Idempotents in Ring Extensions

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Abstract

The aim of the paper is to study idempotents of ring extensions $R \subseteq S$ where S stands for one of the rings $R[x_1, x_2, \dots, x_n]$, $R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$, $R[[x_1, x_2, \dots, x_n]]$. We give criteria for an idempotent of S to be conjugate to an idempotent of R. Using them we show, in particular, that idempotents of the power series ring are conjugate to idempotents of the base ring and we apply this to get a new proof of the result of P.M. Cohn (Theorem 7, [5]) that the ring of power series over a projective-free ring is also projective-free. We also get a short proof of the more general fact that if the quotient ring R/J of a ring R by its Jacobson radical J is projective-free then so is the ring R.

Introduction

By the ring extension S of an associative unital ring R we mean, in this article, one of the following rings: the polynomial ring $R[x_1, \dots x_n]$ in finite number of commuting indeterminates $x_1, \dots, x_n$, the Laurent polynomial ring $R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ and the power series ring $R[[x_1, \dots x_n]]$. The aim of the paper is to study relations between idempotents of R and those of S. One of the motivations of our study is the Quillen and Suslin's solution of Serre's problem which says that every finitely generated projective module over a polynomial ring $K[x_1, \dots, x_n]$, where K is a field, is free (cf. [9] for more details). Since any finitely generated projective module is associated with an idempotent of a matrix ring, the above result can be translated in terms of idempotents as follows: every idempotent $e^2 = e \in M_l(K)[x_1, \dots, x_n]$ is conjugate to an idempotent in the base ring $M_l(K)$.

$E(R)$ will denote the set of all idempotents of a ring R. At the beginning, we present necessary and sufficient conditions for the equality $E(S) = E(R)$ (Corollary 6). As a byproduct of our investigations we obtain a short proof of a result of H. Bass on idempotents in commutative group rings.

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Two elements a, b in a ring R are said to be *conjugate* if there exists an invertible element $u \in R$ such that $b = uau^{-1}$. We provide, in Corollary 11, a sufficient condition for two idempotents in a ring to be conjugate. With the help of this condition we show, in Theorem 13, that any idempotent of $R[[x_1, \ldots, x_n]]$ is conjugate to an idempotent of R. We also study situations when a similar result holds for a polynomial ring.

Let us recall a few definitions: a ring R is *2-primal* if the set of its nilpotent elements is exactly the prime radical. A ring R is *abelian* if all idempotents of R belong to its center R. It is well-known that reduced rings are abelian (cf. [7]) . We show that every idempotent of a polynomial ring $R[x]$ is conjugate to an idempotent of R in the following cases: R is abelian, R is the matrix ring $M_n(A)$, where A is either a division ring or a polynomial ring $K[x_1, \ldots, x_m]$ over a field K. It is also shown that any idempotent of degree one in $R[x]$ is conjugate to an idempotent of R. Based on a result of Ojanguren-Sridharan we give an explicit example of a polynomial e of degree two with coefficients in $R = M_2(\mathbb{H}[y])$, which is an idempotent of $R[x]$ not conjugate to any idempotent of R. In fact, there are uncountably many non conjugate such idempotents.

We also show that, for any ring R, the semicentral idempotents of $R[x]$ are conjugate to idempotents of R (Theorem 18).

A ring R is *projective free* if every finitely generated left (equivalently right) R-module is free of unique rank. As a consequence of our investigations, we give new short proofs of a series of classical results. a theorem of P.M. Cohn saying that the projective-free property lifts up from R/J to R, where J is the Jacobson radical of R (Theorem 21(a)); a particular case of a result of I. Kaplansky which says that local rings are projective-free (Theorem 21(b)); another theorem of P.M. Cohn stating that if R is projective-free then so is $R[[x]]$ (Theorem 22) and a theorem of G. Song and X. Guo saying that two idempotents in a ring are equivalent if and only if they are conjugate (Corollary 20).

