σ,δ)-stable left ideal $\widetilde{A}$ of $\widetilde{R}$. Notice that in our situation $y^{n-1}f\in\widetilde{R}^{(\delta)}$ and $Ry^{n-1}f$ is a (σ,δ)-stable left ideal of $\widetilde{R}$. Hence if $y^{n-1}f\neq 0$, then $0\neq\theta(Ry^{n-1}f)=Ty^{n-1}f$. Consequently $y^{n-1}f=0$. □

Lemma 3.4. *If R is left artinian and $\mathcal{B}$ is the prime radical of $R^{(\delta)}$, then $\mathcal{B}$ is nilpotent and $y^{n-1}\mathcal{B}=\mathcal{B}y^{n-1}=0$.*

Proof. It is well known that nil subrings of artinian rings are nilpotent. Hence $\mathcal{B}$ is nilpotent. Notice that $\theta(R\mathcal{B}) = \theta(R)\mathcal{B} \subseteq \mathcal{B}$. Since $\mathcal{B}$ is nilpotent, there exists an integer l such that $\theta(R\mathcal{B})^l = 0$. Then

$$0 = \theta(R\mathcal{B})^l y^{n-1} = y^{n-1}R\mathcal{B}y^{n-1} \cdots y^{n-1}R\mathcal{B}y^{n-1}.$$

Hence $(R\mathcal{B}y^{n-1})^{l+1} = 0$. On the other hand $R\mathcal{B}y^{n-1}$ is a (σ, δ)-stable left ideal of $\widetilde{R}$, so $R\mathcal{B}y^{n-1} = 0$ and in particular $\mathcal{B}y^{n-1} = 0$. Since for any $b \in R^{(\delta)}$ $yb = \sigma(b)y$, we have also that $y^{n-1}\mathcal{B} = 0$. □

It is easy to check, using the formula (3.VI) defining θ and $\sigma(y) = q^{-1}y$, that $\theta\sigma = q^{n-1}\sigma\theta$ and $\theta(arb) = \sigma^{n-1}(a)\theta(r)b$ for all $a, b \in R^{(\delta)}$ and $r \in R$. This shows that T is a nonzero σ-stable ideal of $R^{(\delta)}$. Let $\bar{T} = T/\mathcal{B}(T)$, where $\mathcal{B}(T) = T \cap \mathcal{B}(R^{(\delta)})$ is the prime radical of T. $^{-}$will denote the canonical homomorphism from T to $\bar{T}$. Keeping the above notation we have:

Lemma 3.5. *Suppose R is left artinian. Then $\bar{T}$ is a semiprime artinian algebra.*

Proof. First notice that $\bar{T} \neq 0$. Indeed, RT is a nonzero (σ, δ)-stable left ideal of R. Thus, by Lemma 3.1, $0 \neq \theta(RT) = T^2$. Hence, using simple inductive argument, we get $T^{k+1} = \theta(RT^k) \neq 0$ for any $k \in \mathbb{N}$. On the other hand, Lemma 3.4 yields that $\mathcal{B}(T)$ is nilpotent. Thus $T \neq \mathcal{B}(T)$.

Let $\bar{L}$ denote a nonzero left ideal of $\bar{T}$. Then $\bar{T}\bar{L} \neq 0$, because $\bar{T}$ is semiprime. We claim that $\bar{L}$ contains a minimal left ideal. Indeed, if $\bar{L}$ is not minimal then there exists a left ideal L_1 of T such that $L \supset L_1 \supset \mathcal{B}(T)$ and $\bar{L} \supset \bar{T}\bar{L}_1 \neq 0$. If $\bar{T}\bar{L}_1$ is not minimal, we can pick a left ideal L_2 of T such that $TL_1 \supset L_2 \supset \mathcal{B}(T)$. Then $\bar{T}\bar{L}_1 \supset \bar{T}\bar{L}_2 \neq 0$. Thus, continuing this process we can find a descending chain $\{L_k\}$ of left ideals of T such that

$$TL_1 \supset TL_2 \supset \ldots \supset \mathcal{B}(T). \tag{3.VII}$$

Consider the descending chain $RL_1 \supseteq RL_2 \supseteq \ldots$ of left ideals of R. Since R is left artinian, there is $k \in \mathbb{N}$ such that $RL_k = RL_{k+1}$. Hence, applying θ, we obtain $TL_k = TL_{k+1}$. This contradicts (3.VII) and shows every nonzero left ideal $\bar{L}$ of $\bar{T}$ contains a minimal left ideal. This means that the socle $\mathrm{Soc}(\bar{T})$ of $\bar{T}$ is essential in $\bar{T}$.

