REMARKS ON THE JACOBSON RADICAL

ANDRÉ LEROY AND JERZY MATCZUK

Abstract. The aim of the paper is to investigate the set $\Delta(R) =: \{r \in R \mid r + U(R) \subseteq U(R)\}$ of a ring R. This set is a ring closely related to the Jacobson radical of R. It is shown that $\Delta(R)$ is the largest Jacobson radical subring of R which is closed with respect to multiplication by units of R. The behavior of Δ under ring constructions is studied, some families of rings for which $\Delta(R) = J(R)$ are presented. Methods of constructing rings with $\Delta(R) \neq J(R)$ are also described.

Introduction

It is well-known that if r is an element of the Jacobson radical $J(R)$ of R, then $r + u \in U(R)$ for every $u \in U(R)$, i.e.

$$J(R) \subseteq \Delta(R) =: \{r \in R \mid r + U(R) \subseteq U(R)\}.$$

It was already observed in [5, Exercise 4.24] that the above inclusion can be strict and that equality holds in the case of rings of stable range 1 (cf. [5, Exercise 20.10B]).

The aim of this paper is to investigate the set $\Delta(R)$ in details. We offer various characterizations of $\Delta(R)$ and then study the behavior of the operator Δ under some standard ring constructions. We also extend the definition and properties of Δ to rings without unity.

We show, in Theorems 1.3 and 1.12, that $\Delta(R)$ is the largest Jacobson radical subring of R which is closed with respect to multiplication by all units (quasi-invertible elements) of R, $\Delta(R) = J(T)$, where T is the subring of R generated by units of R, and the equality $\Delta(R) = J(R)$ holds if and only if $\Delta(R)$ is an ideal of R. As a consequence, we obtain that Δ is a closure operator, i.e. $\Delta(\Delta(R)) = \Delta(R)$ and that $\Delta(M_n(R)) = J(M_n(R)) = M_n(J(R))$, for any ring R and $n \geq 2$. Some further instances of rings for which $\Delta(R) = J(R)$ are presented in Theorem 1.10. Various examples exhibiting the differences in the behavior of Δ and the Jacobson radical are given in Examples 1.16 (including more instances of rings where the inclusion $J(R) \subseteq \Delta(R)$ is strict).

We also investigate Δ of polynomial rings and corners of rings. It appears (cf. Corollary 1.15) that for any ring R with 2 invertible, $\Delta(eRe) \subseteq e\Delta(R)e$, where e is denotes an idempotent of R. Moreover, contrary to the Jacobson radical, this inclusion can be strict. We conclude this short note by showing that, for a polynomial ring $R[x]$ over 2-primal ring R, we have $\Delta(R[x]) = \Delta(R) + J(R[x])$.

Key words and phrases. Jacobson radical.

In all the text R will stand for a ring usually with identity and $U(R)$ will denote the group of units of R. $J(R)$ and $B(R)$ stand for Jacobson and the prime radicals of R, respectively.

1. Description and properties of $\Delta(R)$

We begin with the following lemma which collects basic properties of

$$\Delta(R) = \{r \in R \mid \forall_{u \in U(R)} \ \ r + u \in U(R)\}.$$

Lemma 1.1. *For any ring R, we have:*

1. $\Delta(R) = \{r \in R \mid \forall_{u \in U(R)} \ \ ru + 1 \in U(R)\} = \{r \in R \mid \forall_{u \in U(R)} \ \ ur + 1 \in U(R)\}$*;*
2. *For any $r \in \Delta(R)$ and $u \in U(R)$, $ur, ru \in \Delta(R)$;*
3. *$\Delta(R)$ is a subring of R;*
4. *$\Delta(R)$ is an ideal of R iff $\Delta(R) = J(R)$;*
5. *For any rings R_i, $i \in I$, $\Delta(\prod_{i \in I} R_i) = \prod_{i \in I} \Delta(R_i)$.*

Proof. Notice that, for any $u \in U(R)$ and $r \in R$, $ru^{-1} + 1 \in U(R)$ iff $r + u \in U(R)$ iff $u^{-1}r + 1 \in U(R)$. This yields (1). The statement (2) is an easy consequence of (1).

(3) Let $r, s \in \Delta(R)$. Then $-r + s + U(R) \subseteq -r + U(R) = -r - U(R) \subseteq U(R)$, i.e. Δ is a subgroup of the additive group of R. Then also $rs = r(s + 1) - r \in \Delta(R)$, as $r(s + 1) \in \Delta(R)$ by (2).

