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Chains of prime ideals and primitivity of $\mathbb{Z}$-graded algebras

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Abstract In this paper we provide some results regarding affine, prime, $\mathbb{Z}$-graded algebras $R = \bigoplus_{i \in \mathbb{Z}} R_i$ generated by elements with degrees $1, -1$ and 0, with R_0 finite-dimensional. The results are as follows. These algebras have a classical Krull dimension when they have quadratic growth. If $R_k \neq 0$ for almost all k then R is semiprimitive. If in addition R has GK dimension less than 3 then R is either primitive or PI. The tensor product of an arbitrary Brown-McCoy radical algebra of Gelfand-Kirillov dimension less than three and any other algebra is Brown-McCoy radical.

Keywords Graded algebras · primitive rings · semiprimitive rings · Brown-McCoy radical · chains of prime ideals · GK dimension · growth of algebra

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1 Introduction

In this paper, we investigate affine, prime, $\mathbb{Z}$-graded algebras $R = \oplus_{i\in\mathbb{Z}} R_i$, which are generated by elements with degrees $1, -1$ and 0, and with finite-dimensional R_0. This class contains all $\mathbb{N}$-graded, affine, prime algebras generated in degree one.

One of the main objects of interest to ring theorists is the Jacobson radical of a ring. This is equal to the intersection of all (right) primitive ideals of R. In Sections 2 and 3, we investigate the semiprimitivity of affine graded prime algebras and the primitivity of graded algebras. The main result of Section 2 is Theorem 1, and the main result of Section 3 is Theorem 13.

By Bergman's Gap Theorem, a finitely generated algebra cannot have a GK dimension strictly between 1 and 2. In Section 4, we prove Theorem 14, which can be seen as a counterpart of Bergman's Gap Theorem for ideals. This section is the most technical in the paper, but with thorough analysis of the proofs we can quite clearly see the structure of homogeneous ideals of the considered algebras.

In Section 5, we study chains of prime ideals in graded domains of GK dimension 3 (see Theorem 25) . This investigation was inspired by Artin's proposed classification of domains with GK dimension 3 (see [2]). A brief, and slightly simplified, description of this classification for non-specialists can be found in [9] (Introduction, and page 4). We also study chains of prime ideals in graded prime algebras with quadratic growth. In particular, here we consider graded algebras with quadratic growth, instead of PI algebras (though our result was inspired by Schelter's theorem for PI algebras). Recall that Schelter's theorem says that ascending chains of prime ideals in affine PI algebras are finite. We do not know if our result also holds for ungraded affine algebras with quadratic growth. In the context of the material contained in Section 5, let us also mention the reference [10], where Bell proved that if A is a finitely generated prime Goldie algebra over an uncountable field K, and A has quadratic growth, then either A is primitive or A satisfies a polynomial identity (this answers a question by Lance Small in the affirmative). Some interesting related results can also be found in [12].

Recall that a ring R is said to be *Brown-McCoy radical* if it cannot be homomorphically mapped onto a simple ring with identity. It is well known that if R is a Jacobson radical ring then R is Brown-McCoy radical. In [26] it was proved that if R is a nil ring then the polynomial ring $R[x]$ in one variable is Brown-McCoy radical. Then, Beidar et al. showed that $R[x]$ cannot be even mapped onto a ring with a nonzero idempotent. In [26], the question was posed as to whether for every n and a nil ring R the polynomial ring in n commuting indeterminates over R is Brown-McCoy radical. In [30] Smoktunowicz showed that if $R[x]$ is Jacobson radical then $R[x, y]$ is Brown-McCoy radical. Another interesting result obtained in [11] says that if R is a nil ring with $pR = 0$ for some prime p then the polynomial ring $R[x, y]$ in two commuting indeterminates is Brown-McCoy radical.

Recall that a graded ring is called *graded-nil* if every homogeneous element r of R is nilpotent. Recently, Smoktunowicz (see [32]) showed that if R is a ring graded by the additive semigroup of positive integers and R is graded-nil, then R is Brown-McCoy radical. Then, in [20], Lee and Puczyłowski proved that every $\mathbb{Z}$-graded ring which is graded-nil is Brown-McCoy radical. Motivated by these results related to Brown-McCoy radical, we will consider in Section 6 the tensor product of two algebras over a field such that one of them is an affine Brown-McCoy radical algebra with Gelfand-Kirillov dimension less than 3.

In the final section we present some open questions.

If a ring R satisfies a polynomial identity, as usual we say that R is PI. We recall that the Jacobson radical in $\mathbb{Z}$-graded rings is homogeneous. If A and B are algebras over a field K, then, where it does not cause confusion, we will abbreviate $A \otimes_K B$ by $A \otimes B$. If R is a ring and I an ideal of R then (if needed) we use bar notation $\overline{a}$ for the image of an element $a \in R$ in R/I.

For information about the GK dimension and the growth of algebras we refer the reader to Krause and Lenagan [16].

All rings in this paper are associative, but do not necessarily have unity. Recall that a ring R is $\mathbb{Z}$-graded (for short, graded) if there exist additive subgroups R_i of R, such that $R = \bigoplus_{i\in\mathbb{Z}} R_i$ and $R_iR_j \subseteq R_{i+j}$ for any i, j. If $r \in R_i$ for some i, then we say that r is an *homogeneous element* of R.

2 On the semiprimitivity of $\mathbb{Z}$-graded algebras

In this section, we investigate the semiprimitivity of $\mathbb{Z}$-graded algebras. Our aim is to prove the following.

Theorem 1. *Let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be an affine, prime, $\mathbb{Z}$-graded algebra over a field K. Suppose that R_0 is finite-dimensional, and that R is generated in degrees $1, -1$ and 0. Suppose that $R_k \neq 0$, for almost all k. Then R has no nonzero graded-nil ideals. In particular, the Jacobson radical of R is zero, so R is semiprimitive. Moreover, R_0 is semiprimitive.*

The proof of Theorem 1 will be presented later in this section, but first we introduce some lemmas.

Lemma 2. *Let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be an affine algebra over a field K, generated by elements with degrees $1, -1$ and 0, and such that R_0 is finite-dimensional. Then for every $i, j > 0$, we have $R_{i+j} = R_iR_j$ and $R_{-i-j} = R_{-i}R_{-j}$. Moreover, all linear spaces R_i are finite-dimensional.*

Proof. Let $V \subseteq R_{-1} + R_0 + R_1$ be a generating space of R. We show, by induction, that for any $k \geq 1$ and for any $i > 0$, $R_i \cap V^k \subseteq R_1^i$. For $k = 1$, this is clear. Let $k > 1$ and consider $c = c_1 \cdots c_k \in R_i \cap V^k$, with $c_1, \ldots, c_k \in R_{-1} \cup R_0 \cup R_1$. If $c_1 \in R_{-1}$ then by the induction hypothesis, $c_2 \cdots c_k \in R_{i+1} \cap V^{k-1} \subseteq R_1^{i+1}$. Hence $c = c_1 \cdots c_k \in R_{-1}R_1^{i+1} \subseteq R_1^i$. When $c_1 \in R_0$ or $c_1 \in R_1$, a similar argument works and we conclude that, for any $i \geq 1$, monomials and hence also elements from R_i belong to R_1^i. Now, if $i, j > 0$ we have $R_{i+j} = R_1^{i+j} = R_1^iR_1^j = R_iR_j$. The similar formulas for negative indices are proved in the same way.

The second part follows because, by the above, $R_i = R_1^i$, $R_{-i} = R_{-1}^i$, and R_0 is finite-dimensional. □

Lemma 3. *Let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be an affine, prime algebra over a field K generated by elements with degrees $0, -1$ and 1, with R_0 finite-dimensional. If $R_kR_{-k} = 0$ for some $k \neq 0$, then either $R_{4k} = 0$ or $R_{-4k} = 0$.*

Proof. Suppose on the contrary that $R_{4k} \neq 0$ and $R_{-4k} \neq 0$. By Lemma 2, $R_{4k} = R_k^4$, and $R_{-4k} = R_{-k}^4$, and since R is prime $R_{4k}R_sR_{-4k}$ is not zero for some integer s (notice also that $R_kR_sR_{-k} \neq 0$).

Using Lemma 2, we first want to show that $-|k| \leq s \leq |k|$. To do so, consider the case $k > 0$. If $s > k$, then $R_sR_{-k} = R_{s-k}R_kR_{-k} = 0$, which is impossible. Similarly, if $s < -k$, then $R_kR_s = R_kR_{-k}R_{s+k} = 0$, and we get a contradiction. So we must have $-k \leq s \leq k$. Similarly, in the case $k < 0$ we get $k \leq s \leq -k$.