On the other hand $\mathrm{Soc}(\bar{T}) = \bigoplus_{i\in I} \bar{L}_i$, where L_i, for $i \in I$, are left ideals of T minimal over $\mathcal{B}(T)$. In particular, $TL_i = L_i$ for every i. We show that the above direct sum must be finite. Assume $\bar{L}_1 \oplus \bar{L}_2 \oplus \ldots \oplus \bar{L}_k \oplus \ldots \subseteq \mathrm{Soc}(\bar{T})$. For every $k \in \mathbb{N}$ set $M_k = L_k + L_{k+1} + \ldots$ and $\tilde{M}_k = RL_k + RL_{k+1} + \ldots$. Then $\{M_k\}_{k\in\mathbb{N}}$ is a strictly descending chain such that $\theta(\tilde{M}_k) = \sum_{j=k}^{\infty} TL_j = M_k$ for all $k \in \mathbb{N}$. However this is impossible since R is left artinian, thus $\tilde{M}_k = \tilde{M}_{k+1}$ for $k \gg 1$. This means that $\mathrm{Soc}(\bar{T})$ is a finite direct sum of minimal left ideals of $\bar{T}$ and shows that $\mathrm{Soc}(\bar{T})$ is a semisimple artinian ring. In particular it possesses a unit element. Therefore $\mathrm{Soc}(\bar{T})$ is a direct summand of $\bar{T}$. Consequently, $\bar{T} = \mathrm{Soc}(\bar{T})$ is a semiprime artinian algebra, since $\mathrm{Soc}(\bar{T})$ is essential in $\bar{T}$. □

Lemma 3.6. *Suppose that R is left artinian, q is a primitive m-th root of unity and $\sigma^m = id_R$. Then there exists an idempotent $e \in T$ such that $\sigma(e) = e$ and $\bar{e} = 1$ in $\bar{T}$.*

Proof. We know, by Lemma 3.5, that $\bar{T}$ has the unit element $\bar{a}$ for some $a \in T$. Since $\sigma(T) = T$, σ induces an automorphism on $\bar{T} = T/\mathcal{B}(T)$ and $\sigma(\bar{a}) = \bar{a}$. It means that $\sigma^i(a) - a \in \mathcal{B}(T)$ for every $i \geq 0$. Since q is a primitive m-th root of unity and $\sigma^m = \mathrm{id}_R$, m is invertible in the base field K and the element $a_1 = \frac{1}{m}\sum_{i=0}^{m-1} \sigma^i(a)$ is fixed by σ. Clearly $\bar{a_1} = \bar{a}$. Thus, using [1, Proposition 27.1], we can lift the idempotent $\bar{a_1}$ to an idempotent e of T such that $\sigma(e) = e$ and $\bar{e} = \bar{a}$. □

Now we are ready to prove the following main result of this section:

Theorem 3.7. *Let R be a σ-simple algebra and δ be a nilpotent q-skew σ-derivation such that q is a primitive m-th root of unity and $\sigma^m = id_R$. Then $R^{(\delta)}$ is left artinian provided R is left artinian.*

Proof. Suppose R is left artinian. We will proceed by induction on $n = n(R) = \min\left\{k \mid \mathrm{r.ann}_R(\delta^k(R)) \neq 0\right\}$. If $n = 1$, then $\delta = 0$ and the thesis follows.

Suppose $n > 1$. By Lemmas 3.5 and 3.6, $\bar{T} = T/\mathcal{B}(T)$ is semiprime artinian and there exists an idempotent $e \in T$ such that $\sigma(e) = e$ and $\bar{e} = e + \mathcal{B}(T)$ is the unity of $\bar{T}$. We have:

$$T = Te + \mathcal{B}(T) = eT + \mathcal{B}(T) = eTe + \mathcal{B}(T). \tag{3.VIII}$$

Let $f = 1 - e$. Since $f \in R^{(\delta)}$ and $\sigma(f) = f$, the algebra fRf is (σ, δ)-stable. Moreover, by [8, Theorem 21.11], fRf is σ-simple and artinian, as R has these properties.