Idempotents

We begin with the following elementary result (cf. [1], Proposition 2.5) .

Lemma 1. *Let R be a ring and $e(x) = \sum_{i=0}^{\infty} e_i x^i \in R[[x]]$ be an idempotent. If $e_0 e_i = e_i e_0$, for every $i \geq 1$, then $e(x) = e_0$. In particular, if R is abelian, then $E(R[[x]]) = E(R[x]) = E(R)$.*

Proof. It is clear that e_0 is an idempotent of R. Assume that $e(x) \neq e_0$ and let $k > 0$ be the least index such that $e_k \neq 0$. Comparing the degree k coefficients of $e(x)^2$ and $e(x)$ we get $2e_0 e_k = e_k$. Multiplying this equality by e_0 we obtain $e_0 e_k = 0$ and hence also $e_k = 0$. This contradiction yields the result. □

The above observation is also contained in Lemma 8 of [6]. As a first application we prove the following result which can also be obtained by combining Propositions 2.4 and 2.5 in [1] and Lemma 1.7 in [2].

Proposition 2. *Let S_n denote one of the following rings $R[x_1, \ldots, x_n]$, $R[[x_1, \ldots, x_n]]$ and $R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$. If e is a central idempotent of S_n, then $e \in R$.*

Proof. Lemma 1 gives the result for rings $R[x]$ and $R[[x]]$. Since the element $1-x$ is invertible in $R[[x]]$, the R-monomorphism $\phi \colon R[x] \to R[[x]]$ which sends x onto $1-x$ has a natural extension to an R-monomorphism $\phi \colon R[x^{\pm 1}] \to R[[x]]$ given by $\phi(x^{-1}) = (1-x)^{-1}$. Let

$e \in R[x, x^{-1}]$ be a central idempotent. Since e commutes with elements of R the same property holds for the idempotent $\phi(e) \in R[[x]]$. This implies that $\phi(e) \in R$ and $e \in R$ follows. Hence the result holds for $R[x^{\pm 1}]$. Now it is standard to complete the proof by induction on n. □

Remark 3. As in the above proposition, let S_1 denote either $R[x], R[x, x^{-1}]$ or $R[x]$. It is easy to check that if δ is a derivation of a ring R then, for any central idempotent $e \in R$, we have $\delta(e) = 0$. Thus, if the additive group of R is torsion-free, taking the standard derivation $\delta = \delta/\delta x$ of the ring S_1 we directly get $e \in R$, for any central idempotent $e \in S_1$.

Let R be a ring and G a group.

If G is a finitely generated abelian group an R is a ring, we can write $RG \cong (RH)F$ where H is the torsion part of G and F is free abelian group of finite rank. From Proposition 2 we thus easily get:

Corollary 4. *(H.Bass) Let K be a commutative ring and G an abelian group with the torsion part H. Then any idempotent of KG belongs to KH.*

The following theorem offers a characterization of rings such that every idempotent of $R[x]$ belongs to R.

Theorem 5. *For a ring R, the following conditions are equivalent:*

(1) R is abelian;

(2) Idempotents of R commute with units of R;

(3) $E(R[[x]]) = E(R)$;

(4) $E(R[x^{\pm 1}]) = E(R)$;

(5) $E(R[x]) = E(R)$;

(6) There exists $n \geq 1$ such that $R[x]$ does not contain idempotents which are polynomials of degree n.

Proof. The implication (2) $\Rightarrow$ (1) is a direct consequence of the fact that for any $e \in E(R)$ and $r \in R$, $(er - ere)^2 = 0 = (re - ere)^2$. Hence $1 + (er - ere)$ and $1 + (re - ere)$ are units in R.

The implication (1) $\Rightarrow$ (3) is given by Lemma 1.