Observe that $R_{4k}R_sR_{-4k} \neq 0$ implies $R_k^4R_sR_{-k} \neq 0$. Moreover, $R_k^4R_sR_{-k} \subseteq R_kR_{s'}R_{-k}$, where $s' = s + 3k$. By the same argument as above (applied for s' instead of s), we get that $R_kR_{s'}R_{-k} \neq 0$ implies $-|k| \leq s' \leq |k|$. Consider the case $k > 0$. Since $s' = s + 3k$, we get $s + 3k \leq k$, so $s < -k$, a contradiction. If $k < 0$, for $s' = s + 3k$ we get $k \leq s' = s + 3k$ so $-2k \leq s$, a contradiction since $k < 0$. Thus we conclude that either $R_{-4k} = 0$ or $R_{4k} = 0$. □

Lemma 4. *Let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be an affine, prime algebra over a field K generated by elements with degrees $0, -1$ and 1, with R_0 finite-dimensional. Suppose that $R_k \neq 0$ for almost all $k \in \mathbb{Z}$. Then $R_kR_{-k} \neq 0$ for every $k \neq 0$.*

Proof. Aiming for a contradiction, suppose that $R_kR_{-k} = 0$, for some $k > 0$. Then, by Lemma 3 either $R_{4k} = 0$ or $R_{-4k} = 0$. By Lemma 2, for any $j > 4k$ we have $R_j = R_{j-4k}R_{4k} = 0$ in the first case and $R_{-j} = R_{-j+4k}R_{-4k} = 0$ if the latter holds, a contradiction. We arrive at the same conclusion when we consider the case $k < 0$. □

Lemma 5. *Let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be an affine, prime algebra over a field K generated by elements with degrees $0, -1$ and 1, with R_0 finite-dimensional. Suppose that $R_k \neq 0$ for almost all $k \in \mathbb{Z}$; then there is $i > 0$ such that $R_iR_{-i} = R_jR_{-j}$ and $R_{-i}R_i = R_{-j}R_j$ for every $j > i$. Moreover, $R_iR_{-i} \neq 0$ and $R_{-i}R_i \neq 0$.*

Proof. We let C_j denote the product R_jR_{-j} for $j > 1$. Observe that $C_j \subseteq C_i$ for $i < j$. Indeed, by Lemma 2, $R_j = R_iR_{j-i}$ and $R_{-j} = R_{-j+i}R_{-i}$, so $R_jR_{-j} = R_iR_{j-i}R_{-j+i}R_{-i} \subseteq R_iR_0R_{-i} \subseteq R_iR_{-i}$. As $C_1, C_2, \ldots$ form a descending chain of linear subspaces of a finite-dimensional space R_0, we get that there exists i such that $C_i = C_{i+1} = \ldots$. Therefore, $R_iR_{-i} = R_jR_{-j}$ for $j \geq i$, as required.

To prove the second assertion, consider the sets $C_j = R_{-j}R_j$ for $j = 1, 2 \ldots$ and proceed in the same way. The last statement follows from the previous lemma. □

Lemma 6. *Let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be an affine, prime algebra over a field K generated by elements with degrees $0, -1$ and 1, with R_0 finite-dimensional. Moreover, let $R_k \neq 0$ for almost all $k \in \mathbb{Z}$, and let i be the positive integer produced in the previous lemma. Let I be the ideal generated in R by R_iR_{-i} and let I' be the ideal generated in R by $R_{-i}R_i$. Then $I \subseteq R_sR \cap RR_{-s}$ and $I' \subseteq RR_s \cap R_{-s}R$ for any $s > 0$. Moreover, $II' \subseteq R_sR$ and $II' \subseteq RR_s$, for any $s \in \mathbb{Z}$.*

Proof. Firstly, we show that $I \subseteq R_sR \cap RR_{-s}$ and $I' \subseteq RR_s \cap R_{-s}R$, for any $s > 0$. In fact, we present only a proof of $I \subseteq R_sR$ for any $s > 0$; the remaining facts are similarly proved.

Observe that
$$I \subseteq \sum_{j\in\mathbb{Z}} R_jR_iR_{-i}R.$$

Fix an integer j, and let $p > \max\{i, s-j, j\}$. Then $p = s-j+t$ for some $t > 0$. By the previous lemma, $R_iR_{-i} = R_pR_{-p}$ and we get

$$R_jR_iR_{-i}R = R_jR_{s-j+t}R_{-s+j-t}R \subseteq R_{s+t}R_{-s+j-t}R = R_sR_tR_{-s+j-t}R \subseteq R_sR$$

by Lemma 2. Therefore $I \subseteq R_sR$, as required.

Now we will show that $II' \subseteq R_sR$, for any $s \in \mathbb{Z}$. Notice that if $s > 0$ the result follows by what was shown above. On the other hand if $s \leq 0$ then $II' \subseteq R_1RI' \subseteq R_1I' \subseteq R_1R_{-1+s}R \subseteq R_sR$. Inclusion $II' \subseteq RR_s$ holds for any s by the same arguments. □

Lemma 7. *Let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be an affine, prime algebra over a field K generated by elements with degrees $0, -1$ and 1, with R_0 finite-dimensional. Moreover, let $R_k \neq 0$ for almost all $k \in \mathbb{Z}$. If P is an ideal of R containing a nonzero homogeneous element u, then $P \cap R_l \neq 0$ for any $l \in \mathbb{Z}$.*

Proof. Let I, I' be as in Lemma 6. We first show that $(u) \cap R_0 \neq 0$. Without loss of generality we can assume that $u \in \bigoplus_{i\geq 0} R_i$. Taking t, which is the smallest nonnegative integer such that $R_t \cap (u) \neq 0$, we have some nonzero $v \in R_t \cap (u)$. We will show that $t = 0$. Indeed if $t > 0$ then we have $vR_{-1} = 0$ so also $vR_{-k} = vR_{-1}^k = 0$ for any $k > 0$. Consider any nonzero element $vw \in (v) \subseteq (u)$ such that $w \in R_f$. Then $f \geq 0$ and for $h = -f-1$ by Lemma 6 we have $II' \subseteq R_hR$. Thus $vwII' \subseteq vR_fR_{-f-1}R \subseteq vR_{-1}R = 0$ which implies $(v)II' = 0$, a contradiction, as R is prime. Thus we have $v \in R_0$.

Now consider a nonzero element $v \in (u) \cap R_0$. We will show that $vR_k \neq 0$ and $vR_{-k} \neq 0$ for any $k > 0$. Clearly it is enough to show the first fact. Suppose for a contradiction that there exists $k > 0$ such that $vR_k = 0$. As $R_{i+j} = R_iR_j$ for any $i, j > 0$ we have $vR_p = 0$ for any $p \geq k$. Let $q \in R_l$ for some l, and let $h > k + |l|$. Then $II' \subseteq R_hR$ and finally $vqII' = 0$ which implies $(v)II' = 0$, a contradiction. □

The following Lemma 8 coincides with Theorem 22.6 on page 225 in [23]. We present a different proof of this result below.

Lemma 8 (Theorem 22.6, [23])**.** *Let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be a Jacobson radical, $\mathbb{Z}$-graded algebra over a field K, with R_0 finite-dimensional. Then R is graded-nil.*

Proof. First we will show that R_0 is a Jacobson radical ring. Consider any $a \in R_0$. Then there exists $b = \sum_{k=m}^{n} b_k \in R$ with $b_k \in R_k$ and $m, n \in \mathbb{Z}$ such that $a + b - ab = 0 = a + b - ba$. Now considering degrees of components we get $a + b_0 - ab_0 = 0 = a + b_0 - b_0a$, so b_0 is the quasi-inverse of a in R_0. As R_0 is finite-dimensional it follows that it is nilpotent.

Consider an element $a \in R_s$ for some $s > 0$ and injective homomorphism of rings $\psi : R \to R[[x, x^{-1}]]$ (here, $R[[x, x^{-1}]]$ is the Laurent power series ring over R) given by the rule $\psi(r_k) = r_kx^k$, where r_k is an homogeneous element of R of degree k.

Let $b = \sum_{k=m}^{n} b_k$ with $b_k \in R_k$ be the quasi-inverse of a. Then $a + b - ab = 0 = a + b - ba$. Thus using ψ we get

$$ax^s + \sum_{k=m}^{n} b_kx^k - ax^s \cdot \sum_{k=m}^{n} b_kx^k = 0 = ax^s + \sum_{k=m}^{n} b_kx^k - \sum_{k=m}^{n} b_kx^k \cdot ax^s.$$

As also for $f = \sum_{i=1}^{\infty}(-1)^i(ax^s)^i \in R[[x, x^{-1}]]$ we have $ax^s + f - ax^s f = 0 = ax^s + f - fax^s$, we get $\sum_{k=m}^{n} b_k x^k = f$, and it follows that a is nilpotent. Since in the same way we can show that for any $s < 0$ an element $a \in R_s$ is nilpotent, the proof is complete. □

Now we are almost ready to prove the main result of this section. We need only mention the following result by Posner, which we will use several times.

Lemma 9 ([24, Corollary]). *If R is an algebra satisfying a polynomial identity and such that every element of R is a sum of nilpotent elements, then R is nil.*

Proof of Theorem 1. **Part 1 - R is semiprimitive.** Suppose that $R_k \neq 0$ for almost all k. Let N be a graded-nil ideal in R. Observe that $N \cap R_0$ is finite-dimensional and nil, hence it is nilpotent. Let n be such that $(N \cap R_0)^n = 0$. Let I, I' be as in Lemma 6. We will first show that $(II'NII')^n = 0$. Let $c_1, c_2, \ldots, c_n \in II'NII'$ be homogeneous elements, where $c_i \in R_{p_i}$. By Lemma 6, we get $c_1 \in NII' \subseteq NRR_{p_1}$, and by considering the gradation on both sides of this inclusion we get that $c_1 \in (N \cap R_0)R_{p_1}$. Similarly, by Lemma 6 we get that for every i, $c_i \in II'NII' \subseteq R_{-p_1-p_2-\ldots-p_{i-1}}RNRR_{p_1+p_2+\ldots+p_i}$, and by comparing the degrees we get that $c_i \in R_{-p_1-p_2-\ldots-p_{i-1}}(N \cap R_0)R_{p_1+p_2+\ldots+p_i}$. Thus we obtain $\prod_{i=1}^{n} c_i \in \prod_{i=1}^{n} R_{-p_1-p_2-\ldots-p_{i-1}}(N \cap R_0)R_{p_1+p_2+\ldots+p_i} \subseteq \left(\prod_{i=1}^{n} N \cap R_0\right)R_{p_1+\ldots+p_n} = 0$. Since R is prime and $(II'NII')^n = 0$, and $I, I' \neq 0$ by Lemmas 5 and 6, we get $N = 0$, as required.