(4) Clearly $J(R) \subseteq \Delta(R)$. Suppose that $\Delta(R)$ is an ideal and $r \in R$. Then $rx+1 \in U(R)$, for any $x \in R$ and $\Delta(R) \subseteq J(R)$ follows, i.e. $\Delta(R) = J(R)$. The reverse implication is clear.

We leave the easy proof of (5) to the reader. $\square$

When e is an idempotent of a ring R, then the element $1 - 2e$ is a unit of R. This observation and Lemma 1.1(2) give immediately the following corollary.

Corollary 1.2. *For any ring R:*

1. *$\Delta(R)$ is closed by multiplication by nilpotent elements;*
2. *If $2 \in U(R)$, then $\Delta(R)$ is closed by multiplication by idempotents.*

Recall that if R is a ring (not necessarily with 1), then the circle monoid $R_\circ = (R, \circ)$ of R is the set R with the circle operation $\circ$ defined by $x \circ y = x + y - xy$. If R is a unital ring, then the monoid $R_\circ$ is isomorphic to the multiplicative monoid $(R, \cdot)$ of R, the isomorphism is given by assigning to every x of $R_\circ$ the element $1 - x$. Moreover $y \in R$ is invertible as an element of the monoid $R_\circ$ (such elements are called quasi-invertible or quasi-regular) if and only if $1 + y$ is invertible as an element of the ring R and its inverse in $R_\circ$ is called the quasi-inverse of y. Thus the group of units $U(R)$ of R is isomorphic to the group $U_\circ(R)$ of quasi-invertible elements of R. It is known that $I = J(R)$ is the largest ideal of R such that $U_\circ(I) = I$.

Theorem 1.3. *Let R be a unital ring and T be the subring of R generated by $U(R)$. Then:*

1. *$\Delta(R) = J(T)$ and $\Delta(S) = \Delta(R)$, for any subring S of R such that $T \subseteq S$;*

(2) $\Delta(R)$ *is the largest Jacobson radical ring contained in* R *which is closed with respect to multiplication by units of* R.

Proof. (1) Notice that the subring T consists of all finite sums of units of R. Thus the statements (2) and (4) of Lemma 1.1 imply that $\Delta(T)$ is an ideal of T and $\Delta(T) = J(T)$, respectively.

If $r \in \Delta(R)$, then $r + U(R) \subseteq U(R)$. This means that r can be presented as a sum of two units. In particular, $r \in T$ and $\Delta(R) \subseteq T$ follows.

Let S be a subring of R such that $T \subseteq S$. Then $U(S) = U(R)$, so $\Delta(S) = \{r \in S \mid r + U(S) \subseteq U(S)\} = \{r \in S \mid r + U(R) \subseteq U(R)\} = S \cap \Delta(R) = \Delta(R)$, as $\Delta(R) \subseteq T \subseteq S$.

(2) By (1), $\Delta(R)$ is a Jacobson radical subring of R and Lemma 1.1(2) shows that $\Delta(R)$ is closed by left and right multiplication by units of R.

Now let S be a Jacobson radical ring contained in R which is closed by multiplication by units. If $s \in S$ and $u \in U(R)$, then $su \in S = J(S)$. Thus su is quasi-regular in S and hence $1 + su \in U(R)$. Now, Lemma 1.1(1) shows that $s \in \Delta(R)$, i.e. $S \subseteq \Delta(R)$, for any Jacobson radical subring of R. This proves (2). $\square$

The characterization of $\Delta(R)$ given in Theorem 1.3(2) gives immediately:

Corollary 1.4. *Let* R *be a ring such that every element of* R *is a sum of units. Then* $\Delta(R) = J(R)$.

The classical theorem of Amitsur states that the Jacobson radical of an F-algebra R over a field F is nil, provided $\dim_F R < |F|$. Applying statement (1) of Theorem 1.3 we get the following corollary.

Corollary 1.5. *Let* R *be an algebra over a field* F. *If* $\dim_F R < |F|$, *then* $\Delta(R)$ *is a nil ring.*

One can also apply directly the Amitsur's argument to prove the above corollary. In particular, when R is an algebra over a field F, Amitsur's argument shows that algebraic elements from $\Delta(R)$ are nilpotent.