(3) $\Rightarrow$ (4) Let $\phi\colon R[x^{\pm 1}] \to R[[x]]$ denote the R-homomorphism defined in Proposition 2. Then $E(\phi(R[x^{\pm 1}])) \subseteq E(R[[x]]) = E(R)$. Since ϕ is injective and $\phi|_R = \mathrm{id}_R$, we get $E(R[x^{\pm 1}]) = E(R)$.

The implication (4) $\Rightarrow$ (5) is clear as $R[x] \subseteq R[x^{\pm 1}]$ and (5) $\Rightarrow$ (6) is a tautology.

(6) $\Rightarrow$ (2) Suppose (6) holds and let $e \in E(R)$. If e is not central in R, then either $eR(1-e)$ or $(1-e)Re$ is nonzero. Suppose $0 \neq r \in eR(1-e)$. Then $(1 + rx^n)(1 - rx^n) = 1$. Hence $1 + rx^n$ is an invertible element and consequently $(1 + rx^n)e(1 - rx^n) = e - rx^n$ is an idempotent of degree n, a contradiction. Hence e has to be central in R. Thus, in particular, (2) holds. □

If S is an abelian ring then so is any of its subrings. This observation and the above theorem yield that we could add in Theorem 5 the following next equivalent statement "the ring S is abelian", where S stands for any of the rings $R[x]$, $R[[x]]$, $R[x^{\pm 1}]$. Therefore, using induction with respect to the number of indeterminates we obtain the following:

Corollary 6. *Let S denote one of the rings $R[x_1,\ldots,x_n]$, $R[x_1^{\pm 1},x_2^{\pm 1},\ldots,x_n^{\pm 1}]$, $R[[x_1,\ldots,x_n]]$. Then R is is abelian if and only if S is abelian if and only if $E(S)=E(R)$.*

A weaker version of the above corollary holds in the more general setting of graded rings. Let M denote an additive monoid with a neutral element 0.

Proposition 7. *Suppose $R=\bigoplus_{m\in M} R_m$ is an M-graded ring. Then:*

(1) If R_0 is abelian and $E(R)=E(R_0)$, then R is an abelian ring;

(2) Suppose $M=\mathbb{Z}$. Then R is abelian if and only if R_0 is abelian and $E(R)=E(R_0)$;

Proof. (1) Suppose R_0 is abelian and $E(R)=E(R_0)$. For showing that R is abelian, it is enough to prove that for every $e\in E(R)$ and homogeneous element $r\in R$, $er(1-e)=(1-e)re=0$. One can check that $e+er(1-e)$ is an idempotent. Thus, as $E(R)=E(R_0)$, $er(1-e)\in R_0$. This implies that $er(1-e)=0$ if $r\in R_m$ with $m\neq 0$. If $r\in R_0$, then $er(1-e)=0$ as R_0 is abelian. Replacing e by $1-e$ we get $(1-e)re=0$.

(2) Let $R=\bigoplus_{m\in\mathbb{Z}} R_m$ be $\mathbb{Z}$-graded. One implication is given by the statement (1).

Suppose R is abelian. Then R_0 is abelian as a subring of R. Moreover, by Corollary 6, the ring $R[x^{\pm 1}]$ is also abelian and $E(R)=E(R[x^{\pm 1}])$.

Let $\psi\colon R\to R[x^{\pm 1}]$ be the monomorphism defined by $\psi(r_i)=r_ix^i$, for any homogeneous element $r_i\in R_i$, $i\in\mathbb{Z}$. Let $e=\sum_{i\in\mathbb{Z}} e_i\in E(R)$, where $e_i\in R_i$ and $e_i=0$, for almost all $i\in\mathbb{Z}$. Then $\psi(e)=\sum_{i\in\mathbb{Z}} e_ix^i\in R$. This implies that $e=e_0\in R_0$ and proves that $E(R)\subseteq E(R_0)$. The reverse inclusion is clear. □