As stated in the Introduction, the Jacobson radical of a $\mathbb{Z}$-graded ring is homogeneous. Thus, using Lemma 8 we can see that the Jacobson radical of R must be graded-nil, so R is semiprimitive, by the above.

Part 2 - R_0 is semiprimitive. To show that R_0 is semiprimitive, let J_0 be the Jacobson radical of R_0. As R_0 is finite-dimensional, J_0 is nilpotent. Let n be such that $J_0^n = 0$. Consider the ideal $J = R^1 J_0 R^1$ (here, R^1 denotes the usual extension with an identity of the ring R) of R and subalgebra $J \cap R_0$ of R_0, which is clearly finite-dimensional. Notice that every element of $J \cap R_0$ is a sum of elements from $R_k J_0 R_{-k}$ for some $k \in \mathbb{Z}$. Clearly, $(R_k J_0 R_{-k})^{n+1} = 0$, since $J_0 R_{-k} R_k \subseteq J_0$. Thus it follows that every element in $J \cap R_0$ is a sum of nilpotent elements. As $J \cap R_0$ is finite-dimensional it is PI, so using Lemma 9 we get the information that $J \cap R_0$ is nil, and finally $J \cap R_0$ is nilpotent and $(J \cap R_0)^m = 0$, for some m.

Let I, I' be as in Lemma 6. Consider homogeneous elements $c_1, c_2, \ldots, c_m \in II'JII'$ with $c_i \in R_{p_i}$. Since J is an ideal in R, similarly as in the first part of the proof we get that $c_1 c_2 \ldots c_m \in \prod_{i=1}^{m} R_{-p_1-\ldots-p_{i-1}}(J \cap R_0)R_{p_1+p_2+\ldots+p_i} = 0$, because $(J \cap R_0)^m = 0$.

Therefore $(II'JII')^m = 0$, but R is prime and $I, I' \neq 0$ by Lemmas 5 and 6, so $J = RJ_0R = 0$, and the proof is complete.

3 On the primitivity of $\mathbb{Z}$-graded algebras

Recall that a right ideal Q of a ring R is *modular* if and only if there exists an element $a \in R$, such that $r - ar \in Q$, for any $r \in R$. If an ideal P is the maximal two-sided ideal contained in Q for some modular maximal right ideal Q of R, then we say that P is (*right*) *primitive*. In the case where 0 is a right primitive ideal of R, we say that R is a (*right*) *primitive ring.*

Using the main result of the previous section, we want to show that $\mathbb{Z}$-graded algebras satisfying certain additional conditions are either PI or primitive.

We first recall Lemma 3.1 and Corollary 3.2 from [8], and, since their proofs hold also for $\mathbb{Z}$-graded algebras (whereas in [8] $\mathbb{N}$-graded algebras were considered), we state it as follows.

Lemma 10 ([8, Lemma 3.1])**.** *Let K be a field, let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be a graded prime K-algebra, and let Z denote the extended centre of R. Suppose that I is an ideal in R that does not contain a nonzero homogeneous element, and $z \in Z$, $x, y \in R$ are such that:*

1. *x is a nonzero homogeneous element;*
2. *y is a sum of homogeneous elements of degree smaller than the degree of x;*
3. *$x + y \in I$;*
4. *$zx = y$.*

Then z is not algebraic over K.

Proof. The proof is the same as the one of [8, Lemma 3.1]. □

Lemma 11 ([8, Corollary 3.2])**.** *Let K be a field, and let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be a finitely generated prime graded K-algebra of quadratic growth. If P is a nonzero prime ideal of R, then either P is homogeneous or R/P is PI.*

Proof. The proof is the same as the proof of [8, Corollary 3.2] when we use Lemma 10 instead of Lemma 3.1, [8]. □

Using [7, Corollary 1.2] and reformulating a sentence that is contained in the proof of Lemma 11 we can see that the following holds.

Lemma 12. *Let K be a field, and let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be an affine, prime, graded K-algebra of GK dimension less than 3. Assume also that R has no nonzero locally nilpotent ideals. If P is a nonzero prime ideal of R which does not contain nonzero homogeneous elements, then R/P is PI.*

We now present the main result of this section.

Theorem 13. *Let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be an affine, prime, $\mathbb{Z}$-graded algebra. Suppose that R_0 is finite-dimensional, and that R is generated in degrees $1, -1, 0$. Suppose that $R_k \neq 0$, for almost all k. If R has GK dimension less than 3, then either R is primitive or R satisfies a polynomial identity.*

Proof. Assuming that R is not PI, we will show that R is primitive. By Theorem 1, we know that R and R_0 are semiprimitive algebras. As the Jacobson radical is zero, then the intersection of all primitive ideals in R is zero. Therefore, to show that zero is a primitive ideal (and hence R is primitive) it suffices to show that the intersection of all nonzero primitive ideals in R is nonzero.

Let P be a nonzero primitive ideal in R. We will first show that $P \cap R_0$ is nonzero. Since R is not PI, if R/P is PI an argument of Small (see proof of [8, Lemma 2.6]) shows that P has a nonzero homogeneous element. On the other hand, if R/P is not PI, then by Lemma 12 the ideal P contains a nonzero homogeneous element. Thus, in each case there exists nonzero $c \in R_t \cap P$ for some t. By Lemma 7 we have $P \cap R_0 \neq 0$.

Observe that $P \cap R_0$ is an ideal in R_0 (nonzero by above). Since R_0 is finite-dimensional and semiprimitive, R_0 has only a finite number of nonzero ideals $P_1, P_2, \dots, P_n$. Let $0 \neq c_i \in P_i$. Then $Rc_1Rc_2 \dots c_nR \neq 0$ and $Rc_1Rc_2 \dots c_nR \subseteq P$, since $c_i \in P_i = P \cap R_0$ for some i. Therefore the intersection of all nonzero primitive ideals is nonzero and contains the ideal $Rc_1Rc_2 \dots c_nR$. □

4 Analogue of Bergman's Gap theorem for ideals

Recall that, by Bergman's result, algebras with growth less than $n(n-1)/2$ have linear growth. The main result for this chapter is closely related to this fact. We also note that this result generalizes [8, Theorem 1.3].

Theorem 14. *Let R be a prime algebra with quadratic growth, $\mathbb{Z}$-graded and finitely generated in degrees 1, -1 and 0. We write $R = \bigoplus_{i\in\mathbb{Z}} R_i$, and assume that R_0 is finite-dimensional. Let $u \neq 0$ be an homogeneous element of R, and let (u) denote the ideal generated by u in R. Then there is a number m, such that*

$$\dim_K((u) \cap (\bigoplus_{i=-n}^{n} R_i)) \geq \frac{(n-m)(n-m-1)}{2}$$

for all sufficiently large n.

To prove this theorem, we first present some supporting lemmas and generalize some results from [8] to the case of $\mathbb{Z}$-graded rings.

Let K be a field. We denote by $K(X) = K\{x_{i,j}, x_{-i,j}\}_{i,j>0}$ the field of rational functions in commuting variables $x_{i,j}, x_{-i,j}$. We now reword [8, Theorem 1.3].

Lemma 15 ([8, Theorem 1.3])**.** *Let $R = \bigoplus_{i=1}^{\infty} R_i$ be an algebra with quadratic growth and finitely generated by elements of degree 1. Let u be an homogeneous element in R. Let $a_1, a_2, \dots, a_n$ be a basis of R_1. For any $i > 0$, let*

$$c_i = \sum_{j=1}^{n} x_{i,j}a_j,\; d_i = \sum_{j=1}^{n} x_{-i,j}a_j,$$

$$S = \{c_1c_2 \dots c_iud_jd_{j-1} \dots d_1 : i, j > 0\}.$$

Then we have the following:

(i) If elements of S are linearly independent over $K(X)$, then there is $m > 0$ such that

$$\dim_K((u) \cap \bigoplus_{k=1}^{n} R_k) \geq \frac{(n-m)(n-m-1)}{2},$$

for almost all n.

(ii) If elements of S are linearly dependent over $K(X)$, then there are integers p_u and k_u such that

$$\dim_K \sum_{i+j<n} R_{i+k_u}uR_{j+k_u} < p_u n,$$

for every $n > 0$.

(iii) Moreover, if S is the set of linearly dependent elements over $K(X)$, there exists an integer t_u, such that the ideal generated by u in R is contained in

$$\sum_{k=1}^{\infty}(R_k u + uR_k) + \sum_{j=1}^{\infty}\sum_{i=1}^{t_u}(R_i uR_j + R_j uR_i).$$

Proof. Part (i) is the same as the first part of the proof of [8, Theorem 1.4]: we observe that $\dim_{K(X)} K(X)S \geq \frac{(n-m)(n-m-1)}{2}$, hence $\dim_K KS \geq \frac{(n-m)(n-m-1)}{2}$.

Although the proof of (ii) is done in the second part of the proof of [8, Theorem 1.3], we present a slightly different, more detailed approach showing at the same time (iii).

