COMMUTATIVELY CLOSED SETS IN RINGS

DILSHAD ALGHAZZAWI AND ANDRÉ LEROY

Abstract. Encompassing many standard notions such as Dedekind finite and reversible rings we introduce and study a new property for subsets of a ring. We give many examples and characterize some rings such as 2-primal rings with the aid of this notion.

Introduction

Motivated by standard notions like reversible rings and Dedekind finite rings, we say that *a subset S of a ring R is commutatively closed if for any $a, b \in R$ such that $ab \in S$, we also have $ba \in S$.* In particular, the set $\{1\}$ (resp. $\{0\}$) is commutatively closed if and only if R is a Dedekind finite (resp. reversible) ring. If $Reg(R) = \{a \in R \mid \exists x \in R \text{ such that } a = axa\}$ denotes the set of von Neumann regular elements, then $Reg(R) - 1$ is commutatively closed (cf. Example 1.1[5]). Many other examples of this kind are given showing the interesting nature of this concept. Every subset S of R is contained in a commutatively closed subset $\overline{S}$, its commutative closure. A natural equivalence relation on R appears related to this definition. The equivalence classes are bigger than the usual similarity classes and give rise to a topology. These equivalence classes are analyzed in particular in the case of elements such as $\{0\}$, $\{1\}$, units, zero divisors, idempotents and regular elements. The set of nilpotent elements is always commutatively closed and, in the case of a matrix ring over a field, the nilpotent elements form exactly the commutative closure of $\{0\}$ (cf. Proposition 3.7). We recall that an element c in a ring R is clean (resp. strongly clean) if there exist an idempotent $e = e^2 \in R$ and a unit $u \in R$ (resp. with $eu = ue$) such that $c = e + u$. A ring is clean if all its elements are clean. A ring is 2-primal if its set of nilpotent elements coincides with the prime radical. The commutatively closed notion naturally appears in the context of 2-primal rings (cf. Proposition 3.8) or clean rings (cf. Proposition 3.2). The notion of a symmetric subset of a ring is introduced towards the end of the paper. The obvious connection of this notion with the one of commutatively closed subsets is given in Proposition 3.11. In this proposition it is also shown that a subset S is symmetric if any permutation of a factorisation of an element $s \in S$ is still a factorisation of s. Throughout the text R will be a unital ring, $U(R)$, $Z(R)$ and $N(R)$ will stand for the invertible, central and nilpotent elements of R, respectively. For an element $a \in R$ we denote $l(a)$ (resp. $r(a)$ the set of left (resp. right) annihilators of a, i.e. $l(a) = \{x \in R \mid xa = 0\}$ (resp. $r(a) = \{x \in R \mid ax = 0\}$).

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1. Some definitions, examples and first properties

The notion of a commutatively closed subset of a ring R defined in the introduction will be the center of our concerns in this short note.

Let us give some commented examples of commutatively closed subsets of a ring R.

Examples 1.1.
(1) A ring R is commutative if and only if any subset of R is commutatively closed.
(2) The set $\{0\}$ is commutatively closed if and only if R is reversible.
(3) The set $\{1\}$ is commutatively closed if and only if R is Dedekind finite.
(4) The ring R is symmetric if and only if for every $a, b, c \in R$, $abc = 0$ implies that $acb = 0$. This can be translated into asking that for any $a \in R$, $r(a) := \{b \in R \mid ab = 0\}$ is commutatively closed. We refer the reader to [4] for more information on this kind of rings.
(5) $U(R) - 1$ is always commutatively closed. This is due to the well-known fact that, for any $a, b \in R$, $1 - ab \in U(R)$ if and only if $1 - ba \in U(R)$. More generally this is true for (von Neumann) regular elements. It is easy to check (and well-known) that if $1 - ab = (1 - ab)x(1 - ab)$ then $1 - ba = (1 - ba)(1 + bxa)(1 - ba)$. This shows that, $1 - ab$ is regular if and only if $1 - ba$ is also regular. In other words, denoting the set of regular elements by $Reg(R) = \{r \in R \mid \exists x \in R \text{ such that } r = rxr\}$ we have that $Reg(R) - 1$ is always commutatively closed. A similar result is true for the set of unit regular elements and also for the set of strongly π-regular elements. For proofs of these facts we refer the reader to [8].
(6) We consider $Z_l(R) = \{a \in R \mid r(a) \neq 0\}$ the set of left zero divisors. We define similarly $Z_r(R)$ the set of right zero divisors. Similarly as in the item above, we claim that $Z_r(R) + 1$ (resp. $Z_l(R) + 1$) is commutatively closed. We must show that for any $x, y \in R$, $1 - yx \in Z_r(R)$ implies that $1 - xy \in Z_r(R)$. Let $0 \neq r$ be such that $r(1 - yx) = 0$. Note that $ry \neq 0$. Since $ry(1 - xy) = r(1 - yx)y = 0$, we have that $1 - xy \in Z_r(R)$.
(7) The set $N(R)$ of nilpotent elements of a ring R is easily seen to be commutatively closed.
(8) Recall that an element $r \in R$ is strongly clean if there exist an invertible element $u \in U(R)$ and an idempotent $e^2 = e \in R$ such that $eu = ue$ and $r = e + u$. It is proved in Theorem 2.5 of [4] that the set of strongly clean element is commutatively closed. In the same paper the author also shows that the set of Drazin (resp. almost, pseudo) invertible elements is also commutatively closed.
(9) A ring is UJ (cf. [6]) if its set of units is exactly the set $1 + J(R)$ where $J(R)$ is the Jacobson radical of R. If R is UJ then $J(R)$ is commutatively closed.
(10) If the center of a ring is commutatively closed then the ring is Dedekind finite. Indeed if $a, b \in R$ are such that $ab = 1$ then ba is in the center of R and hence $a = aba = ba^2$. This gives $1 = ab = ba^2b = ba$.
(11) A pair (α, β) is a Jacobson pair if there exist $a, b \in R$ such that $\alpha = 1 - ab$ and $\beta = 1 - ba$. One can easily obtain that a subset S is commutatively closed if and only if for any Jacobson pair (α, β), if $\alpha \in 1 - S$, then also $\beta \in 1 - S$. For the definition and properties of Jacobson pairs the reader may consult [8].