For a unital ring R and its, not necessary unital, subring S, $\hat{S}$ will denote the subring of R generated by $S \cup \{1\}$. For further applications we will need the following

Proposition 1.6. *Let* R *be a unital ring. Then:*

(1) *Let* S *be a subring of* R *such that* $U(S) = U(R) \cap S$. *Then* $\Delta(R) \cap S \subseteq \Delta(S)$.

(2) $U(\widehat{\Delta(R)}) = U(R) \cap \widehat{\Delta(R)}$;

(3) *Let* I *be an ideal of* R *such that* $I \subseteq J(R)$. *Then* $\Delta(R/I) = \Delta(R)/I$.

Proof. The statement (1) is an easy consequence of the definition of Δ.

(2) Let us notice that if $r \in \Delta(R)$, then $v = 1 + r \in U(R)$ and $v^{-1} = 1 - rv^{-1} \in \widehat{\Delta(R)} \cap U(R)$, as by Lemma 1.1, $-rv^{-1} \in \Delta(R)$.

Let $u = r + k \cdot 1 \in \widehat{\Delta(R)} \cap U(R)$, where $r \in \Delta(R)$ and $k \in \mathbb{Z}$. We claim that $\bar{k} = k \cdot 1 \in U(R)$. We have $u - \bar{k} = r \in \Delta(R)$ and using Lemma 1.1, we obtain $1 - \bar{k}u^{-1} = (u - \bar{k})u^{-1} = ru^{-1} \in \Delta(R)$. Therefore $\bar{k}u^{-1} = 1 - (1 - \bar{k}u^{-1}) \in U(R)$ and $\bar{k} \in U(R)$

follows. As $\Delta(R)$ is closed by multiplications by units we can apply the first part of the proof to $v = u\bar{k}^{-1} = 1 + r\bar{k}^{-1}$ to obtain $u^{-1}\bar{k} = v^{-1} \in \widehat{\Delta(R)}$, i.e. $u^{-1}\bar{k} = s + \bar{l}$, for some $s \in \Delta(R)$ and $l \in \mathbb{Z}$. This implies, as $s\bar{k}^{-1} \in \Delta(R)$, $u^{-1} = s\bar{k}^{-1} + \bar{k}^{-1}\bar{l} \in \widehat{\Delta(R)}$ and shows that $U(R) \cap \widehat{\Delta(R)} \subseteq U(\widehat{\Delta(R)})$. The reverse inclusion $U(\widehat{\Delta(R)}) \subseteq U(R) \cap \widehat{\Delta(R)}$ is clear and (2) follows.

(3) Let $^-$ denote the canonical epimorphism of R onto R/I. Notice that, as $I \subseteq J(R)$, $U(\bar{R}) = \overline{U(R)}$.

Let $\bar{r} \in \Delta(\bar{R})$ and $u \in U(R)$. Then $\bar{r} + \bar{u} \in U(\bar{R})$ and there are elements $v \in U(R)$ and $j \in I$ such that $r + u = v + j$. Moreover $v + j \in U(R)$, as $I \subseteq J(R)$. This implies that $\Delta(\bar{R}) \subseteq \overline{\Delta(R)}$. Due to equality $U(\bar{R}) = \overline{U(R)}$, the reverse inclusion is clear, i.e. (3) holds. □

As an application of the above proposition we get the following

Corollary 1.7. *For any unital ring R, $\Delta(\widehat{\Delta(R)}) = \Delta(R)$, i.e. Δ is a closure operator.*

Proof. Notice that $\Delta(R)$ is a Jacobson radical ideal of $T = \widehat{\Delta(R)}$. Hence, $\Delta(R) \subseteq T$.

Since $\Delta(R)$ contains all central nilpotent elements, $T/\Delta(R)$ is either isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$, for some $n > 1$ which is square free. Therefore, by Proposition 1.6(3) and Corollary 1.4 we have $\Delta(T)/\Delta(R) = \Delta(T/\Delta(R)) = J(T/\Delta(R)) = 0$. This means that $\Delta(T) = \Delta(R)$, as required. □

Proposition 1.6(1) applies to $S = Z(R)$ - the center of R. Therefore, as for the Jacobson radical, we have:

Corollary 1.8. $\Delta(R) \cap Z(R) \subseteq \Delta(Z(R))$.

We will see in Example 1.16(4) that the inclusion from the above corollary can be strict even when $J(R) = J(Z(R)) = 0$.