Since $R(X)$ is a graded $K(X)$ algebra and S is linearly dependent over $K(X)$, there is in fact some natural number q such that the elements $c_1c_2...c_iud_jd_{j-1}\dots d_1$ with $i+j=q$ are linearly dependent over $K(X)$. Hence there is some k with $0 \leq k \leq q$ such that

$$c_1c_2\dots c_k ud_{q-k}d_{q-k-1}\dots d_1 \in c_1\dots c_k \sum_{j=1}^{q-k-1} K(X)c_{k+1}\dots c_{k+j}ud_{q-k-j}\dots d_1. \tag{1}$$

Let $K' = K\{x_{i,j}\}_{i,j>0}$ be the field of rational functions in commuting variables $x_{i,j}$, and let $P = K'[x_{-i,j}]_{i,j>0}$ be the polynomial ring in variables $x_{-i,j}$ over K' (for $i \geq 1$). As $c_1c_2\dots c_k ud_{q-k}d_{q-k-1}\dots d_1 \in P$, it follows that

$$c_1c_2\dots c_k ud_{q-k}d_{q-k-1}\dots d_1 \in L \cap P,$$

where

$$L = c_1\dots c_k \sum_{j=1}^{q-k-1} K(X)c_{k+1}\dots c_{k+j}ud_{q-k-j}\dots d_1.$$

Therefore,

$$c_1c_2\dots c_k ud_{q-k}d_{q-k-1}\dots d_1 \in c_1\dots c_k \sum_{j=1}^{q-k-1} Pc_{k+1}\dots c_{k+j}ud_{q-k-j}\dots d_1. \tag{2}$$

We claim that, for any $\alpha \geq q-k$, we have

$$c_1c_2\dots c_k uR_\alpha \subseteq (c_1c_2\dots c_k)\sum_{\gamma=1}^{q-k-1} K(X)c_{k+1}\dots c_{\alpha+k-\gamma}uR_\gamma. \tag{3}$$

For $\alpha = q-k$ this comes from equation (2) by evaluating some of the $x_{-i,j}$ in $d_1,\dots,d_{q-k}$ (note that evaluation is possible by (2)). Indeed, we then get

$$c_1c_2\dots c_k uR_{q-k} \subseteq (c_1c_2\dots c_k)\sum_{j=1}^{q-k-1} K(X)c_{k+1}\dots c_{k+j}uR_{q-k-j}, \tag{4}$$

which is easily seen to coincide with the required equation (3) when $\alpha = q-k$.

We now suppose that the equation (3) has been proved up to some $\alpha \geq q-k$. Using the induction hypothesis, we have:

$$\begin{aligned} c_1 \dots c_k u R_{\alpha+1} &= (c_1 \dots c_k) u R_\alpha R_1 \\ &\subset (c_1 \dots c_k) \textstyle\sum_{\gamma=1}^{q-k-1} K(X) c_{k+1} \dots c_{\alpha-\gamma+k} u R_\gamma R_1 \\ &\subseteq (c_1 \dots c_k) \textstyle\sum_{\gamma=1}^{q-k-2} K(X) c_{k+1} \dots c_{\alpha-\gamma+k} u R_{\gamma+1} + \\ &\quad + (c_1 \dots c_k) K(X) c_{k+1} \dots c_{\alpha-q+2k+1} u R_{q-k} \\ &\subseteq (c_1 \dots c_k) \textstyle\sum_{\gamma=2}^{q-k-1} K(X) c_{k+1} \dots c_{(\alpha+1)-\gamma+k} u R_\gamma + \\ &\quad + (c_1 \dots c_k) K(X) c_{k+1} \dots c_{(\alpha+1)-q+2k} u R_{q-k}. \end{aligned} \tag{5}$$

For $\alpha \geq q-k$ there exists a K-automorphism $\sigma \in Aut_K R(X)$ that permutes some of the c_i's as follows: $\sigma(c_1) = c_{\alpha-q+k+2}$, $\sigma(c_2) = c_{\alpha-q+k+3}, \dots, \sigma(c_k) = c_{\alpha-q+2k+1}, \dots, \sigma(c_{q-\gamma}) = c_{\alpha-\gamma+k+1}$ and $\sigma(c_{\alpha-\gamma+1}) = c_1$ (this can be easily obtained by using the ad-hoc permutation on the first indices of the indeterminates x_{ij}, and extending this to $R(X) = K(X) \otimes R$). Let us now apply this automorphism to the equation (4); we get

$$a u R_{q-k} \subseteq a \sum_{\gamma=1}^{q-k-1} K(X) b_\gamma u R_\gamma \tag{6}$$

where $a = c_{\alpha-q+k+2} \dots c_{\alpha-q+2k+1}$ and $b_\gamma = c_{\alpha-q+2k+2} \dots c_{\alpha-\gamma+k+1}$.

Multiplying the left side of this equation by $c_1 c_2 \dots c_{\alpha-q+k+1}$, we get

$$c_1 c_2 \dots c_{\alpha-q+2k+1} u R_{q-k} \subseteq c_1 c_2 \dots c_{\alpha-q+2k+1} \sum_{\gamma=1}^{q-k-1} K(X) b_\gamma u R_\gamma.$$

With this, the equation (5) can now be rewritten as

$$\begin{aligned} c_1 \dots c_k u R_{\alpha+1} &\subseteq (c_1 \dots c_k) \textstyle\sum_{\gamma=2}^{q-k-1} K(X) c_{k+1} \dots c_{\alpha+1-\gamma+k} u R_\gamma + \\ & c_1 c_2 \dots c_{\alpha-q+2k+1} \textstyle\sum_{\gamma=1}^{q-k-1} K(X) c_{\alpha-q+2k+2} \dots c_{\alpha-\gamma+k+1} u R_\gamma. \end{aligned} \tag{7}$$

This proves the claim.

Evaluating in equation (3) some of the $x_{i,j}$ in $c_1, \dots, c_k$ (using an easy-to-formulate counterpart of (2)) we get

$$R_k u R_\alpha \subseteq \sum_{\gamma=1}^{q-k-1} K(X) R_k c_{k+1} \dots c_{\alpha-\gamma} u R_\gamma.$$

Therefore, for every $\alpha \geq q-k$ we get

$$\dim_{K(X)} K(X) R_k u R_\alpha \leq (\dim_{K(X)} K(x) R_k) (\sum_{\gamma=1}^{q-k-1} \dim_{K(X)} K(X) R_\gamma) \leq c,$$

where c is a sufficiently large constant. This implies

$$\dim_K R_k u R_\alpha \leq c. \tag{8}$$

Also, if we evaluate all $x_{i,j}$, we get that for every $\alpha \geq q-k$

$$R_k u R_\alpha \subseteq \sum_{\gamma=1}^{q-k-1} R_{\alpha-\gamma+k} u R_\gamma. \tag{9}$$

Observe now that if we apply the same arguments starting with (1), but with respect to the left side instead of the right side, we will get some similar facts, as follows: for some k' and for the same q (for every $\alpha > q - k'$) we have

$$\dim_K R_\alpha u R_{k'} \leq (\sum_{\xi=1}^{q-k'} \dim_{K(X)} K(X)R_\xi)(\dim_{K(X)} K(x)R_{k'}) \leq c', \tag{10}$$

for sufficiently large constant c', and

$$R_\alpha u R_{k'} \in \sum_{\gamma=1}^{q-k'} R_\gamma u R_{\alpha-\gamma+k'}. \tag{11}$$

Observe now that, for any t_u such that $t_u > max\{k, q-k\}$, taking $\beta > t_u$, $\sigma > t_u$ and using (9) and (11) we get

$$R_\beta u R_\sigma = R_{\beta-k}(R_k u R_\sigma) \subseteq R_{\beta-k} \sum_{\gamma=1}^{q-k} R_{\sigma-\gamma+k} u R_\gamma.$$

Therefore, the ideal generated by u in R is contained in

$$\sum_{k=1}^{\infty}(R_k u + u R_k) + \sum_{j=1}^{\infty}\sum_{i=1}^{t_u}(R_i u R_j + R_j u R_i),$$

and from this and from (8), (10) it follows that

$$\dim_K \sum_{i+j<n} R_{i+k_u} u R_{j+k_u} < p_u n,$$

for some constant p_u not depending on n, and some k_u. □

Lemma 16. *Let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be a prime, affine algebra over a field K generated in degrees $1, -1, 0$, with finite-dimensional R_0. Let u be an homogeneous element of R. Let (u) denote the ideal generated by u in R. If, for any m*

$$\dim_K \left((u) \cap \bigoplus_{i=-n}^{n} R_i\right) < \frac{(n-m)(n-m-1)}{2}$$

for infinitely many n, then there is a number $s_u > 0$ such that the ideal generated by u in R is contained in

$$\sum_{p\in\mathbb{Z}}(R_p u + u R_p) + \sum_{p\in\mathbb{Z}}\sum_{q=-s_u}^{s_u}(R_q u R_p + R_p u R_q).$$

Proof. We let $a_1, \ldots, a_r$ denote a K-basis for R_{-1}, and $b_1, \ldots, b_s$ be a K-basis for R_1, and consider the following elements of $R(X) = K(X) \otimes R$:

$$c_i = \sum_{j=1}^{r} x_{ij} a_j; \quad d_i = \sum_{j=1}^{s} x_{-i,j} b_j; \quad c_i' = \sum_{j=1}^{s} y_{i,j} b_j; \quad d_i' = \sum_{j=1}^{s} y_{-i,j} a_j.$$

Let u be the fixed homogeneous element from the statement. We also denote

$$S = \{c_1 \cdots c_i u d_j \cdots d_1 \mid i, j > 0\},\ S' = \{c'_1 \cdots c'_i u d'_j \cdots d'_1 \mid i, j > 0\},$$

$$T = \{c_1 \cdots c_i u d'_j \cdots d'_1 \mid i, j > 0\},\ T' = \{d_1 \cdots d_i u c'_j \cdots c'_1 \mid i, j > 0\}.$$

Case 1. Here we prove that, if any of the above family S, S', T, T' is linearly independent, then the lemma holds, since the assumption on dimension of (u) is not satisfied. This is done in a similar way to the first part of the proof of [8, Theorem 1.3].