Remarks 8. 1. Using the full strength of Corollary 6, one can replace $M=\mathbb{Z}$ in statement (2) of Proposition 7, by $M=\mathbb{Z}^n$.
2. The analogue of Proposition 7(2) for rings graded by finite groups clearly does not hold. Indeed, let M is a finite abelian group of order $n\geq 2$. Then the group algebra $\mathbb{C}M$ over the field $\mathbb{C}$ of complex numbers is clearly abelian and M-graded. Moreover $E(R_0)=\{1\}$ is not equal to $E(R)$ as R is isomorphic to a direct product of n copies of $\mathbb{C}$.
3. We have seen in Corollary 6, that the polynomial ring $R[x]$ is abelian if and only if R is abelian. The analogue of this statement does not hold for $\mathbb{Z}$-graded rings. Indeed, let K be a field and $e=(1,0)\in K\times K=R_0$. Then R_0 is commutative but the idempotent e is not central in $R=R_0[x;\sigma]$, where σ is the automorphism of $R_0=K\times K$ switching components.

Theorem 5 describes the situation when $E(R[x])=E(R)$. Now we will present certain sufficient conditions for all idempotents of $R[x]$ to be conjugate to idempotents of R. For this we need some preparation. If $e(x)=\sum_{i=0}^n e_ix^i\in R[x]$ is an idempotent then $e(x)=e_0+b$ where e_0 is an idempotent in R. Motivated by this observation we state the following proposition.

Proposition 9. *Let e,b,u be elements of R such that $e^2=e$ and $u=2e-1$. Then the following conditions are equivalent:*

(i) $e+b$ is idempotent;

(ii) $be + eb + b^2 = b$;

(iii) $(1 + bu)e = (e + b)(1 + bu)$.

Moreover, if one of the equivalent statements holds then:

(iv) $bu + ub = -2b^2$;

(v) $b^2u = ub^2$ *and* $(1 + bu)(1 + ub) = (1 + ub)(1 + bu) = 1 - b^2$;

(vi) $1 + bu$ *is invertible iff* $1 + ub$ *is invertible iff* $1 - b^2$ *is invertible;*

(vii) $(1 + 2ub)(1 + 2bu) = 1$ *and* $b^2u = ub^2$.

Proof. $(i) \Leftrightarrow (ii)$ This is easy to check.

$(ii) \Rightarrow (iii)$ First notice that multiplying the equality in (ii) by e on the left and on the right we easily get $ebe = -eb^2 = -b^2e$. We then have $(e+b)(1+bu) = e+b+eb(2e-1)+b^2(2e-1) = e + b - eb - b^2 = e + be = (1 + b)e = (1 + bu)e$.

$(iii) \Rightarrow (ii)$ Observe that $u^2 = 1$. Using the equality given in (iii) multiplied by u on the right, we have $e + be = (1 + bu)e = (1 + bu)eu = (e + b)(1 + bu)u = (e + b)(u + b) = eu + bu + eb + b^2 = e + b(2e - 1) + eb + b^2 = e + 2be - b + eb + b^2$. This gives $be - b + eb + b^2 = 0$ as desired.

Suppose now that one of the equivalent statements $(i) - (iii)$ holds. One can directly check, with the help of (ii), that $bu + ub = 2(be + eb - b) = -2b^2$.

(v) Using (iv) and the fact that $u^2 = 1$, one can get $(1 + bu)(1 + ub) = 1 - b^2$. It was shown, in the proof of $(ii) \Rightarrow (iii)$, that e commutes with b^2. Hence also $b^2u = ub^2$. Thus $(1 + ub)(1 + bu) = 1 + bu + ub + ubbu = 1 - 2b^2 + ub^2u = 1 - b^2$.

(vi) It is known (see Ex.1.6, [8]) that the element $1 + bu$ is invertible if and only if $1 + ub$ is invertible (even when $u, b \in R$ are arbitrary). Thus, if $1 + ub$ is invertible then, by (v), $1 - b^2$ is invertible. Finally, suppose that $1 - b^2$ is invertible. Then the statement (v) shows that $1 + bu$ is invertible.