We claim that $n(fRf) < n = n(R)$. Recall that δ is an inner σ-derivation of $\widetilde{R}$ adjoint to y and $\sigma(y) = q^{-1}y$. Hence, it is easy to check that for any K-linear subspace U of R and $k \in \mathbb{N}$ we have:

$$\delta^k(U) \subseteq \sum_{i=0}^{k} y^{k-i}\sigma^i(U)y^i. \tag{3.IX}$$

Using decomposition (3.VIII) and Lemma 3.4 we obtain:

$$Ty^{n-1}f = (Te + \mathcal{B}(T))y^{n-1}f = Tefy^{n-1} + \mathcal{B}(T)y^{n-1}f = 0.$$

Now, it follows from Corollary 3.3, that $y^{n-1}f = 0$. Notice also that $yf = \sigma(f)y + \delta(f) = fy$. Hence, using the formula (3.IX) we obtain:

$$\delta^{n-1}(fRf) \subseteq y^{n-2}fRfy + y^{n-3}fRfy^2 + \ldots + yfRfy^{n-2}. \tag{3.X}$$

If $n = 2$, then $fy = yf$ and the above formula yields that $\delta(fRf) = 0$. If $n > 2$, then for any sequence $i_1, i_2, \ldots, i_{n-1}$, where $1 \leq i_j \leq n-1$ for $j = 1, 2, \ldots, n-1$, there exists $1 \leq k < n-1$ such that $i_k \leq i_{k+1}$. Using this observation together with the formula (3.X) one can check that $(\delta^{n-1}(fRf))^{n-1} = 0$. This shows that for any $n \geq 2$ $\mathrm{r.ann}_{fRf}(\delta^{n-1}(fRf)) \neq 0$. In particular, this means that $n(fRf) < n = n(R)$. Therefore the inductive hypothesis applied to fRf yields that $(fRf)^{(\delta)}$ is left artinian. Moreover, since $f = f^2 \in R^{(\delta)}$ and $\sigma(f) = f$, $(fRf)^{(\delta)} = fR^{(\delta)}f$.

Remark that $\bar{T} = T/\mathcal{B}(T) \simeq T + \mathcal{B}(R^{(\delta)})/\mathcal{B}(R^{(\delta)})$ is naturally included in $R^{(\delta)}/\mathcal{B}(R^{(\delta)})$, $\bar{e}\overline{R^{(\delta)}} = \bar{T}$ and $\bar{T}\bar{f} = \bar{f}\bar{T} = 0$. Thus, the Pierce decomposition of $R^{(\delta)}/\mathcal{B}(R^{(\delta)})$ reduces to

$$R^{(\delta)}/\mathcal{B}(R^{(\delta)}) = \bar{T} \oplus (fR^{(\delta)}f + \mathcal{B}(R^{(\delta)}))/\mathcal{B}(R^{(\delta)}). \tag{3.XI}$$

Therefore $R^{(\delta)}/\mathcal{B}(R^{(\delta)})$ is semiprime artinian.

Recall that, by Lemma 3.4, $\mathcal{B} = \mathcal{B}(R^{(\delta)})$ is nilpotent. Hence, in order to complete the proof, it is enough to show that $\mathcal{B}^k/\mathcal{B}^{k+1}$ is an artinian left $R^{(\delta)}/\mathcal{B}$-module for any $k \in \mathbb{N}$. Clearly we can decompose $\mathcal{B}^k/\mathcal{B}^{k+1}$ into direct sum of additive subgroups as follows:

$$\mathcal{B}^k/\mathcal{B}^{k+1} = e\mathcal{B}^k + \mathcal{B}^{k+1}/\mathcal{B}^{k+1} \oplus f\mathcal{B}^k + \mathcal{B}^{k+1}/\mathcal{B}^{k+1}.$$

In fact, due to (3.XI), components of the above decomposition are left $R^{(\delta)}/\mathcal{B}$-modules. Thus it is enough to prove that both components are artinian.

Since R is σ-simple and $\sigma(f) = f$, we have $RfR = R$. In particular, $1 = \sum_{i=1}^{k} a_i f b_i$ for some suitable $a_i, b_i \in R$, $1 \leq i \leq k$. Then, for any $r \in R$, we have:

$$fr = fr\sum_{i=1}^{k} a_i f b_i = \sum_{i=1}^{k} (fra_i f)(fb_i).$$

This yields that fR is a finitely generated left fRf-module with generators fb_i, $i \in \{1, 2, \ldots, k\}$.

Applying observations form the beginning of Section 2 to the algebra fRf we can conclude that fRf is artinian as left $fR^{(\delta)}f$-module.

Hence, by the remark above, fR is a finitely generated as left $fR^{(\delta)}f$-module. This shows that fR and, consequently, $f\mathcal{B}^k + \mathcal{B}^{k+1}/\mathcal{B}^{k+1}$ are artinian as left $fR^{(\delta)}f$-modules. This together with (3.XI) imply that $f\mathcal{B}^k + \mathcal{B}^{k+1}/\mathcal{B}^{k+1}$ is artinian over $R^{(\delta)}/\mathcal{B}$.