Case 2. Let $V \subseteq R_{-1} + R_0 + R_1$ be a generating space of R. Due to the first case we may assume that the families S, S', T, T' are each linearly dependent. We remark that

$$V^n u V^n = \sum_{0\le i,j\le n} R_{-1}^i u R_1^j + \sum_{0\le i,j\le n} R_1^i u R_{-1}^j + \sum_{0\le i,j\le n} R_{-1}^i u R_{-1}^j + \sum_{0\le i,j\le n} R_1^i u R_1^j. \tag{12}$$

We will now treat the first term occurring above. Assume that the set

$$\{c_1 \cdots c_i u d_j \cdots d_1 \mid i, j \ge 0\}$$

is linearly dependent. Let

$$\sum f_{i,j}(X) c_1 \dots c_i u d_j \dots d_1 = 0$$

be a non-trivial dependence relation. Since $R(X) = K(X) \otimes R$ is $\mathbb{Z}$-graded, looking at terms of the same degree in this relation we get that there exists an integer h, such that a non trivial relation as the one above holds with $j - i = h$. Hence we get for some l

$$\sum_{i=1}^{l} f_{i,h+i}(X) c_1 \dots c_i u d_{h+i} \dots d_1 = 0.$$

We may assume that $f_{l,h+l} \neq 0$, and hence

$$c_1 \dots c_l u d_{h+l} \dots d_1 \in \sum_{i=1}^{l-1} K(X) c_1 \dots c_i u d_{h+i} \dots d_1.$$

Notice that elements from the right hand side are contained in

$$\sum_{i=1}^{l-1} K(X) R_{-i} u R_{h+i}.$$

Therefore

$$c_1 \dots c_l u d_{h+l} \dots d_1 \in \sum_{i=1}^{l-1} K(X) R_{-i} u R_{h+i}.$$

As the left hand side belongs to $K[X]R$, we get

$$c_1 \dots c_l u d_{h+l} \dots d_1 \in \sum_{i=1}^{l-1} K[X] R_{-i} u R_{h+i}.$$

Evaluating all of the $x_{-i,j}$ and $x_{i,j}$, we finally get

$$R_{-l}uR_{h+l} \subseteq \sum_{i=1}^{l-1} R_{-i}uR_{h+i} \tag{13}$$

Let $c > l$, $d > h + l$; then by (13) and by Lemma 2, we get

$$R_{-c}uR_d = R_{-c+l}(R_{-l}uR_{h+l})R_{d-h-l} \subseteq \sum_{i=1}^{l-1} K(X)R_{-c-i+l}uR_{d+i-l}.$$

Continuing this process we can see that as long as $c > l$ and $d > h + l$, then we can always decrease the degrees in $R_{-c}uR_d$. It is not hard to see that for any n we have

$$\sum_{0\leq i,j\leq n} R_{-1}^i uR_1^j \subseteq \sum_{p\in\mathbb{Z}} \sum_{q=-\alpha}^{\alpha} R_q uR_p + R_p uR_q,$$

where $\alpha = max\{l-1, h+l-1\}$.

Working in a similar manner, with term S' appearing in (12), and for terms T, T', using the same methods as in the proof of Lemma 15 (iii), we will produce β, γ, δ that play the role of α in the relevant parts of the proof. Taking $s_u = max\{\alpha, \beta, \gamma, \delta\}$, we get

$$RuR \subseteq \sum_{p\in\mathbb{Z}}(R_p u + uR_p) + \sum_{p\in\mathbb{Z}} \sum_{q=-s_u}^{s_u} (R_q uR_p + R_p uR_q),$$

which was our goal. □

Regarding Lemma 7, it seems that the next lemma is interesting in its own right.

Lemma 17. *Let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be a prime, affine algebra generated in degrees $1, -1, 0$, with finite-dimensional R_0. Suppose that R is infinite-dimensional and that $R_i = 0$ for almost all $i < 0$. Let u be a nonzero homogeneous element of R. Then $R_s u \neq 0$, for every $s > 0$.*

Proof. We will use Lemma 2 throughout. Let $s > 0$, and let $t > 0$ be such that $R_{-i} = 0$ for all $i > t$. Take $q > s + t$ such that there is a nonzero monomial $0 \neq m \in R_q$ (as R is infinitely dimensional, $R_q \neq 0$ for almost all positive q). Since R is a prime and graded ring, it follows that for some j, $mR_j u \neq 0$. By assumption, $j \geq -t$. If $j < 0$, then we can write $m = m_1m_2$, where $m_2 \in R_{-j}$ and $m_1 \in R_v$ where $v > s$. Then $mR_j u = m_1m_2R_j u \subseteq R_v u$, so $R_v u \neq 0$. Since $v > s$, it follows that $R_s u \neq 0$. If $j \geq 0$, then $mR_j v \subseteq R_k v$, where $k \geq q > s + t$, and so by Lemma 2 $R_k v \neq 0$ implies $R_s v \neq 0$. □

Lemma 18. *Let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be a prime, affine algebra generated in degrees $1, -1, 0$, with finite-dimensional R_0. Suppose that R is infinite-dimensional and that $R_i = 0$ for almost all $i < 0$. Then the algebra $R' = \bigoplus_{i>0} R_i$ is prime.*

Proof. Let $t > 0$ be such that $R_{-i} = 0$ for all $i > t$. Let $0 \neq p \in R_i, 0 \neq q \in R_j$ be homogeneous elements in R. By Lemma 17, $R_{t+1}q \neq 0$. Since R is prime and graded, $pR_l(R_{t+1}q) \neq 0$ for some $l \geq -t$. Therefore $pR_{l+t+1}q \neq 0$. As $l+t+1 > 0$, $pR'q \neq 0$, as required. □

We will also need the following part of [7, Lemma 2.3].

Lemma 19. *Let K be a field. If A is a finitely generated, prime algebra of GK dimension at least 2, and if V is a frame for A and $z \in A$ is nonzero, then there exists a positive constant C such that*

$$\dim_K(V^m z V^m) > Cm^2,$$

for all sufficiently large m.

Lemma 20. *Let R be a prime algebra with quadratic growth, which is $\mathbb{Z}$-graded and finitely generated in degrees 1, -1 and 0. Moreover, let $R_k \neq 0$ for almost all $k \in \mathbb{Z}$. We write $R = \bigoplus_{i\in\mathbb{Z}} R_i$, and assume that R_0 is finite-dimensional. Let $u \neq 0$ be an homogeneous element of R, and let (u) denote the ideal generated by u in R. Then there is a positive number B such that*

$$\dim_K((u) \cap \bigoplus_{i=-n}^{n} R_i) > Bn^2,$$

for almost all $n > 0$.

Proof. By Lemma 7, we can assume that $u \in R_0$. Clearly, we can consider $V = R_0 + R_{-1} + R_1$ as a frame of R. By Lemma 19, we have $\dim_K(V^m u V^m) > Cm^2$ for some positive integer C and all sufficiently large m. As we have

$$V^m u V^m \subseteq (\bigoplus_{i=-m}^{m} R_i)\, u\, (\bigoplus_{i=-m}^{m} R_i) \subseteq (u) \cap \bigoplus_{i=-2m}^{2m} R_i,$$

then taking $n = 2m$ we get

$$\dim_K((u) \cap \bigoplus_{i=-n}^{n} R_i) > Bn^2$$

for all sufficiently large n and $B = \frac{1}{8}C$. The result is proved. □

Proof of Theorem 14. We consider three cases.

Case 1. $R_i = 0$ for almost all $i < 0$. By Lemma 17, there is $u' \in R_j \cap (u)$ for some $j > 0$. By Lemma 18, the ring $R' = \bigoplus_{i>0} R_i$ is prime, so by [8, Theorem 1.3] the ideal (u') generated by u' in R' satisfies

$$\dim_K((u') \cap \bigoplus_{i=1}^{n} R_i) \geq \frac{(n-m)(n-m-1)}{2}$$

for all sufficiently large n and some m. The result follows, as $(u') \subseteq (u)$.

Case 2. $R_i = 0$ for almost all $i > 0$. This case is done by analogy with **Case** 1.

Case 3. By Lemma 2, we are left with the case $R_i \neq 0$ for almost all $i \in \mathbb{Z}$. By Lemma 7, we can assume that $u \in R_0$. Suppose, on the contrary, that for any m

$$\dim_K((u) \cap \bigoplus_{i=-n}^{n} R_i) < \frac{(n-m)(n-m-1)}{2}$$

for infinitely many n.