Using the statement (iv) and the fact that $b^2u = ub^2$, it is easy to complete the proof of (vii). □

Remarks 10. 1. One can check that all statements from Proposition 9 are equivalent, provided R is 2-torsion free.

2. Let $e, b \in R$ be such that $e, e' = e + b \in E(R)$. Set $u = 2e - 1$. We state without proof a few relevant facts (which will not be used in the sequel):

(a) If $k \in \mathbb{N}$ is odd, then $(eb^k + eb^{k+1})e = 0$;

(b) If $k \in \mathbb{N}$ is odd, then $e + eb + eb^2 + \cdots + eb^{k-1}$ is an idempotent;

(c) If $b^k = 0$, then $e + eb + eb^2 + \cdots + eb^{k-1}$ is always an idempotent;

(d) $e - (1 + 2ub)b$ is an idempotent and we have $(1 + ub)e = (e - (1 + 2ub)b)(1 + ub)$;

(e) $e + 2b(1 + ub)$ is an idempotent and we have $(e + b)(1 + ub) = (1 + ub)(e + 2b(1 + ub))$;

(f) If $be = eb$, then $b = b^3$. In particular, b^2 is an idempotent.

The statements (iii) and (vi) of Proposition 9 gives directly the following corollary.

Corollary 11. *Let $e, b \in R$ be such that $e, e+b \in E(R)$. If $1-b^2$ is invertible, then e and $e+b$ are conjugate. In particular, this holds when either b is nilpotent or $b \in J(R)$, the Jacobson radical of R.*

In the following example we will exhibit two idempotents e and $e+b$ that are conjugate although $1-b^2$ is not invertible.

Example 12. Let R be the ring of 2×2 matrices over a ring S. One can easily check that the matrices

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad b = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are such that $e^2 = e$, $(e+b)^2 = e+b$ and $P \in GL_2(S)$ is such that $P(e+b)P^{-1} = e$ but $1-b^2 = 0$ is not invertible.

With the help of Corollary 11 we have:

Theorem 13. *Any idempotent f of $R[[x]]$ is conjugate to its constant term. Thus, in particular, any idempotent of $R[[x_1, \ldots, x_n]]$ is conjugate to an idempotent of R.*

Proof. Let $f = \sum_{i=0}^{\infty} e_i \in R[[x]]$ be an idempotent. Then $f = e+b$, where $e = e_0$ is an idempotent and $b = \sum_{i=1}^{\infty} e_i$. Since $1+bv$ is an invertible element of $R[[x]]$ for any element v of $R[[x]]$ we get $b \in J(R)$. Now, Corollary 11 yields that f and e are conjugate. The second part of the theorem follows by induction. □

There are many other ring extensions where Corollary 11 can be applied. In the following corollary we present some of them, that seem to be interesting:

Corollary 14. *Let R be a ring. Then:*
(1) Any idempotent of $R[x]/(x^n)$ is conjugate to an idempotent of R;
(2) Any idempotent of the upper triangular matrix ring $A_n(R)$ of $n \times n$ matrices over R is conjugate to a diagonal idempotent matrix;
(3) If S is another ring and ${}_RM_S$ is an (R, S)-bimodule, then any idempotent of the ring $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is conjugate to an idempotent of $R \oplus S$.

Proof. It is easy to see that every idempotent of $R[x]/(x^n)$ (resp. of $A_n(R)$ or of T) can be presented in a form $e+b$, where e is an idempotent of R (resp. a diagonal idempotent of $A_n(R)$, or of $R \oplus S$) and b is a nilpotent element. Thus, Corollary 11 gives statements (1) and (2) and (3). □

Corollary 15. *Let R be any ring and $e(x) = e + cx^n \in R[x]$ be an idempotent, where $e, c \in R$ and $n \geq 1$. Then $e(x)$ is conjugate in $R[x]$ to $e = e^2 \in R$. In particular, every idempotent of $R[x]$ having degree one is conjugate to an idempotent of R.*

Proof. Since $e(x)$ is an idempotent, e has to be an idempotent and $b = cx^n$ a nilpotent element. Now the corollary is a direct consequence of Corollary 11 applied to the ring $R[x]$. □

Recall that a ring R is called 2-primal if the set of all nilpotent elements of R coincides with the prime radical $B(R)$ of R (equivalently, all minimal prime ideals of R are completely prime).