We claim that $e\mathcal{B}^k+\mathcal{B}^{k+1}/\mathcal{B}^{k+1}$ is also artinian as left $R^{(\delta)}/\mathcal{B}$-module. Using the decomposition (3.XI), it is clear that this is equivalent to showing that $e\mathcal{B}^k + \mathcal{B}^{k+1}/\mathcal{B}^{k+1}$ is artinian over $\bar{T}$. To this end, recall that $\bar{T}$ is semiprime artinian, so $e\mathcal{B}^k + \mathcal{B}^{k+1}/\mathcal{B}^{k+1}$ is a semisimple left $\bar{T}$-module. Thus, it is enough to show that $e\mathcal{B}^k + \mathcal{B}^{k+1}/\mathcal{B}^{k+1}$ does not contain infinite direct sum of nonzero $\bar{T}$-submodules. Assume that $\bigoplus_{i\in\mathbb{N}} \bar{L}_i \subseteq e\mathcal{B}^k + \mathcal{B}^{k+1}/\mathcal{B}^{k+1}$ where $0 \neq \bar{L}_i = L_i + \mathcal{B}^{k+1}/\mathcal{B}^{k+1}$ for some suitable $L_i \subseteq e\mathcal{B}^k + \mathcal{B}^{k+1} \subseteq R^{(\delta)}$. Since $\bar{T}\bar{L}_i = \bar{L}_i$, we also have $\bar{L}_i = TL_i + \mathcal{B}^{k+1}/\mathcal{B}^{k+1}$. For every $j \in \mathbb{N}$ define $M_j = \mathcal{B}^{k+1} + \sum_{i=j}^{\infty} TL_i$ and $\tilde{M}_j = \sum_{i=j}^{\infty} RL_i$. Then $\{M_j\}_{j\in\mathbb{N}}$ is a strictly descending chain such that

$$\mathcal{B}^{k+1} + \theta(\tilde{M}_j) = \mathcal{B}^{k+1} + \sum_{i=j}^{\infty} \theta(R)L_i = \mathcal{B}^{k+1} + \sum_{i=j}^{\infty} TL_i = M_j$$

for all $j \in \mathbb{N}$. Notice that this is impossible since $\tilde{M}_j = \tilde{M}_{j+1}$ for $j \gg 1$, as R is left artinian. This yields that $e\mathcal{B}^k + \mathcal{B}^{k+1}/\mathcal{B}^{k+1}$ is artinian as a left $R^{(\delta)}/\mathcal{B}$-module and completes the proof of the theorem. □

4. Going down, the general case

Throughout this section we assume that R is semiprime artinian. Then R is a direct product of simple artinian algebras. Let $1 = e_1 + \ldots + e_r$ be the decomposition of 1 into the sum of orthogonal centrally primitive idempotents. Then σ permutes the set $\{e_1, \ldots, e_r\}$. If $\mathcal{O}$ denotes an orbit of this action, then $R_{\mathcal{O}} = \bigoplus_{e\in\mathcal{O}} eR$ is σ-simple and the restriction of δ to $R_{\mathcal{O}}$ is a skew derivation of $R_{\mathcal{O}}$. Thus, while proving artinian property of $R^{(\delta)}$ we may assume, without loss of generality, that $R = R_{\mathcal{O}}$ is σ-simple.

Proposition 4.1. *Suppose that δ is an algebraic q-skew σ-derivation of R such that q is a primitive m-th root of unity and $\sigma^m = id_R$. Then $R^{(\delta)}$ is left artinian.*

Proof. Since q is a root of unity, we can apply Proposition 1.4 to obtain $T = t(R)$ is left artinian. Lemma 1.2 and assumptions on σ yield that δ^m is a usual derivation and $T = t_{\delta^m}(R)$. Hence, by [6, Lemma 5], T is semiprime. Therefore we may replace R by T and assume that δ is nilpotent. Moreover, as we have seen at the beginning of the section, we may also assume that R is σ-simple. Now the proposition is a direct consequence of Theorem 3.7. □

As a direct consequence of the above proposition and Theorem 2.4 we obtain:

Corollary 4.2 (Theorem 4.8, [5])**.** *If R is a semiprime algebra and δ is an algebraic derivation of R then $R^{(\delta)}$ is left artinian if and only if R is left artinian.*

The following lemma is a generalization of its classical counterpart, when δ is a usual derivation.