Let $a_1, \ldots, a_e$ be generators of R_1 and let $u_1 = a_1u, \ldots, u_e = a_eu, u_{e+1} = ua_1, \ldots, u_{2e} = ua_e$. Then $u_l \in R_1$ for any l, and by Lemma 15(ii) there exist k_{u_l} and p_{u_l} such that

$$\dim_K \sum_{\substack{i,j>0\\ i+j<n}} R_{i+k_{u_l}} u_l R_{j+k_{u_l}} < p_{u_l} n \tag{14}$$

for every positive integer n. Considering $\overline{k} = max\{k_{u_1}, \ldots, k_{u_{2e}}\}$, for any l we get

$$\sum_{\substack{i,j>\overline{k}\\ i+j<n}} R_i u_l R_j \subseteq \sum_{\substack{i,j>\overline{k}\\ i+j<n+2\overline{k}}} R_i u_l R_j \subseteq \sum_{\substack{i,j>0\\ i+j<n}} R_{i+\overline{k}} u_l R_{j+\overline{k}} \subseteq \tag{15}$$

$$\subseteq \sum_{\substack{i,j>0\\ i+j<n+2\overline{k}-2k_{u_l}}} R_{i+k_{u_l}} u_l R_{j+k_{u_l}}.$$

Thus, by (14) and (15), we have

$$\dim_K \sum_{\substack{i,j>\overline{k}\\ i+j<n}} R_i u_l R_j \leq \dim_K \sum_{\substack{i,j>0\\ i+j<n+2\overline{k}-2k_{u_l}}} R_{i+k_{u_l}} u_l R_{j+k_{u_l}} < p_{u_l}(n + 2\overline{k} - 2k_{u_l})$$

and finally

$$\dim_K \sum_{\substack{i,j>\overline{k}\\ i+j<n}} R_i u_l R_j < \bar{p}n, \tag{16}$$

for all n and some $\bar{p}$.

Similarly, we consider generators $b_1, \ldots, b_d$ of R_{-1} and elements $v_l \in R_{-1}$ for $l \in \{1, \ldots, 2d\}$ such that $v_l = b_lu$ for $l = 1, \ldots, d$ and $v_l = ub_l$ for $l = d+1, \ldots, 2d$. Using the appropriate "negative" version of Lemma 15(ii), we get positive integers $\overline{k}'$ and $\overline{p}'$ such that, for any l,

$$\dim_K \sum_{\substack{i,j>\overline{k}'\\ i+j<n}} R_{-i} v_l R_{-j} < \overline{p}' n \tag{17}$$

for all n.

By Lemma 16, there is a number $s_u > 0$ such that the ideal generated by u in R is contained in the linear space

$$\sum_{p\in\mathbb{Z}} (R_p u + uR_p) + \sum_{p\in\mathbb{Z}} \sum_{q=-s_u}^{s_u} (R_q u R_p + R_p u R_q).$$

We can assume that $s_u > \overline{k}$ and $s_u > \overline{k}'$ (we can take a bigger s_u if necessary).

Using ideals I, I' defined in Lemma 6 and information therein we have $II' \subseteq IRR_{6s_u} \subseteq IR_{6s_u} \subseteq RR_{-3s_u}R_{6s_u}$, and by similar arguments $II' \subseteq R_{6s_u}R_{-3s_u}R$. Therefore $II'(u)II' \subseteq R_{6s_u}R_{-3s_u}(u)R_{-3s_u}R_{6s_u}$.

Observe now that we have

$$II'(u)II' \subseteq R_{6s_u}R_{-3s_u}(u)R_{-3s_u}R_{6s_u} \subseteq$$

$$\subseteq R_{6s_u}R_{-3s_u}\big(\sum_{p\in\mathbb{Z}}(R_pu+uR_p)+\sum_{p\in\mathbb{Z}}\sum_{q=-s_u}^{s_u}(R_quR_p+R_puR_q)\big)R_{-3s_u}R_{6s_u} \subseteq$$

$$\subseteq R_{6s_u}\big(\sum_{p\in\mathbb{Z}}\sum_{q=-4s_u}^{-2s_u}R_quR_p+R_puR_q\big)R_{6s_u}. \tag{18}$$

Consider

$$W_1=\sum_{p\leq 0}\sum_{q=-4s_u}^{-2s_u}R_quR_p+R_puR_q \subseteq \bigoplus_{z=-1}^{-n}R_z$$

and let

$$J=\sum_{p<-s_u-1}\sum_{q=-4s_u}^{-2s_u}R_quR_p+R_puR_q=\sum_{p<-s_u}\sum_{q=-4s_u}^{-2s_u}R_quR_{-1}R_p+R_pR_{-1}uR_q$$

(obviously $J \subseteq \bigoplus_{k<0}R_k$). Then

$$W_1=V+J$$

where $V=\sum_{p=0}^{-s_u}\sum_{q=-4s_u}^{-2s_u}(R_quR_p+R_puR_q)$ is a finite-dimensional linear space. Using all of the above, for all sufficiently large n we have

$$J\cap\bigoplus_{z=-1}^{-n}R_z \subseteq \sum_{l=1}^{2d}\sum_{\substack{i,j>\overline{k}'\\ i+j<n}}R_{-i}v_lR_{-j}. \tag{19}$$

Thus, by (17) and (19),

$$\dim_K(J\cap\bigoplus_{z=-1}^{-n}R_z)<2d\overline{p}'n$$

for almost all n, which gives

$$\dim_K(W_1\cap\bigoplus_{z=-1}^{-n}R_z)=\dim_K\big((V+J)\cap\bigoplus_{z=-1}^{-n}R_z\big)<qn \tag{20}$$

for some q and almost all n.

In order to make the next step, we need to observe that $R_{6s_u}W_1R_{6s_u}\cap\bigoplus_{z=0}^{\infty}R_z$ is finite-dimensional (as $W_1\subseteq\bigoplus_{z<0}R_z$) and that

$$\dim_K(R_{6s_u}W_1R_{6s_u}\cap\bigoplus_{z=-1}^{-n}R_z)\leq\dim_K(R_{6s_u}(W_1\cap\bigoplus_{z=-1-12s_u}^{-n-12s_u}R_z)R_{6s_u}\leq$$

$$\leq \dim_K R_{6s_u} \cdot\ dim_K(W_1 \cap \bigoplus_{z=-1-12s_u}^{-n-12s_u} R_z) \cdot \dim_K R_{6s_u} <$$

$$< \dim_K R_{6s_u} q(n+12s_u) \dim_K R_{6s_u} \text{ (by (20))},$$

which together imply that there is α such that

$$\dim_K(R_{6s_u} W_1 R_{6s_u} \cap \bigoplus_{z=-n}^{n} R_z) < \alpha n \tag{21}$$

for all sufficiently large n.

Consider now the linear space

$$W_2 = \sum_{p>0} \sum_{q=-4s_u}^{-2s_u} R_q u R_p + R_p u R_q.$$

Then

$$R_{6s_u} W_2 R_{6s_u} \subseteq \sum_{p>0} \sum_{q=2s_u}^{4s_u} R_q u R_p + R_p u R_q \subseteq \bigoplus_{z>0} R_z,$$

and as in (20), using elements u_l we can show that there is β such that

$$\dim_K(R_{6s_u} W_2 R_{6s_u} \cap \bigoplus_{z=-n}^{n} R_z) < \beta n, \tag{22}$$

for almost all n.

Notice that by (4) we have

$$II'(u)II' \subseteq R_{6s_u} R_{-3s_u}(u) R_{-3s_u} R_{6s_u} \subseteq R_{6s_u}(W_1 + W_2) R_{6s_u},$$

and using (21), (22) we get

$$\dim_K(II'(u)II' \cap \bigoplus_{z=-n}^{n} R_z) \leq \tag{23}$$

$$\leq\ dim_K\big(R_{6s_u} W_1 R_{6s_u} \cap \bigoplus_{z=-n}^{n} R_z\big) + \dim_K \big(R_{6s_u} W_2 R_{6s_u} \cap \bigoplus_{z=-n}^{n} R_z\big) < (\alpha+\beta)n$$

for all sufficiently large n.

As R is prime by assumption, the homogeneous ideal $II'(u)II'$ is nonzero. Thus, taking a nonzero homogeneous element $a \in II'(u)II'$ and considering the ideal (a), we can see that by (23) we get a contradiction with Lemma 20, which completes the proof.

5 Chains of prime ideals

Following on from our notes in the Introduction regarding this section, we add that our research was also motivated by an example of Bergman (see [22]), which assures that there are affine prime algebras with infinite ascending chains of prime ideals, and also by [14], where Greenfeld, Rowen and Vishne gave other interesting related results and examples.

Recall that an algebra R over a field K has a *classical Krull dimension* equal to m if there exists a chain of prime ideals $P_m \subsetneq P_{m-1} \subsetneq \ldots \subsetneq P_0$ of length m and there is no such chain longer than m. If R has chains of prime ideals of arbitrary length then the classical Krull dimension of R is equal to ∞.

Remark 21. Note that by [27, Theorem 2], if R is a prime PI algebra then any nonzero ideal P of R has a regular element. Thus, using [16, Proposition 3.15], we find that $GKdim(R/P) < GKdim(R)$. We will use this fact below.

Theorem 22. *Let R be an affine, prime algebra over a field K with quadratic growth, which is $\mathbb{Z}$-graded and generated in degrees $1, -1$ and 0. We write $R = \bigoplus_{i\in\mathbb{Z}} R_i$, and assume that R_0 is finite-dimensional. Then R has finite classical Krull dimension.*

Proof. We first remark that if P is a prime ideal of R such that R/P is not PI, then P is homogeneous, by Lemma 11. Moreover, as R has quadratic growth there is a number $p > 0$ such that $\dim_K(V + \ldots + V^n) < pn^2$ for every n, where $V \subseteq R_{-1} + R_0 + R_1$ is a generating space of R.