(12) Although we have $ab \in J$ implies $1 + ab \in U$, we don't have that $J(R)$ is always commutatively closed since R/J is not always reversible. Consider $M_2(D)$, with D a division ring, as a counter example.

(13) Let us mention another source of commutatively closed sets. We say that a multiplicatively closed subset of a ring R is saturated if for any element $s \in S$, the factors of s are also in S. Any saturated multiplicatively closed set is also commutatively closed.

Definition 1.2. *For a subset $S \subseteq R$ we define recursively a collection of subsets $S_i \subseteq R$, $i \geq 0$, containing S as follows:*

$$S_0 = S \quad \text{and, for } i > 0, \quad S_i = \{ab \mid ba \in S_{i-1}\}.$$

We denote $\overline{S} = \bigcup_{i \geq 0} S_i$. This set is called the commutative closure of S. Clearly a set S is commutatively closed if and only if $\overline{S} = S$.

Let us notice that this definition makes sense for subsets of a semigroup and might be interesting for people working in this area.

With the above notations we have the following lemma.

Lemma 1.3. *Let S be a non-empty subset of a ring R. The subset $\overline{S}$ constructed above has the following properties:*

(a) *The chain S_n, $n \geq 0$ is ascending.*
(b) *For $n, m \in \mathbb{N}$ and $S \subseteq R$, we have $(S_n)_m = S_{n+m}$. Moreover, if $S_n = S_{n+1}$ then $S_n = S_{n+k}$ for any $k \geq 0$ and $S_n = \overline{S}$.*

Proof. (a) Since $1 \in R$, we immediately get, for any $i \in \mathbb{N}$, $S_i \subseteq S_{i+1}$.
(b) The definition shows that S_n is in fact $(\ldots((S_1)_1 \ldots)_1$ where there are n subscripts 1. The additional statements follow. $\square$

Associating $\overline{S}$ to a subset S of R is a (finitary) closure operation. We state this more explicitly in the next proposition.

Proposition 1.4. *Let S be a non-empty subset of a ring R. The subset $\overline{S}$ has the following properties:*

(a) *$\overline{S}$ is commutatively closed i.e. $\overline{\overline{S}} = \overline{S}$.*
(b) *$\overline{S} = \bigcup_{s \in S} \overline{\{s\}}$.*

Proof. (a) We must show that if $a = bc \in \overline{S}$ then $cb \in \overline{S}$. Since $a \in \overline{S}$, there exists $n \in \mathbb{N}$ such that $a = bc \in S_n$, but then $cb \in S_{n+1} \subseteq \overline{S}$.
The proof of Statement (b) is left to the reader. $\square$

Let us remark that for any subset $A, B \subseteq R$ such that $A \subseteq B$ we have that $\overline{A} \subseteq \overline{B}$. It might be worth pointing out that the notion of commutative closure leads to a topology on R as follows from the next result.

Proposition 1.5. *Let A be a subset of a ring R.*

(a) *If A is commutatively closed then its complement $R \backslash A$ is also commutatively closed.*

(b) *A union (resp. an intersection) of commutatively closed sets is commutatively closed.*

(c) *The collection of commutatively closed subsets defines a topology on the ring R. For this topology the open sets are also closed.*

(d) *If $S_\lambda \subseteq R$ with $\lambda \in \Lambda$ are subsets of a ring R then*
- $\cup\overline{S_\lambda} = \overline{\cup S_\lambda}$.
- $\overline{\cap S_\lambda} \subseteq \cap\overline{S_\lambda}$.

Proof. (a) If $x = ab \in R \setminus A$ and $ba \notin R \setminus A$, then $ba \in A$ and, since A is commutatively closed, $x = ab \in A$, a contradiction. This shows that $(R\setminus A)_1 \subseteq R\setminus A$ and hence $R \setminus A$ is commutatively closed.
(b) Let S_λ be a collection of commutatively closed sets ($\lambda \in \Lambda$). If $x \in \cup_{\lambda\in\Lambda} S_\lambda$ and $x = ab$ for some $a, b \in R$, then, since each S_λ is commutatively closed, we get that $ba \in \cup_{\lambda\in\Lambda} S_\lambda$ and hence $(\cup_{\lambda\in\Lambda} S_\lambda)_1 \subseteq \cup_{\lambda\in\Lambda} S_\lambda$. This implies that $\cup_{\lambda\in\Lambda} S_\lambda$ is indeed commutatively closed. This shows that a union of commutatively closed sets is commutatively closed and the analogue statement for the intersection follows by using the item (a).
(c) First notice that the empty set is commutatively closed (its complement is the ring R itself). The statement (c) is then a direct consequence of the items (a) and (b).
(d) Statement (b) implies that $\cup\overline{S_\lambda}$ is closed and hence $\overline{\cup S_\lambda} \subseteq \cup\overline{S_\lambda}$. On the other hand, for any $\lambda \in \Lambda$, we have $S_\lambda \subseteq \overline{\cup S_\lambda}$ and this leads to $\cup\overline{S_\lambda} \subseteq \overline{\cup S_\lambda}$.
Since Statement (b) implies that $\cap\overline{S_\lambda}$ is commutatively closed, it is clear that $\cap_\lambda S_\lambda \subseteq \cap\overline{S_\lambda}$ leads to the required inclusion. □

Remark 1.6. If two elements $a, b \in R$ are in the same class but are different then $\overline{\{a\} \cap \{b\}} = \emptyset$ but $\overline{\{a\}} \cap \overline{\{b\}} = \overline{\{a\}}$. This shows that in Proposition [1.5 (d)] the containment can be strict.

Let us now look at the behavior of the commutatively closed notion with respect to morphisms of rings.