Lemma 4.3. *Suppose that $a \in R$ is a nilpotent element such that $\sigma(a) = q^{-1}a$. Let δ be the inner σ-derivation adjoint to a. Then δ is a nilpotent q-skew derivation.*

Proof. It is standard to check that $\delta\sigma = q\sigma\delta$, i.e. δ is q-skew.

Let $\mathrm{End}_K(R, +)$ stand for the algebra of all endomorphisms of K-vector space R and $l_a, r_a \in \mathrm{End}_K(R, +)$ denote the left and right multiplications by a, respectively. Then l_a commutes with r_a and, making use of $\sigma(a) = q^{-1}a$, it is easy to compute that $\sigma l_a = q^{-1}l_a\sigma$ and $\sigma r_a = q^{-1}r_a\sigma$.

Using the above, it is standard to check that

$$\delta^{2n-1} = (l_a - r_a\sigma)^{2n-1} = 0,$$

where $n \in \mathbb{N}$ is such that $a^n = 0$. This shows that δ is nilpotent. □

Let $R[x, \sigma]$ denote the skew polynomial ring of automorphism type. It is well-known that σ can be extended to an automorphism of $R[x, \sigma]$ by setting $\sigma(x) = x$. The following lemma seems to be known, we present only the sketch of its proof.

Lemma 4.4. *Suppose that R is σ-simple. Let M be a nonzero σ-stable ideal of $R[x, \sigma]$ such that $R \cap M = 0$. Then there exist $n \geq 0$ and a monic polynomial $f = x^{n+1} + c_nx^n + \ldots + c_0 \in R[x, \sigma]$ such that*

(1) *$M = fR[x, \sigma] = R[x, \sigma]f$.*
(2) *$\sigma(c_i) = c_i$ for all $0 \leq i \leq n$. Moreover, if $c_i \neq 0$ for some $0 \leq i \leq n$ then c_i is invertible in R and $\sigma^{n+1-i}(r) = c_irc_i^{-1}$ for all $r \in R$.*
(3) *Suppose that $c_0 \neq 0$ and let $s \in \mathbb{N}$ be the smallest number such that σ^s is an inner automorphism adjoint to a σ-stable element $c \in R$. Then s divides both $n + 1$ and i, $0 \leq i \leq n$, provided $c_i \neq 0$.*

Proof. Let A denote the set of all leading coefficients of polynomials from M of minimal nonzero degree, say $n+1$. Then $I = A \cup \{0\}$ is an ideal of R. Moreover $\sigma(I) = I$ since M is σ-stable. Now, σ-simplicity of R yields $I = R$. Thus M contains a unique polynomial f of minimal degree which is monic. Let $f = x^{n+1} + c_n x^n + \ldots + c_0$.

By making use of division algorithm, we obtain (1).

$\sigma(f) \in \sigma(M) = M$ is monic. Hence $\sigma(f) = f$, i.e. $\sigma(c_i) = c_i$ for all $0 \le i \le n$. For any $r \in R$ $fr - \sigma^{n+1}(r)f \in M$ is of degree smaller than $n+1$. Therefore $fr = \sigma^{n+1}(r)f$ for all $r \in R$. This implies the statement (2).

Suppose $c_0 \neq 0$. Then the choice of s and statement (2) yield that s divides both $n+1$ and $n+1-i$ for all $0 \le i \le n$ such that $c_i \neq 0$. This gives (3). □

Theorem 4.5. *Suppose both σ and δ are algebraic and R is left artinian. Then $R^{(\delta)}$ is left artinian.*

Proof. As it was noted at the beginning of the section, we may additionally assume that R is σ-simple.

Notice that R has a natural structure of a left module over the skew polynomial ring $R[x, \sigma]$, which is given by

$$\sum r_i x^i \rightharpoonup r = \sum r_i \sigma^i(r).$$

Let $\widetilde{M} = \mathrm{ann}_{R[x,\sigma]}(R)$. Clearly $\widetilde{M} \cap R = 0$. Moreover, since σ is algebraic, $\widetilde{M} \neq 0$. Obviously $\sigma(\widetilde{M}) \subseteq \widetilde{M}$. Therefore $\sigma(\widetilde{M}) = \widetilde{M}$, since $R[x, \sigma]$ is noetherian as R is such. Let M be a maximal ideal of $R[x, \sigma]$ in the class of σ-stable ideals such that $\widetilde{M} \subseteq M$ and $M \cap R = 0$. Since R is σ-simple and M is σ-stable, we can apply Lemma 4.4 to obtain $M = fR[x, \sigma] = R[x, \sigma]f$ for some monic polynomial $f \in R[x, \sigma]$, say of degree $n+1$. It is clear that the minimal polynomial for σ has a nonzero free term and is divisible by f. Therefore the free term c_0 of f is nonzero and the same lemma gives also that $\sigma(c_0) = c_0$ and σ^{n+1} is an inner automorphism adjoint to c_0.