We now show that if $P_1 \subsetneq P_2 \subsetneq \ldots \subsetneq P_k$ is a chain of proper prime ideals of R such that R/P_i is PI for every i, then $k \le 3$. Indeed, by Remark 21 it is not hard to see that $GKdim(R/P_k) < GKdim(R/P_{k-1}) < \ldots < GKdim(R/P_1)$. Thus by the Small-Warfield theorem [29] and by Bergman's gap theorem it follows that $k \le 3$.

In the second part of the proof we show that there are no chains of nonzero proper homogeneous prime ideals in R which are longer than $2p+1$, with the condition that R/P is not PI for any P appearing in this chain. Suppose on the contrary that $P_1 \subsetneq P_2 \subsetneq \ldots \subsetneq P_{2p+2} \subsetneq R$ is a chain of nonzero proper homogeneous prime ideals of R and R/P_i is not PI for any i. Next time, using the Small-Warfield theorem [29] and Bergman's gap theorem, we get that R/P_i has quadratic growth for any i. Observe now that for every i, P_{i+1}/P_i is a nonzero homogeneous prime ideal in the ring R/P_i and R/P_i satisfies all of the assumptions appearing in Theorem 14. Thus, by this theorem P_{i+1}/P_i has more than $\frac{n^2-n^{\frac{3}{2}}}{2}$ linearly independent elements with degrees between $-n$ and n, for all sufficiently large n. This holds for $i = 1, 2, \ldots, 2p+2$. By the above, it follows that for every i the ideal P_{i+1} has more than $i \cdot (\frac{n^2-n^{\frac{3}{2}}}{2})$ linearly independent elements of degrees between $-n$ and n for almost all n (summing all elements in P_1, $P_2/P_1, P_3/P_2, \ldots P_{i+1}/P_i$). Therefore the ideal P_{2p+2} has more than $(2p+2) \cdot (\frac{n^2-n^{\frac{3}{2}}}{2})$ linearly independent elements of degrees between $-n$ and n for almost all n, and for sufficiently large n we have $(2p+2) \cdot (\frac{n^2-n^{\frac{3}{2}}}{2}) > pn^2$. This is impossible by assumption on growth of R.

We are now ready to show that there are no chains of nonzero proper prime ideals in R which are longer than $2p+4$. Indeed, consider a chain $P_1 \subsetneq P_2 \subsetneq \ldots \subsetneq$

P_k of proper prime ideals of R. Notice that if R/P_t is PI for some t then R/P_l is PI for any $l \geq t$. Thus, by the first part of the proof in the chain $P_1 \subsetneq \ldots \subsetneq P_{k-3}$, we have only homogeneous prime ideals such that for any i, R/P_i is not PI, which implies $k-3 \leq 2p+1$, and the result follows.

Clearly the classical Krull dimension of R is finite (less than or equal to $2p+4$), and it depends upon p. □

Now we consider domains with cubic growth. At the same time we need to notice that if $R = \bigoplus_{i\in\mathbb{Z}} R_i$ is a domain, with R_0 finite-dimensional generated in degrees $1, -1$ and 0 and $R_1, R_{-1} \neq 0$, then $R_i c \subseteq R_0$, for any $c \in R_{-i}$, hence, $\dim_K R_i \leq \dim_K R_0$, and in the discussed case the growth of the algebra is at most quadratic. By the above, we need only consider domains graded by non-negative integers. Firstly, we want to prove the following.

Lemma 23. *Let R be a domain (also an algebra over a field K) with cubic growth, graded by non-negative integers, and finitely generated in degrees 1 and 0. We write $R = \bigoplus_{i=0}^{\infty} R_i$, and assume that R_0 is finite-dimensional. If I is a nonzero homogeneous prime ideal of R, such that R/I is not PI, then R/I has quadratic growth. Moreover, there exists D such that*

$$\dim_K((R/I)_1 + \ldots + (R/I)_n) \leq Dn^2$$

for almost all n, and D does not depend on I, only on R.

Proof. We need to establish some information to work with. Firstly, by Lemma 2, R_1 is finite-dimensional and $R_n = R_1^n$, for any $n > 0$. Secondly, by assumption there exists $c > 0$ such that $\dim_K(R_1 + \ldots + R_n) \leq cn^3$, for all n. Finally, as R is a domain it is also true that $\dim_K R_t \leq \dim_K R_{t+1}$, for any $t > 0$.

Since I is nonzero homogeneous and R is a domain, there exists a nonzero element $i \in I$ such that $i \in R_k$ for some $k > 0$. Then it is not hard to see that

$$\dim_K(R_m/I_m) \leq \dim_K R_m - \dim_K R_{m-k},$$

for any $m > k$, where $I_m = R_m \cap I$. Thus we can see that for any $n > k$ we have

$$\sum_{j=k+1}^{n} \dim_K(R_j/I_j) \leq \dim_K R_n + \ldots + \dim_K R_{n-k+1} + \sum_{l=1}^{k} \dim_K R_l. \quad (24)$$

Assume for a while that there exists $D > 0$ such that for any n we have $\dim_K R_n \leq Dn^2$. Then, using (24), we can see that R/I has at most quadratic growth. In fact, R/I has exactly quadratic growth.

By the above to prove the first claim, it is enough to show that there exists a positive number D such that $\dim_K R_n \leq Dn^2$, for any n. Suppose, to get a contradiction, that for some n we have $\dim_K R_n > 16cn^2$. As we remarked at the beginning, $\dim_K R_1 \leq \dim_K R_2 \leq \ldots$, so

$$\dim_K R_1 + \ldots + \dim_K R_{2n} \geq \dim_K R_n + \ldots + \dim_K R_{2n} \geq n \dim_K R_n > 16cn^3.$$

On the other hand, we have $\dim_K R_1 + \ldots + \dim_K R_{2n} \leq 8cn^3$, a contradiction. Thus, taking $D = 16c$, we finish the proof. □

Now, we recall a result of Bergman [Theorem 22.5, page 224, [23]]

Theorem 24 (Theorem 22.5, page 224, [23]). *Let S be a $\mathbb{Z}$-graded ring and let P be a nonzero prime ideal. If I is an ideal of S properly containing P, then I has a nonzero homogeneous element.*

We will now continue to consider rings with cubic growth and prove the second main result for this section.

Theorem 25. *Let R be a domain with cubic growth which is graded by non-negative integers, finitely generated in degrees 1 and 0. Write $R = \bigoplus_{i=0}^{\infty} R_i$ and assume that R_0 is finite-dimensional. Then R has a finite classical Krull dimension.*

Proof. Consider a chain of proper prime ideals $P_1 \subsetneq P_2 \subsetneq \ldots \subsetneq P_m$ of R. Using Remark 21 and similar arguments as in the proof of Theorem 22, we can see that the first index t such that R/P_t is PI (if such exists) is at least equal to $m-3$. Observe that by Theorem 24, applied for $S = R$, $P = P_1$, $I = P_2$, the ideal P_2 contains a homogeneous element. Let Q be the set of all homogeneous elements in P_2, then Q is a prime ideal in R (since P_2 is a prime ideal in R). Moreover, by Lemma 23, R/Q has quadratic growth and

$$\dim_K((R/Q)_1 + \ldots + (R/Q)_n) \leq Dn^2$$

for almost all n, and some D which does not depend on Q but only on R.

Consider the chain

$$P_3/Q \subsetneq P_4/P_1 \subsetneq \ldots \subsetneq P_{m-4}/P_1$$

of proper prime ideals of R/P_1. By Theorem 22 and its proof, we have $m - 8 \leq 2D + 4$, which implies that $m \leq 2D + 12$. This completes the proof. □

6 On tensor products of algebras which are Brown-McCoy radical

As recalled in the Introduction, a ring R is Brown-McCoy radical if it cannot be homomorphically mapped onto a simple ring with an identity element. Clearly, if R is Brown-McCoy radical then any homomorphic image of R is Brown-McCoy radical as well. As the first supporting result, we present the following.

Lemma 26. *Let K be a field, R be a K-algebra, and let A be a K-algebra with identity. If $R \otimes A$ can be mapped onto a simple ring with identity, then there is a prime homomorphic image R' of R such that $R' \otimes A$ and $R'C'$ can be mapped onto a simple ring with identity, where C' is the extended centroid of R'. Moreover, R' has no locally nilpotent ideals.*

Proof. Let $f : R \otimes A \to S$ be a homomorphism onto a simple ring S with identity. Let $I = \{c \in R : f(c \otimes 1) = 0\}$ (we leave it for the reader to check that I is a prime ideal of R, and equivalently $I = \{c \in R : f(c \otimes b) = 0 \text{ for every } b \in A\}$), and consider the ring $R' = R/I$. Then $R' \otimes A$ can be homomorphically mapped onto S with the analogous mapping $f' : R' \otimes A \to S$ (where $f'((a+I) \otimes b) = f(a \otimes b)$). The first part of the theorem has been proven.

Observe that if R has a locally nilpotent ideal L then $L \subseteq I$, so R' has no nonzero locally nilpotent ideals. Indeed, it follows because S is a simple ring with identity, so if $f(L \otimes A) \neq 0$ then $f(L \otimes A) = S$, which is impossible.

We consider for some n elements $\overline{a_1}, \ldots, \overline{a_n} \in R'$, $b_1, \ldots, b_n \in A$ such that $f'(\sum_{i=1}^n \overline{a_i} \otimes b_i) = 1$. Consider the central closure $R'C'$ of R'. By [6, Theorem 2.3.3], it follows that we have two possibilities:

(1) $1 \in \sum_i C'\overline{a_i}$.
(2) There are elements p_j, q_j in R' such that $\sum_j p_j \overline{a_i} q_j = 0$ for every i and $\sum_j p_j 1 q_j = \overline{c} \neq 0$ for some $c \in R$.