Theorem 1.7. *Let $\varphi : R \longrightarrow S$ be a ring homomorphism, then*

(a) *For any $X \subseteq R$, $\varphi\left(\overline{X}\right) \subseteq \overline{\varphi(X)}$.*

(b) *If φ is a ring isomorphism, then for any $X \subseteq R$, $\varphi\left(\overline{X}\right) = \overline{\varphi(X)}$*

(c) *If $T \subseteq S$ is commutatively closed in S, then $\varphi^{-1}(T)$ is closed in R.*

(d) *If S is reversible,* $\ker(\varphi)$ *is commutatively closed.*

(e) *If S is Dedekind finite then $\varphi^{-1}(\{1\})$ is commutatively closed.*

Proof. (a) It is enough to check that $\varphi(X_1) \subseteq \varphi(X)_1$. For $x \in \varphi(X_1)$ there exists $y \in X_1$ such that $x = \varphi(y)$. Hence there exists $a, b \in R$ such that $y = ab$ and $ba \in X$. We then have $x = \varphi(y) = \varphi(a)\varphi(b)$ and $\varphi(b)\varphi(a) = \varphi(ba) \in \varphi(X)$. This shows that $x \in \varphi(X)_1$, as desired.
(b) It is enough to prove that $\varphi(X)_1 \subseteq \varphi(X_1)$. Now, if $y \in \varphi(X)_1$, then there exist $a, b \in S$ such that $y = ab$ and $ba \in \varphi(X)$. So there exists $x \in X$ such that $ba = \varphi(x)$. Since φ is onto we obtain $a', b' \in R$ such that $a = \varphi(a')$ and $b = \varphi(b')$. This gives that $\varphi(x) = ba = \varphi(b'a')$ and hence $x = b'a' \in X$ since φ is injective. We thus conclude that $a'b' \in X_1$, so that $y = ab = \varphi(a'b') \in \varphi(X_1)$.
(c) Let $a, b \in R$ be such that $ab \in \varphi^{-1}(T)$ Then $\varphi(a)\varphi(b) = \varphi(ab) \in T$. Since T is commutatively closed, we get $\varphi(b)\varphi(a) \in T$ and this yields $ba \in \varphi^{-1}(T)$, as desired.
(d) and (e) are immedaite consequences of (c). □

Remarks 1.8. (a) Let R and S be two rings endowed with the topology defined in Proposition 1.5 (c), then any ring morphism $\varphi : R \longrightarrow S$ is continuous. This is a consequence of Statement (c) in Theorem 1.7.

(b) Let us remark that Theorem 1.7 remains true if φ is a map from the ring R to the ring S that respects the products of the rings and is such that $\varphi(1) = 1$.

Proposition 1.9. *Let S be a subset of a ring R.*

(1) $$S^U = \{usu^{-1} \mid u \in U(R), s \in S\} \subseteq S_1.$$

(2) *For any $n \geq 1$ we have $(1 + r(S))^n S \cup S(1 + l(S))^n \subseteq S_n$, where $r(S) = \{x \in R \mid Sx = 0\}$ and $l(S) = \{x \in R \mid xS = 0\}$.*

Proof. (1) It is clear that if $s \in S$ and $u \in U(R)$, then $s = u^{-1}us$ and hence $usu^{-1} \in S_1 \subseteq \overline{S}$.

(2) Since $S(1+r(S)) \subseteq S$ we immediately get that $(1+r(S))S \subseteq S_1$. This gives that $(1+r(S))S(1+r(S)) \subseteq (1+r(S))S \subseteq S_1$ and hence we also have $(1+r(S))^2 S \subseteq S_2$. Continuing this process we get that, for any $n \geq 1$, $(1 + r(S))^n S \subseteq S_n$. A similar argument leads to $S(1 + l(S))^n \subseteq S_n$ and combining these two inclusions leads to the desired conclusion. □

Proposition 1.10. (1) *For two idempotents $e = e^2 \in R$ and $f = f^2 \in R$ we have $eR \cong fR$ if and only if $f \in \{e\}_1$.*

(2) *For any $a, x \in R$ for any $i, n \in \mathbb{N} \setminus \{0\}$, if $x \in \{a\}_i$ then $x^n \in \{a^n\}_i$.*

(3) *If $a \in R$ is such that $\{a\}$ is commutatively closed, then its left and right annihilators coincide, i.e. $l(a) = r(a)$.*

(4) *If $\{a, b\} \subseteq R$ is commutatively closed then $r(a) \cup l(b) = r(b) \cup l(a)$.*

Proof.

(1) This is a direct consequence of Statement (1) in Proposition 21.20 in [9].

(2) Fix $n \in \mathbb{N}\setminus\{0\}$. We prove the result by induction on the index i. Let $x, c, d \in R$ be such that $a = cd$ and $x = dc \in \{a\}_1$. Then $x^n = (dc)^n = d(cd)^{n-1}c \in \{(cd)^n\}_1 = \{a^n\}_1$. This proves the statement (2) in case $i = 1$. Assume the statement holds for every integer $i < l$ and let $x \in \{a\}_l$. There exists $u, v \in R$ such that $x = uv$ and $vu \in \{a\}_{l-1}$. The induction shows that $(vu)^n \in \{a^n\}_{l-1}$ and hence we have $x^n = (uv)^n = u(vu)^{n-1}v \in \{(vu)^n\}_1 \subseteq \{\{a^n\}_{l-1}\}_1 = \{a^n\}_l$

(3) This is in fact a simple consequence of the statement (2) of Proposition 1.9, indeed we have for every $x \in r(a)$, $a = a(1 + x)$ hence $a = (1 + x)a$. This leads to $r(a)a = \{0\}$ and hence $r(a) \subseteq l(a)$. The reverse inclusion is obtained similarly.

(4) Since $\{a, b\}$ is commutatively closed, we have that for any $x \in r(a)$, $(1 + x)a \in \{a, b\}$. Hence either $xa = 0$, i.e. $x \in l(a)$ or $(1 + x)a = b$ and hence $br(a) = 0$, so that $r(a) \subseteq r(b)$. This shows that $r(a) \subseteq l(a) \cup r(b)$. Similarly we also obtain $l(b) \subseteq r(b) \cup l(a)$ and the other two necessary inclusions. □

Remark 1.11. In terms of reversible sets as defined in [1] the statement (3) of Proposition 1.10 says that every element $a \in R$ such that $\{a\}$ is commutatively closed is in fact a reversible element.

2. The closure of an element

Definitions 2.1. *An element $a \in R$ is said to be commutatively closed when $\overline{\{a\}} = \{a\}$. If $a, b \in R$ we say that b is a factor of a if there exist elements $c, d \in R$ such that $a = cbd$.*

For instance any central element whose factors are nonzerodivisors is commutatively closed. In the next lemmas we analyze the properties of the factorization of commutatively closed elements.