In the sequel we will make use of the algebra $S = R[x, \sigma]/M$. M is σ-stable, so σ induces an automorphism of S which is also denoted by σ. The choice of M and σ-simplicity of R yield that S is σ-simple, R is naturally embedded in S and S is free as left R-module with basis $1, y, \ldots, y^n$, where y denotes the natural image of x in S. In particular, S is an artinian left R-module and, consequently, S is left artinian. Notice that the element y is algebraic over K and invertible in S, since the minimal polynomial for σ belongs to M and c_0 is invertible in R.

In particular, σ is an inner automorphism of S adjoint to y and S has to be a simple artinian ring.

Case 1: σ is an inner automorphism of R.
Let us remark that in this case we do not use the assumption that σ is algebraic. Because R is σ-simple and σ is inner, R is in fact simple. Let $a \in R$ be an invertible element such that $\sigma(r) = a^{-1}ra$ for $r \in R$. Then for any $r, w \in R$ we have:

$$a\delta(rw) = a\delta(r)w + a\sigma(r)\delta(w) = a\delta(r)w + ra\delta(w).$$

Thus $d = a\delta$ is a derivation of R. For any $l \in \mathbb{N}$ $d^l \in \sum_{i=0}^{l} R\delta^i$ and δ is algebraic over the base field K, thus d is algebraic over R. Now [10, Theorem 1.8] yields that d is algebraic over C – the center of R. Let $F = C^{(d)}$. Since the restriction of d to C is a C-algebraic derivation of the field C, we can apply [4, Proposition 2.1] and conclude that $\dim_F C$ is finite. Now it is standard to see that d is F-algebraic. Applying Corollary 4.2 to the simple artinian F-algebra R, we obtain that $R^{(d)}$ is left artinian. This completes the proof in this case, since $R^{(\delta)} = R^{(d)}$.

Case 2: q is not a root of unity.
In this case, by Lemma 2.3 and Corollary 3.2, there exists a nilpotent element $a \in R$ such that

$$\delta(r) = ar - r^{\sigma}a \quad \text{for all} \quad r \in R.$$

Considering $\delta\sigma(r^{\sigma^{-1}}) = q\sigma\delta(r^{\sigma^{-1}})$ one can easily check that

$$(a - qa^{\sigma})r = r^{\sigma}(a - qa^{\sigma}) \quad \text{for all} \quad r \in R.$$

This implies that σ is either inner or $\sigma(a) = q^{-1}a$. If σ is inner, then $R^{(\delta)}$ is left artinian by Case 1. Thus we will assume that $\sigma(a) = q^{-1}a$.

Let δ also denote the inner σ-derivation of $S = R[x, \sigma]/M$ adjoint to a. The element a is nilpotent and $\sigma(a) = q^{-1}a$ thus, by Lemma 4.3, δ is nilpotent on S. Therefore, as σ is inner on S, we may apply Case 1 to S and conclude that $S^{(\delta)}$ is left artinian.

For any $k \in \mathbb{N}$ and $r \in R$ we have:

$$\begin{aligned}\delta(ry^k) &= ary^k - \sigma(ry^k)a = ary^k - r^{\sigma}y^k a = \\ &= (ar - q^{-k}r^{\sigma}a)y^k\end{aligned}$$

where y denotes the natural image of x in S.

The above equation shows that Ry^k is a δ-stable K-subspace of S for any $k \in \mathbb{N}$. Hence

$$S^{(\delta)} = (\bigoplus_{i=0}^{n} Ry^i)^{(\delta)} = R^{(\delta)} \oplus (Ry)^{(\delta)} \oplus \ldots \oplus (Ry^n)^{(\delta)}.$$

Note that if $L_1 \subset L_2$ are left ideals of $R^{(\delta)}$ then

$$\begin{aligned} S^{(\delta)} L_1 &= L_1 \oplus (Ry)^{(\delta)} L_1 \oplus \ldots \oplus (Ry^n)^{(\delta)} L_1 \\ &\subset L_2 \oplus (Ry)^{(\delta)} L_2 \oplus \ldots \oplus (Ry^n)^{(\delta)} L_2 = S^{(\delta)} L_2. \end{aligned}$$

This implies that $R^{(\delta)}$ is left artinian and completes the proof in this case.