For a ring R, let $S_n(R)$ denote one of the ring extensions $R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$, $R[x_1, \ldots, x_n]$. Keeping this notation we have the following theorem.

Theorem 16. *Suppose R is a 2-primal ring. Then any idempotent of $S_n(R)$ is conjugate to an idempotent of R.*

Proof. Let $B = B(R)$. It is well known and easy to check that $BS_n(R)$ is a nil ideal of $S_n(R)$ and the factor ring $S_n(R)/BS_n(R)$ is isomorphic to $S_n(R/B)$. Moreover, as R is 2-primal, R/B and hence $S_n(R/B)$ is a reduced ring. Since reduced rings are abelian, Corollary 6 shows that $E(S_n(R/B)) = E(R/B)$. This and the fact that idempotents can be lifted modulo nil ideals (see Theorem 21.38 of [7]) imply that every idempotent f of $S_n(R)$ can be presented in a form $f = e + b$ where $e = e^2 \in R$ and $b \in BS_n(R)$. In particular, b is a nilpotent element and Corollary 11 shows that f is conjugate to $e \in R$. This gives the result. □

In the next theorem we give yet another situation where an idempotent of a polynomial ring is conjugate to its constant term. Recall that an idempotent e of a ring R is called right semicentral if $er = ere$, for all $r \in R$. Similarly one defines left semicentral idempotents.

Proposition 17. *Let $T \subseteq S$ be a ring extension and $e, f \in T$ be right semicentral idempotents of S. If e, f are conjugate in S, then:*

(1) e=ef and f=fe;

(2) e and f are conjugate in T.

Proof. (1) Let $v \in S$ be an invertible element of S such that $f = vev^{-1}$. Using the fact that e is right semicentral in S, we get $fr = vev^{-1}r = vev^{-1}re = fre$, for any $r \in R$. Taking $r = 1$, we get $f = fe$. Similarly, changing the roles of e and f we obtain $e = ef$.

(2) Let $b = f - e \in T$, that is, $f = e + b$. By (1), $b^2 = 0$. Thus, the statement is a direct consequence of Corollary 11. □

Theorem 18. *Let f be a right (resp. left) semicentral idempotent of $R[x]$ (resp. of $R[x, x^{-1}]$). Then f is conjugate to the constant term of f.*

Proof. We give the proof in the case $e \in R[x]$ is right semicentral. If e is right semicentral then, with the help of right version of Proposition 17, the same proof works.

Let $f = e + b$, where $e \in R$ is the constant term of f and $b \in R[x]x$. Then, by Theorem 13, idempotents e and f are conjugate in the power series ring $S = R[[x]]$. Observe that, for $n \geq 0$ the coefficient at x^n of the product hp of two power series $p, h \in R[[x]]$ depends only of a finite number of coefficients of h and p. This implies that the semicentral idempotent f of $R[x]$ is also semicentral as idempotent of $R[[x]]$. Hence, by Proposition 17 (2), idempotents e and f are conjugate in $T = R[x]$.

The case of an idempotent element in $R[x, x^{-1}]$ is treated similarly using the emebedding ϕ of $R[x, x^{-1}]$ into $R[[x]]$ given in the proof of the implication (3) $\Rightarrow$ (4) in Theorem 5. □

Remark 19. Let $e, f \in R$ be conjugate, right semicentral idempotents of a ring T. The identities from Proposition 17 were observed first in Propositions 2.4, 2.5 of [1] in the special case when T is either $R[x]$ or $R[[x]]$, f is an idempotent of T and e is the constant term of f.