Proposition 2.2. *Let R be a unital ring and $a \in R$ a commutatively closed element. Then:*

(1) *The element a commutes with units.*
(2) *If 2 is not a zero divisor in R then a commutes with idempotent elements.*
(3) *The element a commutes with its factors.*

Proof. (1) This is clear since if $u \in U(R)$ then $a = u^{-1}ua$ and hence, since a is closed, we also have $a = uau^{-1}$. This gives $au = ua$.
(2) This statement is due to the fact that, if e is an idempotent, $(2e-1)^2 = 1$ so that $2e - 1$ is a unit. Statement (1) leads to the equality $2(ea - ae) = 0$ and this gives the proof of (2).
(3) Let $b, c, d \in R$ be such that $a = bcd$. We then also have $a = dbc = cdb$ and hence $ca = c(dbc) = (cdb)c = ac$. $\square$

Let us mention the following obvious consequence of Statement (3) in Corollary 1.10.

Corollary 2.3. *Let $a \in R$ be a commutatively closed element.*

(1) *If a is nilpotent then RaR is a nilpotent ideal.*
(2) *If a is not a right (or left) zero divisor then R is Dedekind finite.*

Proof. (1) This statement is an easy consequence of the fact that for a commutatively closed element a, $l(a) = r(a)$ (cf. Proposition 1.10).
Let us prove (2). Since a is commutatively closed it commutes with its factors and $l(a) = r(a)$ (cf. loc.cit.). If $x, y \in R$ are such that $xy = 1$ we have $a = axy = xay = yxa$. Since $l(a) = 0$, we get $1 = yx$. $\square$

Remark 2.4. Let us observe that if R is a domain, and a commutes with its factors then $\{a\} = \overline{\{a\}}$. This gives a partial converse of the statement (3) of Proposition 2.2.

While considering factorizations of an element it might be of some interest to consider only factorizations using a specific subset. This is the case in the following lemma. This will be used in the proof of Theorem 2.7.

Lemma 2.5. *Let S and T be multiplicatively closed subsets of R such that $S \cap T = \{0\}$ and $T \subseteq r(S) \cap l(S)$. If, for $s \in S$ and $t \in T$, $s + t$ is commutatively closed then s is commutatively closed in S and t is commutatively closed in T.*

Proof. Let us suppose that $s + t \in S + T$ is commutatively closed in R. If $s = s_1 s_2$ and $t = t_1 t_2$ with $s_1, s_2 \in S$ and $t_1, t_2 \in T$. Then $s + t = (s_1 + t_1)(s_2 + t_2)$ and hence we also have $s+t=(s_2+t_2)(s_1+t_1)$. Our hypothesis leads to $s=s_2s_1$ and $t=t_1t_2$, as required. $\square$

Let us examine the case of Dedekind finite rings separately.

Proposition 2.6. (1) *If R is Dedekind finite and $a \in U(R)$, then $\overline{\{a\}} = \{uau^{-1} \mid u \in U(R)\}$.*

(2) *The set of units $U(R)$ of a ring R is commutatively closed if and only if R is Dedekind finite.*

Proof. (1) If R is Dedekind finite and $a \in U(R)$ then any factor of a will also be a unit. In particular, if $a=bc$ then $cb=b^{-1}bcb=b^{-1}ab$. Denoting a^U the set $\{xax^{-1} \mid x \in U(R)\}$ we then get that $\{a\}_1 \subseteq a^U \subseteq U$. So that we have $a^U=\{a\}_1$. Hence $\{a\}_2=(a^U)_1=a^U=\{a\}_1$. Lemma 1.3 (b) finishes the proof.
(2) If $U=\overline{U}$ and $a,b \in R$ are such that $ab=1$ then $ba=u \in U$ and $a=(ab)a=a(ba)=au$ so that $a(1-u)=0$ left multiplying by b we get $u(1-u)=0$ and hence $u=1$, as desired.
The converse is easy and left to the reader. $\square$

In the next theorem we characterize the regular elements that are closed.

Theorem 2.7. *Let $a=axa \in Reg(R)$ be a regular element of a ring R. Then a is commutatively closed if and only if the following conditions are satisfied:*

(1) *$e=ax=xa$ is a central idempotent.*
(2) *$a \in U(eRe)$ commutes with all units in eRe.*
(3) *eRe is a Dedekind finite ring.*
(4) *$(1-e)R(1-e)$ is a reversible ring.*

In particular, a is strongly regular and the idempotent ax is central.

Proof. Let us first show that the conditions are necessary. We thus suppose that $a=axa$ is a regular commutatively closed element of R. Since x is a factor of a, Proposition 2.2 (3) gives that $e=ax=xa$. Since $xR(1-ax)a=0$ Proposition 1.10 (3) implies that $axR(1-ax)=0$ and hence $er=ere$ for every $r \in R$. Similarly we also get $re=ere$ for every $r \in R$. This gives (1).

Let us first remark that $a=ex=xe \in eRe$ and that $aexe=axe=e^2=e$. This shows that a is a unit in eRe. Since a is also commutatively closed in eRe, we get, by Proposition 2.2, that a commutes with the units of eRe.

Let us now show that eRe is Dedekind finite. Let $u,v \in eRe$ be such that $uv=e$. Proposition 2.2 (3) implies $a=uva=uav=vua$. Thus $e=ax=vuax=vue=vu$, as required.

The fact that $(1-e)R(1-e)$ is a reversible ring is a direct consequence of Lemma 2.5 obtained by considering $a \in S=eRe$ and $0 \in T=(1-e)R(1-e)$.