Case 3: q is a primitive m-th root of unity.
Recall that some nonzero power of σ is an inner automorphism adjoint to a σ-stable element. Let $l \in \mathbb{N}$ and $c \in R$ be such that $\sigma^l(r) = crc^{-1}$ for $r \in R$ and $\sigma(c) = c$. Then, for any $r \in R$ we have:

$$\begin{aligned} \sigma^{-l}\delta\sigma^l(r) &= \sigma^{-l}\delta(crc^{-1}) = c^{-1}(\delta(c)rc^{-1} + c\sigma(r)\delta(c^{-1}) + c\delta(r)c^{-1})c \\ &= c^{-1}\delta(c)r + \sigma(r)\delta(c^{-1})c + \delta(r) \end{aligned}$$

and

$$\sigma^{-l}\delta\sigma^l(r) = q^l\delta(r).$$

Using the above equalities together with $\delta(c^{-1})c = -c^{-1}\delta(c)$ we obtain:

$$(q^l - 1)\delta(r) = c^{-1}\delta(c)r - \sigma(r)c^{-1}\delta(c) \tag{4.I}$$

for every $r \in R$.

If $q^l \neq 1$, then the above equation shows that δ is an inner σ-derivation of R adjoint to $a = (q^l - 1)^{-1}c^{-1}\delta(c)$. Thus, similarly as in Case 2, we may consider δ as a σ-derivation of S. Notice that $\sigma(a) = \sigma(c^{-1})\sigma\delta(c) = q^{-1}c^{-1}\delta(c) = q^{-1}a$ and δ is algebraic on S. Indeed, by Lemma 1.2, δ^m is a 1-skew σ^m-derivation and $\delta^m(y) = 0$. This yields that δ^m is algebraic. Now we can use the same argument as in Case 2 to obtain that $R^{(\delta)}$ is left artinian.

Suppose that $q^l = 1$. Then the equation (4.I) shows that $c^{-1}\delta(c)r = \sigma(r)c^{-1}\delta(c)$ for any $r \in R$. Hence, if $\delta(c) \neq 0$, then $c^{-1}\delta(c)$ is invertible in R since R is σ-simple. Thus σ is inner and Case 1 completes the proof in this case. Therefore we may assume that $\delta(c) = 0$.

Let $f = x^{n+1} + c_n x^n + \ldots + c_0 \in R[x, \sigma]$ be such that $M = fR[x, \sigma] = R[x, \sigma]f$ and s denote the smallest natural number such that σ^s is inner adjoint to a σ-stable element. The above considerations together with Lemma 4.4 imply that, without loss of generality, we may assume that

$q^s = 1$, $\delta(c_i) = 0$, $\sigma(c_i) = c_i$ and s divides both $n+1$ and i provided $c_i \neq 0$, where $0 \leq i \leq n$.

It is standard to check that the automorphism σ of R can be extended to an automorphism $\hat{\sigma}$ of $R[x, \sigma]$ by setting $\hat{\sigma}(x) = qx$.

Let $D\colon R[x, \sigma] \to R[x, \sigma]$ be a map defined as follows:

$$D(\sum r_i x^i) = \sum \delta(r_i) x^i.$$

Then D is additive and for any $u, v \in R$ and $k, l \geq 0$ we have

$$\begin{aligned} D(ux^k \cdot vx^l) &= D(u\sigma^k(v)x^{k+l}) \\ &= \delta(u\sigma^k(v))x^{k+l} = [\delta(u)\sigma^k(v) + \sigma(u)\delta\sigma^k(v)]x^{k+l} \\ &= \delta(u)\sigma^k(v)x^{k+l} + \sigma(u)q^k\sigma^k\delta(v)x^{k+l} \\ &= \delta(u)x^k vx^l + \sigma(u)q^k x^k \delta(v)x^l \\ &= D(ux^k)vx^l + \hat{\sigma}(ux^k)D(vx^l) \end{aligned}$$

This shows that D is $\hat{\sigma}$-derivation of $R[x, \sigma]$.

The properties of f and q described above guarantee that $\hat{\sigma}(f) = f$ and $D(f) = 0$. Therefore $\hat{\sigma}(M) = M$, $D(M) \subseteq M$. Consequently, $\hat{\sigma}$ and D induce an automorphism and a skew derivation (also denoted by $\hat{\sigma}$ and D, respectively) of $S = R[x, \sigma]/M$.