As another application of Corollary 11 we now give a result of Guo and Song ([10]). Recall that two elements e, e' of a ring R are called equivalent if there exist invertible elements $p, q \in R$ such that $e' = peq$.

Corollary 20. *(Theorem 2 of [10]) Let e, e' be two idempotents of a ring R. Then e and e' are equivalent if and only if they are conjugate.*

Proof. It is enough to prove that if $e' = peq$ with p, q invertible in R then pep^{-1} is conjugate to e'. According to Corollary 11 it suffices to prove that $b := pep^{-1} - e'$ is nilpotent. Since $e'^2 = e'$ we get $eqpe = e$ and $(pep^{-1} - peq)^2 = p(e - eqp)^2 p^{-1} = p(e - eqpe - eqp + eqpeqp)p^{-1} = 0$, as required. □

Recall that a ring R is called projective-free if every finitely generated projective R-module is free of unique rank. According to Proposition 0.4.5 in [4], a ring is projective-free precisely when it has invariant basis number (IBN for short) and every idempotent matrix is conjugate to a matrix of the form $diag(1, \dots, 1, 0 \dots, 0)$. As an application of Corollary 11 we obtain a new proof of a classical result by I. Kaplansky: Projective modules over a local ring are free. We give a short proof for the case of finitely generated projective modules.

Theorem 21. *(1) Let I denote an ideal of ring R contained in the Jacobson radical $J(R)$ of R. If R/I is projective-free then R is also projective-free;*

(2) Every local ring R is projective-free.

Proof. (1) We first remark that R has IBN as R/I has IBN. Now, consider $e^2 = e \in M_n(R)$. Let us write $\bar{e} := e + M_n(I) \in M_n(R)/M_n(I) \cong M_n(R/I)$. Since R/I is projective-free, there exists an invertible matrix $\bar{P} \in GL_n(R/I)$ such that $\bar{P}\bar{e}\bar{P}^{-1}$ is a diagonal matrix of the form $\bar{D} = diag(\bar{1}, \dots, \bar{1}, \bar{0} \dots, \bar{0}) \in M_n(R/I)$. Since $I \subseteq J(R)$, units of $M_n(R/I)$ lift to units of $M_n(R)$. Therefore there exists an invertible matrix $P \in GL_n(R)$ such that $P + M_n(I) = \bar{P}$. This leads to $PeP^{-1} - diag(1, \dots, 1, 0, \dots, 0) \in M_n(I) \subseteq J(M_n(R))$. Hence, by Corollary 11, PeP^{-1} is conjugate to $diag(1, \dots, 1, 0, \dots, 0)$. This yields the result.

The statement (2) is a direct consequence of (1), as every division ring is projective-free. □

As a direct application of Theorem 21(1) we obtain the fact that if a ring R is projective free then so is $R[[x]]$. This was first proved by P.M. Cohn (cf. Theorem 7, [5]). We now present another short proof of this result based on Theorem 13.

Theorem 22. *(P.M. Cohn) Let R be any projective-free ring. Then the power series ring $R[[x]]$ is again projective-free.*

Proof. Since R has the IBN property and is an homomorphic image of $R[[x]]$, the ring $R[[x]]$ has the IBN property as well. By Proposition 0.4.5, [4], it is enough to show that, for any $n \geq 1$, every idempotent matrix $e \in M_n(R[[x]])$ is conjugate to a matrix of the form $diag(1, \dots, 1, 0 \dots, 0)$. Notice that $T = M_n(R[[x]])$ is naturally isomorphic to the ring $M_n(R)[[x]]$. Thus, by Theorem 13, the idempotent e is conjugate in T to an idempotent of $M_n(R)$ which, in turn, is conjugate in $M_n(R) \subseteq T$ with an idempotent of the form $diag(1, \dots, 1, 0 \dots, 0)$, as R is projective-free. □

Remark 23. Suppose B is a ring such that the ring $B[x]$ is projective-free. Then, looking at $M_n(B)[x]$ as $M_n(B[x])$, we see that every idempotent of $R[x]$ is conjugate to an idempotent of $R = M_n(B)$.