Let us now prove the converse. Since e is central, we can write $R=eRe \times (1-e)R(1-e)$. Suppose $a=pq$ for some $p,q \in R$ and write $p=p_1+p_2$ and $q=q_1+q_2$ with $p_1,q_1 \in eRe$ and $p_2,q_2 \in (1-e)R(1-e)$. From $a=pq$ we then have both $a=p_1q_1$ and $p_2q_2=0$. Let $y \in eRe$ by the inverse of $a \in U(eRe)$. We then have $e=ay=p_1q_1y$. The Dedekind finite assumption gives $e=q_1yp_1$. Since a commutes with the units of eRe, we have $a=ea=q_1yp_1a=q_1yap_1=q_1ep_1=q_1p_1$. Also, note that the reversible condition guarantees that $q_2p_2=0$. Thus, $qp=(q_1+q_2)(p_1+p_2)=q_1p_1+q_2p_2=a$. This shows that a is commutatively closed. $\square$

Corollary 2.8. *An idempotent e is commutatively closed if and only if e is central, eRe is Dedekind finite and $(1-e)R(1-e)$ is reversible.*

Proposition 2.9. *Let $a \in R$ be a commutatively closed element. Then:*

(1) *If $a = 0$ then R is reversible and hence Dedekind finite (i.e. $\overline{\{1\}} = \{1\}$). In this case, R is abelian and we have, for any idempotent $e^2 = e \in R$, that $\overline{\{e\}} = \{e\}$.*
(2) *If a is a right (or left) invertible element then a and all of its factors are units.*

Proof. (1) As is well-known, a reversible ring is also Dedekind finite and abelian. Indeed, suppose that R is reversible and let $a, b \in R$ be such that $ab = 1$ then $a(ba-1) = 0$ and hence also $(ba-1)a = 0$. Multiplying on the right by b we get $baab = ab = 1$ i.e. $ba = ab = 1$, showing that R is Dedekind finite. To show that R is abelian consider $e^2 = e \in R$ and x any element in R, since R is reversible we have $ex(1-e) = 0 = (1-e)xe$ so that $ex = exe = xe$. This shows that the idempotent e is central. Assume that $e^2 = e = ab$ then $ab(1-ab) = 0$. Since R is reversible, we get $0 = b(1-ab)a = ba - baba$. So ba is an idempotent. Now we have $ba = baba = b(ab)a = (ab)ba = ab(ba) = a(ba)b = abab = ab$.
(2) If $ab = 1$, right multiplying by a, gives $aba = a$ and the fact that a is commutatively closed gives $ba^2 = a$. Multiplying this equality on the right by b we get $ba = 1$, as desired. $\square$

Remarks 2.10. (a) In relation with Theorem 2.7, let us notice that there are strongly regular elements in a von Neumann regular ring that are not commutatively closed e.g. the zero element in a 2×2 matrix ring over a division ring.
(b) In general the closure of the class of 1 is not contained in $U(R)$. Indeed if R is not Dedekind finite, then there exist $a, b \in R$ such that $ab = 1$ and $ba \neq 1$ but then ba is a nontrivial idempotent contained in $\overline{\{1\}}$.

Let us mention some more results in the form of a proposition:

Proposition 2.11. (1) *If a one sided ideal is commutatively closed then it is a two sided ideal. In particular, if the right (left) ideals are all closed then R is right (left) duo.*
(2) *A two sided ideal I of a ring R is commutatively closed if and only if the quotient ring R/I is reversible.*
(3) *A prime ideal is commutatively closed if and only if it is completely prime.*
(4) *A ring is 2-primal if and only if every minimal prime ideal is commutatively closed.*

Proof. (1) If the right ideal I is commutatively closed then, since for any element $a \in I$ and $r \in R$ we have $ar \in I$, we get that also $ra \in I$. The definition of a right (left) duo ring gives immediately the second statement.
(2) This is clear.
(3) If P is a prime ideal and is commutatively closed then for any $a, b \in R$ such that $ab \in P$ we get $abR \subseteq P$ and hence also $bRa \subseteq P$ which gives that either $a \in P$ or $b \in P$.

(4) This is a direct consequence of the fact that a ring is 2-primal if and only if its minimal prime ideals are completely prime ([12]).

□

We now introduce an equivalence relation on the elements of any ring.

Definitions 2.12. (1) *We define a relation between elements in a ring R as follows:*

$$x \sim y \text{ if and only if } y \in \overline{\{x\}}$$

(2) *For any $n \geq 0$, we also define $a \sim_n b$ if and only if $b \in \{a\}_n$.*

We give, without proofs, the first easy observations related to these definitions in the form of a lemma.

Lemma 2.13. (1) *The relation $\sim$ is an equivalence relation, it is the transitive closure of $\sim_1$.*

(2) *For any $a \in R$ and $u \in U(R)$, $uau^{-1} \sim_1 a$. More generally: if $b, c \in R$ are such that $bc = 1$ then $cab \in \{a\}_1$, for any $a \in R$.*

(3) *If $a \in R$ and $b, c \in \overline{\{a\}}$, we put $d(b, c) = n$ if $n \in \mathbb{N}$ is minimal such that $b \sim_n c$. The map d defines a distance on $\overline{\{a\}}$.*

Example 2.14. Consider the algebra $k\langle X_1, X_2, \ldots, X_n, Y_1, \ldots, Y_n\rangle/I$ where k is a field and I is the ideal generated by the elements $Y_1X_1 - X_2Y_2, \ldots, Y_{n-1}X_{n-1} - X_nY_n$. As usual we write x_i, y_i for $X_i + I, Y_i + I$. We work in $\overline{x_1y_1}$ and we have that $d(x_1y_1, y_nx_n) = n$.

Theorem 2.15. (1) *For any $n \geq 1$ and $a, b \in R$, we have $a \sim_n b$ if and only if there exist two sequences of elements in R $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ such that $a = x_1y_1, y_1x_1 = x_2y_2, y_2x_2 = x_3y_3, \ldots, y_nx_n = b$.*

(2) *If $a \sim_n b$, then $a - b$ is a sum of n additive commutators.*

(3) *If $a \sim_n b$ then there exist $x, y \in R$ such that $ax = xb$ and $ya = by$. Moreover, for $l \in \mathbb{N}$, we also have $b^{n+l} = ya^lx$ and $a^{n+l} = xb^ly$. In particular, $b^n = yx$ and $a^n = xy$.*