Recall that y denotes the natural image of x in S, y is invertible in S and σ is inner on S induced by y. Let $\tau = \sigma^{-1}\hat{\sigma}$. Then $\tau(\sum r_i y^i) = \sum q^i r_i y^i$ for any $\sum r_i y^i \in S$. Since q is a primitive m-th root of unity, the order of τ is finite and equal to m. Let $d = y^{-1}D$. Then for any $u, v \in R$ and $k, l \geq 0$ we have:

$$\begin{aligned} d(uy^k \cdot vy^l) &= y^{-1}D(uy^k)vy^l + y^{-1}\hat{\sigma}(uy^k)D(vy^l) \\ &= y^{-1}D(uy^k)vy^l + (y^{-1}\hat{\sigma}(uy^k)y)y^{-1}D(vy^l) \\ &= d(uy^k)vy^l + \tau(uy^k)d(vy^l) \end{aligned}$$

and

$$\begin{aligned} d\tau(uy^k) &= d(uq^k y^k) = q^k y^{-1}D(uy^k) = q^k y^{-1}\delta(u)y^k = q\tau(y^{-1}\delta(u)y^k) \\ &= q\tau d(uy^k) \end{aligned}$$

The above equations shows that d is a τ-derivation of S and $d\tau = q\tau d$.

Let A stand for the K-subalgebra of $\mathrm{End}_K(S, +)$ generated by D and y^{-1}, where we consider $S \subseteq \mathrm{End}_K(S, +)$ as left multiplications by elements from S. Making use of $d(y^{-1}) = 0$ and $\tau(y^{-1}) = q^{-1}y^{-1}$ one can compute that $Dy^{-1} = q^{-1}y^{-1}D$. Let us recall that y^{-1} is algebraic over K and notice that D is algebraic on S, because the restriction of D to R is equal to δ and $D(y) = 0$. Therefore A is finite dimensional over K and $d = y^{-1}D \in A$ is algebraic over K. Now, Proposition 4.1 applied to (S, d, τ) shows that $S^{(d)}$ is left artinian.

Remark that:

$$S^{(d)} = S^{(y^{-1}D)} = S^{(D)} = (\bigoplus_{i=0}^{n} Ry^i)^{(D)} = R^{(\delta)} \oplus R^{(\delta)}y \oplus \ldots \oplus R^{(\delta)}y^n$$

Hence, similarly as in Case 2, we deduce that $R^{(\delta)}$ is left artinian. This completes the proof of the theorem. □

References

[1] F.W. Anderson, K.R. Fuller, *Rings and categories of modules*, Springer-Verlag, GTM 13, 1992.

[2] J. Bergen, P. Grzeszczuk, *Invariants of skew derivations*, Proc. Amer. Math. Soc.**125**(12), (1997), 3481-3488.

[3] K.R. Goodearl, E.R. Letzter, *Prime ideals in skew and q-skew polynomial rings*, Memoirs of Amer. Math. Soc. no. **521** , vol. **109**(1997).

[4] K.R. Goodearl, R.B. Warfield, *Primitivity in differential operator rings*, Math. Z. **108**, (1982), 503-523.

[5] P. Grzeszczuk, *On constants of algebraic derivations and fixed points of algebraic automorphisms*, J. Algebra **171**, (1995), 826-844.

[6] P. Grzeszczuk, *Constants of algebraic derivations*, Comm. Algebra **21**, (1993), 1857-1868

[7] P. Grzeszczuk, J. Matczuk, *Goldie conditions for constants of algebraic derivations of semiprime algebras*, Israel J. Math. **83**, (1993), 329-342.

[8] T.Y. Lam, *A first course in noncommutative rings*, Graduate Texts in Math. **131**, (1991), Springer-Verlag.

[9] T.Y. Lam, A. Leroy, J. Matczuk, *Primeness, semiprimeness and the prime radical of Ore extensions*, Comm. Algebra **25(8)**, (1997), 2459-2506.

[10] A. Leroy, J. Matczuk, *Derivationes et automorphismes algebraiques d'anneaux premiers*, Comm. Algebra **13** (6), (1985), 1245-1266.

[11] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, Regional Conf. Series in Mathematics, No. 82, Amer. Math. Soc., Providence, RI, USA, 1993.

Institute of Mathematics, University of Białystok, Akademicka 2, 15-267 Białystok, Poland

E-mail address: piotrgr@cksr.ac.bialystok.pl

Universite de Artois, Faculte Jean Perrin, Rue Jean Sauvraz, 62 307 Lens, France

E-mail address: leroy@poincare.univ-artois.fr

Institute of Mathematics, Warsaw University, Banacha 2, 02-097 Warsaw, Poland

E-mail address: jmatczuk@mimuw.edu.pl