In particular, the above remark applies to:
1. B is equal to the polynomial ring $K[x_1, \dots, x_m]$ over a field K, where $m \geq 1$ (by Quillen-Suslin solution of Serre's Problem);
2. B is any division ring (by Theorem 2.5 page 73 of [9]).

Theorem of Ojanguren-Sridharan (cf. Theorem 3.1, page 74 of [9]) states that when D is a noncommutative division ring, then $R = D[x, y]$ is not projective-free. Computing the matrix associated to the map $\mathrm{Id} - \psi\phi$ given in the proof of the above mentioned theorem in the case $D = \mathbb{H}$ is the division ring of real quaternions and $a = i$, $b = j$, we get the polynomial $e(x) = A_0(y) + A_1(y)x + A_2(y)x^2 \in M_2(\mathbb{H}[y])[x] \simeq M_2(\mathbb{H}[x, y])$ such that $e(x)$ is an idempotent of $M_2(\mathbb{H}[y])[x]$ which is not conjugate to $diag(\epsilon_1, \epsilon_2)$, where $\epsilon_1, \epsilon_2 \in \{0, 1\}$. Therefore, as $\mathbb{H}[y]$ is projective-free, $e(x)$ is not conjugate to any idempotent of the base ring $M_2(\mathbb{H}[y])$. Coefficients of this polynomial are

$$A_0(y) = \frac{1}{2}\begin{pmatrix} 1 - jy & k + ky^2 \\ -k & 1 - jy \end{pmatrix}, \quad A_1(y) = \frac{1}{2}\begin{pmatrix} i + ky & 0 \\ 0 & i - ky \end{pmatrix}, \quad A_2(y) = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ -k & 0 \end{pmatrix}$$

A result of Parimala-Sridharan (see Theorem 1.17, page 169 of [9]), states that there exist uncountably many mutually nonisomorphic nonfree projective modules P (similar to the one presented above) over the polynomial ring $\mathbb{H}[x, y]$, such that $P \oplus \mathbb{H}[x, y] \simeq \mathbb{H}[x, y]^2$. This implies that there are uncountably many polynomials of degree 2 in $M_2(\mathbb{H}[y])[x]$ which are idempotents not conjugate to each other and not conjugate to any idempotent of $M_2(\mathbb{H}[y])$

We close the paper by a partial generalization of Theorem 16 to matrices over 2-primal rings. We say, following Steger [11], that a ring R is ID if every idempotent matrix over R is conjugate to a diagonal one. Of course, every projective-free ring R is ID.

Proposition 24. *Let R be a 2-primal ring such $R[x]$ is an ID-ring. Then every idempotent $e \in M_n(R)[x]$ is conjugate to a diagonal matrix of the form $diag(e_1, \ldots, e_n) \in M_n(R)$, where e_i's denote idempotents in R.*

Proof. Let $e \in M_n(R)[x]$ be an idempotent. We look at $M_n(R)[x]$ as $M_n(R[x])$. Since $R[x]$ is an ID ring, e is conjugate to a diagonal matrix, that is, we can find an invertible matrix $u \in M_n(R[x])$ such that $ueu^{-1} = \mathrm{diag}(f_1, \ldots, f_n)$, where $f_i \in E(R[x])$. Since R is 2-primal, Theorem 16 shows that, for any $1 \leq i \leq n$ the idempotent f_i is conjugate to a certain $e_i \in E(R)$. Hence e is conjugate to $diag(e_1, \ldots, e_n) \in M_n(R)$ □

Finally we remark that any commutative ring R such that $R/B(R)$ is a principal ideal ring fulfills the assumptions of the above proposition (cf. Theorem 11, [11]).

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