Proof. (1) We prove the assertion by induction on $n \in \mathbb{N}$. If $n = 1$ the assertion is clear: $b \sim_1 a$ means that there exists a factorisation $a = x_1y_1$ of a such that $b = y_1x_1$. If $n > 1$, we have that $b \sim_n a$ so that, for some $x_n, y_n \in R$, $b = x_ny_n$ and $b_1 := y_nx_n \sim_{n-1} a$. The induction hypothesis gives a sequence $x_1, x_2, \ldots, x_{n-1}, y_1, y_2, \ldots y_{n-1}$ such that $a = x_1y_1$, $y_1x_1 = x_2y_2$, $\ldots, b_1 = y_{n-1}x_{n-1}$. This completely describes the desired sequences.
(2) With the notations we just introduced in the proof of statement (1), we have: $a = b + \sum_{i=1}^n [x_i, y_i]$.
(3) If $b \in \{a\}_n$ there exist sequences of elements $x_1, \ldots, x_n \in R$ and $y_1, \ldots, y_n \in R$ such that $a = x_1y_1$ and $y_1x_1 = x_2y_2$, $y_2x_2 = x_3y_3$ and in general for $i = 1, \ldots, n-1$ we have $y_ix_i = x_{i+1}y_{i+1}$ and $b = y_nx_n$. Let us write $x := x_1 \cdots x_n$. We then compute $ax = ax_1x_2 \cdots x_n = x_1(y_1x_1)x_2 \cdots x_n = x_1x_2(y_2x_2)x_3 \cdots x_n = \cdots = x_1x_2 \cdots x_{n-1}x_n(y_nx_n) = xb$. Similarly if we put $y = y_n \cdots y_1$ we get $ya = by$. Let us compute, for $l \in \mathbb{N}$, $b^{n+l} = (y_nx_n)^{n+l} = y_n(x_ny_n)^{n+l-1}x_n = y_n(y_{n-1}x_{n-1})^{n+l-1}x_n$ and hence we have $b^{n+l} = y_ny_{n-1}(x_{n-1}y_{n-1})^{n+l-2}x_{n-1}x_n$. Continuing this procedure leads to $b^{n+l} = y_ny_{n-1} \ldots y_1(x_1y_1)^lx_1x_2 \ldots x_n$. This immediately gives one of the the desired equalities. The other one is obtained similarly. □

The following is a nice and easy corollary of Theorem 2.15.

Corollary 2.16. *If $a, b \in R$ are such that $a \sim_n b$ then $a^n \sim_1 b^n$.*

Proof. This is a direct consequence of the fact that there exist $x, y \in R$ such that $a^n = xy$ and $b^n = yx$ (cf. statement (3) in Theorem 2.15). $\square$

Corollary 2.17. *For idempotents $e, f \in R$ we have $eR \cong fR$ if and only if $e \sim f$*

Proof. Since $e = e^2$ and $f = f^2$ are idempotent elements Corollary 2.16 implies that $e \sim f$ if and only if $e \sim_1 f$ and thus the statement (1) in Proposition 1.10 also gives $eR \cong fR$ if and only if $e \sim f$. $\square$

Example 2.18. Let k be a field and $V = \oplus_{i \geq 0} ke_i$ a vector space over k with basis $\{e_i | i \in \mathbb{N}\}$. In $R = End_k(V)$ we consider the identity map $1 \in R$. We claim that $\{1\}_1 = \{e = e^2 | dim_k(\mathrm{Im}(e)) = \infty\}$. Indeed, if $f \in \{1\}_1$ then there exist $p, q \in R$ such that $f = pq$ and $qp = 1$,in particular $f^2 = pqpq = pq = f$. Moreover since $qp = 1$, we must have that q is onto and p is injective. This implies that $dim_k(\mathrm{Im}(f) = \infty$.

On the other hand if $f = f^2$ is such that $dim\mathrm{Im}(f) = \infty$ then we can decompose V as $V = \mathrm{Im}(f) \oplus \ker f$ and we let $\{v_1, \ldots, v_n \ldots\}$ and $\{w_1, w_2, \ldots\}$ be bases for $\mathrm{Im}(f)$ and $\ker(f)$ respectively. We then have $f(v_i) = v_i$ and $f(w_i) = 0$. We define $p, q \in R$ by the following: $q(v_i) = e_i$, $q(w_j) = 0$ and $p(e_i) = v_i$. We easily conclude that $f = pq$ and $qp = 1$, so we have that $f \in \{1\}_1$, as desired.

Definition 2.19. *Let $a \in Z(R)$ be a central element of R. An element x of a ring R is said to be a-periodic if there exist nonzero natural numbers $n, m \in \mathbb{N}$, $n \neq m$, such that $x^n = ax^m$. If $a = 1$ we just say that x is periodic. The 0-periodic elements are the nilpotent elements.*

The next lemma offers a quick proof of a characterization of periodic elements. There is an analogue more technical characterization for a-periodic elements, but this will not be needed.

Lemma 2.20. *An element x of a ring R is periodic if there exists $s \in \mathbb{N}$ such that x^s is an idempotent.*

Proof. Let us suppose that x is periodic and let positive integers n and l be such that $x^{n+l} = x^n$. Let us write $n = lq - r$ with $0 \leq r < l$. We then have $(x^{n+r})^2 = x^{(n+r)+lq} = x^{n+lq}x^r = x^{n+r}$, as desired. The converse is clear. $\square$

Proposition 2.21. *If $a \in Z(R)$ and $b \sim a$ then b is a-periodic. The set of a-periodic elements is commutatively closed. The class of 1 (resp. $\{0\}$) is contained in the set of periodic (resp. nilpotent) elements.*

Proof. There exists $n \in \mathbb{N} \cup \{0\}$ such that $b \sim_n a$. According to Theorem 2.15, this implies that there exit sequences $x_1, x_2, \ldots, x_n$ and $y_1, y_2 \ldots, y_n$ such that, for $l \in \mathbb{N}$, we have $b^{n+l} = ya^l x$. Since $a \in Z(R)$, this gives for $l = 1$ that $b^{n+1} = yax = ayx = ab^n$. We thus conclude that b is a-periodic. The proof of the other statements are left to the reader. $\square$

3. The commutative closure for some particular rings

In this last section we will examine the commutatively closed property for subsets of particular rings. We start with the ring that can be additively generated by their units i.e. every element of R is a sum of units. This includes matrix rings $M_n(R)$ with $n \geq 2$ and the group rings over a division ring. (cf. [13] for more information on rings generated by their units). These rings are also strongly related to clean rings (cf. Lemma 3.1). Let us recall that a ring is clean if its elements can be written as a sum of a unit and an idempotent. Before establishing connections with these rings and commutatively closed subsets, let us first state the following easy Lemma that is part of folklore (cf. [13])

Lemma 3.1. *A clean ring R such that $2 \in U(R)$ is generated by its units.*

Proof. First remark that any element $a \in R$ can be written as $a = e + u = \frac{2e-1}{2} + \frac{1}{2} + u$ with $e^2 = e$ and $u \in U(R)$. Since $(2e-1)^2 = 1$ we have that any element of R is a sum of units. □