If (1) holds then $R'C'$ has identity so it can be mapped onto a simple ring with identity.

Assume now that (2) holds. Then we get

$$0 = \sum_i \sum_j f'(p_j \otimes 1) f'(\overline{a_i} q_j \otimes b_i) = \sum_i \sum_j f'(p_j \otimes 1) f'(\overline{a_i} \otimes b_i) f'(q_j \otimes 1) =$$

$$= \sum_j f'(p_j \otimes 1) f'(q_j \otimes 1) = f'(\overline{c} \otimes 1) = f(c \otimes 1).$$

The last fact implies that $c \in I$, which gives a contradiction, as $\overline{c} \neq 0$. Thus case (2) cannot happen. □

Remark 27. Notice that we can add an identity to a ring A to have a ring A^1 with identity which has A as an ideal. So Lemma 26 would also work if A does not have an identity.

Lemma 28. *If R is an algebra (over a field K) without an identity element, and $K \subseteq F$ is an algebraic field extension, then $R \otimes_K F$ is Brown-McCoy radical if and only if R is Brown-McCoy radical.*

Proof. If $R \otimes_K F$ is Brown-McCoy radical then clearly R is Brown-McCoy radical.

Suppose that $R \otimes_K F$ is not Brown-McCoy radical, so $R \otimes_K F$ can be homomorphically mapped onto a simple ring S with identity. Let $f : R \otimes_K F \to S$ be a ring homomorphism onto S, and let $f(\sum_{i=1}^m a_i \otimes \xi_i) = 1$. Since we only need a finite number of ξ_i's we can assume that F is a finitely generated algebraic extension of K. We can proceed by induction with respect to the minimal number of generators of F. Therefore it is sufficient to prove that if $F = K[\xi]$ where ξ is algebraic over K then R is not Brown-McCoy radical.

Observe that we can view $R \otimes K[\xi]$ as $R[\xi]$, where ξ is algebraic over K. By assumption there exists a homomorphism $\psi : R[\xi] \to S$ mapping $R[\xi]$ onto a simple ring S with identity. Let $g(x) \in K[x]$ be a minimal polynomial of ξ, and assume that $g(x)$ has degree $n+1$. Then, considering the minimal number k such that $\psi(\sum_{i=0}^k e_i \xi^i) = 1$ and $\psi(e_k \xi^k) \neq 0$ for some $e_i \in R$, we have $k \leq n$. As $\psi(e_k \xi^k) \neq 0$ we clearly have $\psi(e_k) \neq 0$. Let I be the ideal of R generated by e_k. Then $I' = \sum_{p=0}^n I \xi^p$ is a nonzero ideal of $R[\xi]$, and as $\psi(e_k) \neq 0$ we have $S = \psi(I') = \sum_{p=0}^n \psi(I) \psi(\xi)^p$. Observe that by the obvious fact that $S^n \neq 0$ we also have $I^n \neq 0$, and $S = \sum_{p=0}^n \psi(I^n) \psi(\xi)^p$ follows. In particular,

$$\psi(\sum_{l=0}^j d_l \xi^l) = 1$$

for some $d_l \in I^n$ and some $j \leq n$. Observe that since each $d_l \in I^n$ we have

$$d_l \in \sum_{t_l} c_{t_l} e_k I^{n-1}$$

for some $c_{t_l} \in R$. We now use a substitution

$$\psi(e_k \xi^k) = 1 - \psi(\sum_{i=0}^{k-1} e_i \xi^i) = \psi(\sum_{i=0}^{k-1} a_i \xi_i)$$

for $a_0 = 1 - e_0$ and $a_s = e_s$ for $s > 0$. Observe that if $j > k - 1$ then

$$\psi(d_j \xi^j) \in \sum_{t_j} \psi(c_{t_j} e_k \xi^k) \psi(I)^{n-1} \psi(\xi^{j-k}) \subseteq \sum_{t_j} \psi(c_{t_j} \sum_{i=0}^{k-1} a_i \xi^i) \psi(I)^{n-1} \psi(\xi^{j-k}) \subseteq$$

$$\subseteq \sum_{i=0}^{j-1} \psi(I)^{n-1} \psi(\xi^i).$$

In this way we can see that $\psi(d_j \xi^j) \in \sum_{i=0}^{k-1} \psi(I)^{n-1} \psi(\xi^i)$. The same approach can be used toward other coefficients d_l for $l > k - 1$, so we conclude that $\psi(\sum_{i=1}^{j} d_i \xi^i) \in \sum_{i=0}^{k-1} \psi(I)^{n-1} \psi(\xi^i)$. Thus $1 \in \sum_{i=0}^{k-1} \psi(I)^{n-1} \psi(\xi^i)$, which is impossible regarding the minimality of k. Therefore $k = 0$, so $f(e_0) = 1$ for some $e_0 \in R$ and R can be homomorphically mapped onto a ring with identity and it follows that R is not a Brown-McCoy radical. This completes the proof. □

We are now in a position to prove the main result for this section.

Theorem 29. *Let K be a field, and let R be an affine algebra over K with GK dimension less than 3. If R is Brown-McCoy radical, then $R \otimes A$ is Brown-McCoy radical for every algebra A over K.*

Proof. Suppose on the contrary that for some algebra A over K, the algebra $R \otimes A$ can be mapped onto a simple ring with identity. Then by Lemma 26 and Remark 27, $R'C'$ can be mapped onto a simple ring with identity, where R' is a prime homomorphic image of R (without locally nilpotent ideals), and C' is the extended centroid of R'.

We claim that R' is not PI. Indeed, the Jacobson radical $J(R')$ of R' is zero, since R' does not have locally nilpotent ideals and the Jacobson radical of any finitely generated PI algebra is nilpotent. Thus the intersection of all (right) primitive ideals of R' is zero. If R' is primitive (which means that the zero ideal is primitive), then being PI the ring R' is simple with an identity by Kaplansky's Theorem, a contradiction (as R' is Brown-McCoy radical). Thus there exists a nonzero primitive ideal P of R'. But then R'/P is primitive and PI, so as above it is simple with an identity, a contradiction.

As R' is not PI and has GK dimension less than 3, then by [7, Corollary 1.2], the extended centroid C' of R' is algebraic over K. Additionally, the extended centroid of a prime algebra is a field.

As R' is Brown-McCoy radical and C' is an algebraic field extension of the base field K, then using Lemma 28 we can see that $R'C'$ is Brown-McCoy radical (as a homomorphic image of $R' \otimes C'$). The last fact contradicts the assumption from the beginning, which completes the proof. □

By [20, Theorem 5.7], a graded-nil ring which is $\mathbb{Z}$-graded is Brown-McCoy radical. Therefore the following holds.

Corollary 30. *Let K be a field, and let R be an affine, $\mathbb{Z}$-graded algebra over K with Gelfand-Kirillov dimension less than 3. If R is graded-nil, then $R \otimes A$ is Brown-McCoy radical for every algebra A over K.*

Recall that Bartholdi [4] constructed examples of finitely generated graded-nil algebras with quadratic growth which are not Jacobson radical. These algebras are primitive. This provides examples of a primitive algebra R such that R is Brown-McCoy radical, and $R \otimes A$ is Brown-McCoy radical for any algebra A, showing that Brown-McCoy radical may be quite far from the Jacobson radical. On the other hand, a Jacobson radical ring is Brown-McCoy radical.

We would also like to mention that in [31] examples of Jacobson radical algebras with quadratic growth (over any countable field) are constructed.

7 Open questions

We say that a graded ring R is graded-nilpotent if every subring of R which is generated by homogeneous elements of the same degree is nilpotent.

Question 31. *Is every graded-nilpotent ring locally nilpotent (or nil, or Jacobson radical)?*

This can also be asked of prime rings and rings with small growth.

Question 32. *Is there a graded-nilpotent ring with Gelfand-Kirillov dimension two?*

Lemma 33. *If R is a strongly $\mathbb{Z}$-graded Jacobson radical ring, then R is infinitely generated and R_0 is infinitely generated.*

Proof. Since R is strongly $\mathbb{Z}$-graded, $R = R^2$ and $R_0 = R_0^2$. By Lemma 8, R_0 is nil. Suppose though that R is finitely generated; then by Nakayama's Lemma we learn that $R = 0$. Similarly, if R_0 is finitely generated, then by Nakayama's Lemma $R_0 = 0$. □

Question 34. *Is Lemma 33 also true if R is graded-nil but not nil?*

Example 35. *It is known that simple nil rings exist (cf. [33]). For such a ring R, $R[x, x^{-1}]$ is graded-nil, and strongly $\mathbb{Z}$-graded.*

Question 36. *Let R be a graded-nilpotent algebra which is finitely generated as a Lie algebra. Does it follow that R is nilpotent?*

This question is related to the following.

Question 37. *Let R be a nil algebra which is finitely generated as a Lie algebra. Does it follow that R is nilpotent?*

Notice that Question 37 is practically a reformulation of Question 9 from a survey by Amberg and Kazarin [1] (see [15] for an explanation of why these questions are related). Moreover, this question has connections to group theory, and the famous Eggert's conjecture (see [1]).

Question 38. *Is there an affine algebra R over a field K with finite Gelfand-Kirillov dimension such that R is Brown-McCoy radical, and that, for some K-algebra A, $R \otimes A$ can be mapped onto a simple ring with identity?*

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