For a ring R, let $T_n(R)$ denote the ring of all n by n upper triangular matrices over R, $J_n(R)$ the ideal of $T_n(R)$ consisting of all strictly upper triangular matrices and $D_n(R)$ the subring of diagonal matrices. As a direct consequence of Proposition 1.6(3) we get

Corollary 1.9. *For any ring R:*

(1) $\Delta(T_n(R)) = D_n(\Delta(R)) + J_n(R)$;
(2) $\Delta(R[x]/(x^n)) = \Delta(R)[x]/(x^n)$;
(3) $\Delta(R[[x]]) = \Delta(R)[[x]]$.

The following theorem indicates a few classes of rings in which $\Delta(R) = J(R)$.

Theorem 1.10. $\Delta(R) = J(R)$ *if R is a ring satisfying one of the following conditions:*

(1) *$R/J(R)$ is isomorphic to a product of matrix rings and division rings.*
(2) *R is a semilocal ring.*
(3) *R is a clean ring such that $2 \in U(R)$.*
(4) *R is a UJ-ring, i.e. when $U(R) = 1 + J(R)$.*
(5) *R has stable range 1.*

(6) *$R = FG$ is a group algebra over a field F.*

Proof. (1). Let R be as in (1). In virtue of Proposition 1.6(3) it is enough to show that $\Delta(R/J(R)) = 0$. For doing this, we may assume that $J(R) = 0$, i.e. R is a product of matrix rings and division rings. If R is a matrix ring $M_n(S)$, for some unital ring S and $n \geq 2$ then, by Theorem of M. Henriksen [3], every element of R is a sum of three units so by Corollary 1.4 $\Delta(R) = J(R) = 0$. When S is a division ring, then clearly $\Delta(S) = 0$. Now (1) is a direct consequence of Lemma 1.1(5).

The statement (2) is a special case of (1).

(3). Suppose R is a clean ring such that $2 \in U(R)$. If $e \in R$ is an idempotent, then $1-2e \in U(R)$ and $e = (\frac{1}{2} - \frac{1}{2}(1-2e))$ is a sum of two units. This shows that every element of R is a sum of three units, so the thesis is a consequence of Corollary 1.4.

(4). Suppose $U(R) = 1 + U(R)$. Let us now suppose that the ring R is a UJ-ring. Then if $r \in \Delta(R)$ we have $r + U(R) \subseteq U(R)$ i.e. $r + 1 + J(R) \subseteq 1 + J(R)$. This immediately shows that $r \in J(R)$ and $\Delta(R) = J(R)$ follows.

(5) This statement is Exercise 20.10b in the book [5]. We repeat the short argument for the convenience of the reader. Let us assume the ring R has stable range 1. We only have to show that if $r \in R$ is such that $r + U(R) \subseteq U(R)$ then $r \in J(R)$. Now, for any $s \in R$ we have, $Rr + R(1 - sr) = R$. Hence, by the stable range 1 condition, we know that there exists $x \in R$ such that $r + x(1 - sr) \in U(R)$. This gives $x(1 - sr) \in r + U(R) \subseteq U(R)$ and shows that $r \in J(R)$.

If $R = FG$ is as in (6), then clearly every element of R is a sum of units, so (6) follows as above. □

It is known that semilocal rings have stable range 1, thus the statement (2) of the above proposition is also a consequence of (5).

Let us record the following easy observation.

Lemma 1.11. *Let G be a subgroup of the additive group of a unital ring R. Then G is closed with respect to multiplication by invertible elements if and only if it is closed with respect to multiplication of quasi-invertible elements of R.*

Proof. Let $r \in R$. As G is an additive group, $rG \subseteq G$ if and only if $(1 - r)G \subseteq G$. This observation yields the lemma. □

Therefore, using the above we have:

Theorem 1.12. *Let R be a unital ring and G a subgroup of the additive group of R. Then the following conditions are equivalent:*

(1) $G = \Delta(R)$
(2) *G is the largest Jacobson radical subring of R which is closed with respect to multiplication by quasi-invertible elements of R*
(3) *G is the largest additive subgroup of R consisting of quasi-invertible elements and closed with respect to multiplication by quasi-invertible elements of R.*