Proposition 3.2. *In a ring R generated by its units, the commutatively closed elements are central. In particular this holds in the case of a clean ring with $2 \in U(R)$ or for any matrix ring $M_n(S)$ with $n \geq 2$ and S is any ring.*

Proof. If $\{a\} = \overline{\{a\}}$ we know that a commutes with units and, since any elements of R is a finite sum of units, a commutes with any element of R. □

Remarks 3.3.
(a) As a slight generalization of what is mentioned in Proposition 3.2, let us observe that in a clean ring such that 2 is not a zero divisor, it is still true that closed elements are central.
(b) It is easy to check that for a division ring, we always have $S^U = \overline{S}$.
(c) If the ring R is generated by its units we might expect, as in the case of division rings, stronger relation between S^U and $\overline{S}$. For instance let $R = K[X, X^{-1}; \sigma]$ be the Laurent skew polynomial ring, where K is a field and σ is an automorphism of K. In such a ring the units are the nonzero monomials and this ring is generated by its units. It can be checked that in this case $\overline{\{X\}} = \{X\}^U$. On the other hand, considering the set $S = \{0\} \subset R = M_2(K)$, where K is a field, we remark that R is generated by its units, but $S^U = \{0\} \neq \overline{S} = N(R)$ (cf. Proposition 3.7).

As a concrete example we will decompose the ring $M_2(\mathbb{F}_2)$ into its commutatively closed subsets.

Example 3.4. Let $R = M_2(\mathbb{F}_2)$. We describe the different classes:

- $\overline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$
- $\overline{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$
- $\overline{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}} = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$
- $\overline{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$

- $\overline{\begin{pmatrix}1&0\\0&0\end{pmatrix}} = \left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&1\end{pmatrix}, \begin{pmatrix}1&0\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\0&1\end{pmatrix}\right\}.$

We remark that the class of zero consists exactly of nilpotent elements (this will be generalized in Proposition 3.7), the units elements are divided in three classes and the nontrivial idempotent matrices form a class.

Let us mention the following easy proposition.

Proposition 3.5. (1) *Let R be a ring and S be a set. If $\varphi : R \longrightarrow S$ is a map such that $\varphi(ab) = \varphi(ba)$ then $\varphi(a) = \varphi(b)$ whenever $a \sim b$.*
(2) *Let k be a field and let $A, B \in R = M_n(k)$ be two square matrices such that $A \sim B$, then the two matrices A and B have the same characteristic polynomials. In particular $\det(A) = \det(B)$ and $\mathrm{Tr}(A) = \mathrm{Tr}(B)$.*

Proof. (1) This is an easy consequence of the point (1) in Theorem 2.15.
(2) This is now obvious since, denoting $\xi(A)$ the characteristic polynomial of A, it is well-known that $\xi(AB) = \xi(BA)$. □

Remark 3.6. Let k a field and $n \geq 2$, since the matrix ring $M_n(k)$ is Dedekind finite, the identity matrix is commutatively closed. On the other hand any upper triangular matrix with 1 on the diagonal has the same characteristic polynomial viz.: $(X-1)^n$.

Proposition 3.7. *Let k be a commutative field and $n \in \mathbb{N}$, the class of $\overline{\{0\}}$ in $M_n(k)$ is the set of nilpotent matrices.*

Proof. We have seen that, in any ring, $\overline{\{0\}} \subseteq N(R)$ (cf. Proposition 2.21). Conversely, if $A \in M_n(k)$ is nilpotent there exsts an invertible matrix P and a strictly upper triangular matrix $U \in M_n(k)$ such that $PAP^{-1} = U$. Since the class of an element is the same as the class of any of its conjugate, we conclude that $\overline{\{A\}} = \overline{\{U\}}$. We will show that for any strictly upper triangular matrix U we have $\overline{\{U\}} = \overline{\{0\}}$. Since U is nilpotent we only need to prove that $U \in \overline{\{0\}}$. We may assume that $U \neq 0$ and we denote the lines of U by $L_1, L_2, \ldots, L_n$. In fact the last line L_n is zero, and we define $r \in \{1, \ldots, n-1\}$ to be minimal such that L_i is zero for $i > r$. We will prove that $U \in \overline{\{0\}}$ by induction on r. We write

$$U = \begin{pmatrix} I_{r,r} & 0 \\ 0 & 0 \end{pmatrix} U \quad \text{and} \quad B := U \begin{pmatrix} I_{r,r} & 0 \\ 0 & 0 \end{pmatrix} \in \{U\}_1,$$

where $I_{r,r}$ denotes the identity matrix of size $r \times r$.
If $r = 1$, we get that $B = 0 \in M_n(k)$ and this yields the thesis.
If $r > 1$, write $B = (R_1, \ldots, R_n)$ where R_i is the i^{th} row of B. The matrix B is easily seen to be upper triangular and such that the rows $R_r, \ldots, R_n$ are zero. This means that this matrix has at least one more zero row than the matrix U. The induction hypothesis gives that $B \in \overline{\{0\}}$, but then $U \in \{B\}_1 \subseteq \overline{\{0\}}$, as required. □

Proposition 3.8. (a) *Let R be a ring such that $\{0\}_1$ is contained in the center $Z(R)$. Then R is 2-primal.*
(b) *The prime radical $P(R)$ of a ring R is commutatively closed if and only if R is 2-primal.*

Proof. (a) We must show that every nilpotent element is in fact in the prime radical of R. So let $n \in \mathbb{N}$ and $a \in R$ be such that $a^n = 0$. We then have $a^n R = 0$ and hence $a^{n-1}Ra \subseteq Z(R)$ which in turns gives $a^{n-1}RaRa = 0$ and $a^{n-1}(Ra)^2R = 0$. Applying again our hypothesis leads to $a^{n-2}(Ra)^3 \subseteq Z(R)$ and $a^{n-2}(Ra)^4 = 0$. Continuing this process we finally get $(Ra)^{2n} = 0$. This shows that RaR is a nilpotent ideal and hence a belongs to the prime radical, as desired.

(b) If $P(R)$ is commutatively closed then $R/P(R)$ is semiprime and reversible. If an element $a \in R/P(R)$ is nilpotent, say $a^n = 0$ with $n \in \mathbb{N}$, the reversibility of $\overline{R} = R/P(R)$ implies that $a^{n-1}\overline{R}a = 0$ and continuing this process we finally get that $(\overline{R}a\overline{R})^n = 0$. Since $\overline{R}$ is semiprime, we conclude that $a = 0$. This shows that $R/P(R)$ is reduced and hence $P(R) = N(R)$, showing that R is 2-primal.

Conversely, if R is 2-primal, then $P(R) = N(R)$ and hence $P(R)$ is commutatively closed, as desired. $\square$

Example 3.9. The converse of statement (a) of Proposition 3.8 is not true. Indeed if k is a commutative field, the ring $R = k[x][t;\sigma]/(t^2)$, where σ is the k-algebra map defined by $\sigma(x) = 0$, is easily seen to be 2-primal but $xt + (t^2) \in \{0\}_1$ and is not central.

We have seen many instances of factorisation properties that are related to our commutatively closed sets. In the commutative case the order of factors appearing in a factorisation is irrelevant and in this sense looking at factorisations modulo commutatively closed classes generalizes the commutative case.

Classically besides the reversible rings another notion is also studied: the symmetric rings. This leads to the following definition.

Definition 3.10. *We say that a subset $S \subseteq R$ is symmetric if for any $a, b, c \in R$ we have that $abc \in S$ implies that $acb \in S$. In particular, $S = \{0\}$ is symmetric if and only if R is a symmetric ring.*

Since our rings all have unity, it is clear that a symmetric subset is commutatively closed. The next proposition generalizes classical facts obtained in the case when $S = \{0\}$. We write $\mathcal{S}_n$ for the symmetric group of permutations of a set of cardinal n.

Proposition 3.11. *Let $S \subseteq R$ be a subset in a unital ring. The following are equivalent*

(1) *S is symmetric.*
(2) *For any $a \in R$, the set $\{x \in R \mid ax \in S\}$ is commutatively closed.*
(3) *For any $n \in \mathbb{N}$ and for any elements $a_1, \dots, a_n \in R$ and any $\pi \in \mathcal{S}_n$, $a_1 \cdots a_n \in S$ implies that $a_{\pi(1)} \cdots a_{\pi(n)} \in S$.*
(4) *For any $n \in \mathbb{N}$ and $i \in \{1, 2, \dots, n\}$ and for any elements $a_1, \dots, a_n \in R$, we have that $a_1 \cdots a_n \in S$ implies that $a_1 \cdots a_{i+1}a_ia_{i+2} \dots a_n \in S$.*
(5) *For any $a \in R$, the set $\{x \in R \mid xa \in S\}$ is commutatively closed.*

Proof. (1) $\Rightarrow$ (2). This is a direct consequence of the definitions.
(2) $\Rightarrow$(3). Since the transpositions generate S_n, we only have to show that $a_1 \dots a_n \in S$ then for any $1 \leq i < j \leq n$, $a_1 \dots a_j a_{i+1} \dots a_{j-1} a_i a_{j+1} \dots a_n \in S$ We write successively (when $i = 1$, we use the fact that S is also commutatively closed) $(a_1 \dots a_{i-1})(a_j a_{j+1} \dots a_n)(a_i a_{i+1} \dots a_{j-1}) \in S$ regrouping the factors this gives

$(a_1 \dots a_{i-1} a_j)(a_{i+1} \dots a_{j-1})(a_{j+1} \dots a_n a_i) \in S$ and hence regrouping again leads to $a_1 \dots a_{i-1} a_j a_{i+1} \dots a_{j-1} a_i a_{j+1} \dots a_n \in S$. This shows that the action of the transposition (i, j) keeps the words of S in S.
(3) $\Rightarrow$(4) This is clear.
(4) $\Rightarrow$(1) This is also clear since $a_1 a_2 a_3 \in S$ implies, by (4), that $a_1 a_3 a_2 \in S$.
(4) $\Leftrightarrow$(5). Since (2) is equivalent to (3) and the statement (3) is left right symmetric, we conclude that the left right symmetric statement of (2) is also equivalent to (4). The left right symmetric statement corresponding to (2) is obviously the statement (5). This concludes the proof. □

As with the commutatively closed notion, we may construct for any subset $S \subseteq R$, a sequence of subsets of R leading to the closure of S, denoted $\widehat{S}$, which is the smallest symmetric set containing S. We define $S_{(1)}$ to be the set of all elements of R obtained by permuting the factors of any factorisation of an element of S. We then repeat this procedure i.e. we define inductively $S_{(n+1)} = (S_{(n)})_{(1)}$. These subsets form an increasing sequence and we put $\widehat{S} = \bigcup_n S_{(n)}$. Since $1 \in R$, it is easy to check that for any subset $S \subseteq R$, we have $S_1 \subseteq S_{(1)}$ and hence we always have $\overline{S} \subseteq \widehat{S}$.

One advantage of this construction is that it behaves nicely with respect to products:

Proposition 3.12. *Let $a, b \in R$ then $\widehat{\{a\}}\widehat{\{b\}} \subseteq \widehat{\{ab\}}$*

Proof. It is enough to show that $\widehat{\{a\}}_{(1)}\widehat{\{b\}}_{(1)} \subseteq \widehat{\{ab\}}_{(1)}$. Let $x_1 x_2 \dots x_n \in \widehat{\{a\}}_{(1)}$ and $y_1 y_2 \dots y_l \in \widehat{\{b\}}_{(1)}$, then there exist two permutations $\sigma \in S_n$ and $\tau \in S_l$ such that $a = x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(n)}$ and $b = y_{\tau(1)} y_{\tau(2)} \dots y_{\tau(l)}$. This gives a factorisation of ab and it is clear that $u := x_1 x_2 \dots x_n y_1 y_2 \dots y_l$ can be obtained by permuting this factorisation. This shows that $u \in \{ab\}_{(1)}$, as desired. □

We also have an equivalence relation: $a \equiv b \Leftrightarrow b \in \widehat{\{a\}}$. We intend to analyze further this equivalence relation and its connections with factorisations in a forthcoming paper.

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Departement of Mathematics, Université d'Artois, Rue Jean Souvraz, 62307 Lens, France

Email address: `dalghazzawi@kau.edu.sa`

Department of Mathematics, Université d'Artois, Rue Jean Souvraz, 62307 Lens, France

Email address: `andre.leroy@univ-artois.fr`