Proof. Theorem 1.3 (2) and Lemma 1.11 show that $\Delta(R)$ is Jacobson radical subring of R which is closed by multiplication by quasi-invertible elements. Let G be an additive subgroup consisting of quasi-invertible elements and closed with respect to multiplication by quasi-invertible elements of R. In particular G is a Jacobson radical non unital subring of R and, by Lemma 1.11, G is closed by multiplication by units of R. Therefore Theorem 1.3 (2) gives $G \subseteq \Delta(R)$. This yields the theorem. $\square$

One can modify the definition of Δ to work for rings without unity. Namely, let us set $\Delta_\circ(R) = \{r \in R \mid r + U_\circ(R) \subseteq U_\circ(R)\}$. Then it is clear that if R is a unital ring, then $\Delta_\circ(R) = \Delta(R)$. For any ring R, not necessary containing 1, let R^1 denote the ring obtained from R by adjoining unity with the help of $\mathbb{Z}$. Then, making use of the fact that $U_\circ(\mathbb{Z}) = 0$, it is easy to check that

Lemma 1.13. *For any, not necessary unital, ring R, one has $\Delta_\circ(R) = \Delta_\circ(R^1) = \Delta(R^1)$.*

The above lemma says that we can extend the definition of Δ to all, not necessary, unital rings and all statements from Theorem 1.12 remain equivalent for arbitrary rings. Moreover, if one of the equivalent conditions holds, then $\Delta(\Delta(R)) = \Delta(R)$

A well-known classical result about the Jacobson radical $J(R)$ of a ring R states that $J(eRe) = eJ(R)e$, for any idempotent e of R. We will show that such equality is not satisfied in general for $\Delta(R)$. However the inclusion $e\Delta(R)e \subseteq \Delta(eRe)$ always holds under the mild assumptions that $e\Delta(R)e \subseteq \Delta(R)$. We have seen in Corollary 1.2 that this assumption holds automatically provided $2 \in U(R)$.

Proposition 1.14. *For any ring R, the following conditions hold:*
1. *Let $e^2 = e$ be such that $e\Delta(R)e \subseteq \Delta(R)$. Then $e\Delta(R)e \subseteq \Delta(eRe)$.*
2. *$\Delta(R)$ does not contain nonzero idempotents.*
3. *$\Delta(R)$ does not contain nonzero unit regular elements.*

Proof. (1) Let us first remark that if $y \in U(eRe)$, then $y_1 = y + (1-e) \in U(R)$ is such that $y = ey_1e$. Now, let $r \in e\Delta(R)e \subseteq \Delta(R)$, we want to show that for any unit $y \in U(eRe)$ we have that $e - yr \in U(eRe)$. As above, let $y_1 := y + 1 - e \in U(R)$. Since $r \in e\Delta(R)e \subseteq \Delta(R)$, we know that $1 - y_1r \in U(R)$. Hence there exists $b \in R$ such that $b(1 - y_1r) = 1$ and so $e = eb(1 - y_1r)e = eb(e - y_1re)e = eb(e - (y + 1 - e)re = eb(e - yre) + eb(1 - e)re = ebe(e - yre)$, where the last equality comes from the fact that $r \in eRe$. This gives that $e - yre = e - yr$ is left invertible in eRe. Since $1 - y_1r \in U(R)$ we also have $(1 - y_1r)b = 1$ and hence $1 = (1 - (y + 1 - e)r)b = (1 - yr)b$. Multiplying on both sides by e we get $e = e(1 - yr)be = (e - yr)be = (e - yr)ebe$. This shows that ebe is a right and left inverse of $e - yr$, as required.

(2) If $e^2 = e \in \Delta(R)$ then $1 - e = e + (1 - 2e) \in U(R)$, as $1 - 2e$ is a unit. This forces $e = 0$, i.e. (2) holds.

(3) if $a \in \Delta(R)$ is a unit regular element, then there exists an invertible element $u \in U(R)$ such that au is an idempotent. Statement (2) above shows that a must then be zero. $\square$

Corollary 1.2 and Proposition 1.14(1) yield the following:

Corollary 1.15. *Suppose $2 \in U(R)$. Then $e\Delta(R)e \subseteq \Delta(eRe)$, for any idempotent e of R.*

The following examples show instances where $\Delta(R) \neq J(R)$ and indicate limits for obtained results.

Examples 1.16. (1) Let us observe that Theorem 1.12 implies that if A be a subring of a ring R such that $U(R) = U(A)$, then $J(A) \subseteq \Delta(R)$. In particular taking A to be a commutative domain with $J(A) \neq 0$ and $R = A[x]$, we obtain $0 = J(R) \subset J(A) \subseteq \Delta(R)$. (see also [5, Exercise 4.24])

(2) (cf. [2, Example 2.5]) Let $R = \mathbb{F}_2 < x, y > / < x^2 >$. Then $J(R) = 0$ and $U(R) = 1 + \mathbb{F}_2 x + xRx$. In particular $\mathbb{F}_2 x + xRx$ is contained in $\Delta(R)$ but $J(R) = 0$.

(3) Let S be any ring such that $J(S) = 0$ and $\Delta(S) \neq 0$ and let $R = M_2(S)$. Then, by Theorem 1.10(1), $\Delta(R) = J(R) = 0$. Therefore, if $e = e_{11} \in R$, then $e\Delta(R)e = eJ(R)e = J(eRe) = 0$. and $\Delta(eRe) \simeq \Delta(S) \neq 0$. This shows that the inclusion $e\Delta(R)e \subsetneq \Delta(eRe)$ from Proposition 1.14 can be strict, in general.

(4) Let A be a commutative domain with $J(A) \neq 0$ and $S = A[x]$. Then, by (1), $0 \neq J(A) \subseteq \Delta(S)$ and clearly $J(S) = 0$. $R = M_2(S)$, where A is a commutative local domain. By Theorem 1.10, $\Delta(R) = J(R) = 0$. Notice that the center $Z = Z(R)$ of $R = M_2(S)$ is isomorphic to S and $U(Z) = U(R) \cap Z$. Therefore $0 = \Delta(R) \cap Z \subsetneq \Delta(Z) \simeq J(A) \neq 0$. Thus the inclusion from Corollary 1.8 can be strict even when $J(R) = 0 = J(Z(R))$.

The Example 1.16(2) was attributed to Bergman in [2]. Let us mention that this example appeared earlier in [1, Example 6], where prime rings with commuting nilpotent elements were considered.

Recall that a 2-primal ring is a ring such that the set of all nilpotent elements $N(R)$ of R coincides with the prime radical $B(R)$, i.e. $R/B(R)$ is a reduced ring. The following proposition can be considered as a generalization of Example 1.16(1).

Proposition 1.17. *Let R be 2- primal ring. Then $\Delta(R[x]) = \Delta(R) + J(R[x])$.*

Proof. Suppose first that R is a reduced ring. Then $U(R[x]) = U(R)$ (cf. [4]). Thus, by definition of $\Delta(R[x])$, we have $\Delta(R) \subseteq \Delta(R[x])$. Let $a + a_0 \in \Delta(R[x])$, where $a \in R[x]x$ and $a_0 \in R$. Then, for any $u \in U(R)$, $a + a_0 + u \in U(R)$. This forces $a_0 + u \in U(R)$ and $a = 0$ and $\Delta(R) = \Delta(R[x])$ follows in this case.

Suppose now that R is 2-primal. Clearly $B(R[x]) = B(R)[x] \subseteq J(R[x])$. As R is 2-primal, $R/B(R)$ is reduced, so $J(R[x]) = B(R[x]) = B(R)[x]$. By the first part of the proof applied to $R/B(R)$ and Proposition 1.6(2), we have

$$\Delta(R) + B(R)[x] = \Delta(R/B(R)[x]) = \Delta(R[x]/J(R[x])) = \Delta(R[x])/J(R[x]).$$

The above yields the desired equality. □

References

[1] M. Chebotar, P.-H. Lee, and E. R. Puczylowski, On prime rings with commuting nilpotent elements, Proc. AMS **137**(9) (2009) , 2899-2903.

[2] P. V. Danchev, T. Y Lam, Rings with unipotent units, Publ. Math. Debrecen 88 (2016).

[3] M. Henriksen, Two classes of rings generated by their units, J. Algebra 31 (1974), 182-193

[4] P. Kanwar, A. Leroy, J. Matczuk, Clean elements in polynomial rings, Contemporary Math. 634 (2015), 197-204.

[5] T.Y. Lam, Exercises in classical ring theory, Springer-Verlag, New York (2003), second edition.

Department of Mathematics, Université d'Artois, Rue Jean Souvraz, 62307 Lens, France
Email address: andre.leroy@univ-artois.fr

Departement of Mathematics, University of Warsaw, Ul. Banacha, 2, Warszawa, Poland
Email address: jmatczuk@mimuw.edu.pl