INVARIANT AND SEMI-INVARIANT POLYNOMIALS IN SKEW POLYNOMIAL RINGS

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§1. Introduction

Let K be a ring with a given endomorphism S. An additive map $D : K \longrightarrow K$ is called an S-derivation on K if $D(ab) = S(a)D(b) + D(a)b$ for all $a, b \in K$. Given the triple (K, S, D), we can form the *skew polynomial ring* $R = K[t, S, D]$ whose elements are (left) polynomials of the form $\sum a_i t^i$ $(a_i \in K)$, with multiplication induced by $ta = S(a)t + D(a)$ $(\forall a \in K)$. The study of the ring $K[t, S, D]$ was inaugurated by Ore [O], and gained popularity subsequently through the influential work of Jacobson [J_1], [J_2: Ch. 3] and Amitsur [A_1], [A_2]. Modern treatments of the basic theory of $K[t, S, D]$ have appeared in [Co], [MR] and [JS].

A major concern in the theory of skew polynomial rings is the investigation of the ideal structure in such rings. For this investigation, the first important case is naturally that in which K is a division ring. In this case, assuming that $D = 0$ and S is an automorphism of K, Jacobson determined the structure of the 2-sided ideals of $R = K[t, S, D]$ in [J_2: Ch. 3]. In the more general case when K is a simple ring, assuming that $S = I$, Amitsur also determined the structure of the 2-sided ideals in R [A_2]. More recently, this line of investigation was continued in the work of Carcanague [C], Cauchon [Ca], Leroy-Tignol-van Praag [L_1], Lemonnier [Le] and Leroy [L_2], among others. In [C], K was assumed to be a division ring but there was no restriction on (S, D); in [Ca, L_1, Le], K was allowed to be a simple ring, but S was assumed to be an automorphism of K. In these papers and theses, various results were obtained on the ideal structure of the ring R.

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In the present work, *we shall first assume K to be a division ring*, and investigate the relationship between two classes of polynomials in $R = K[t, S, D]$, namely the invariant polynomials and the semi-invariant polynomials. By definition, a polynomial $p(t) \in R$ is *right invariant* if $p(t)R \subseteq R \cdot p(t)$, and *right semi-invariant* if $p(t)K \subseteq K \cdot p(t)$. (To simplify language, we shall drop the adjective "right" in the following.) The significance of the notion of an invariant polynomial lies in the fact that, if $p(t)$ is invariant, then $R \cdot p(t)$ is a 2-sided ideal in R, and conversely, any 2-sided ideal in R arises in this manner. Thus, the determination of the ideal structure in R is equivalent to the determination of its invariant polynomials; in particular, R is not a simple ring iff it has a nonconstant invariant polynomial. The notion of a semi-invariant polynomial, on the other hand, serves as a first approximation to the notion of an invariant polynomial: in fact, $p(t)$ is invariant iff $p(t)$ is semi-invariant and $p(t)t \in R \cdot p(t)$. In addition, semi-invariant polynomials arise naturally in the theory of semi-linear transformations [J_1], and in the study of minimal polynomials of algebraic sets [LL]. In the case when S is an automorphism, Lemonnier [Le] proved the remarkable fact that the following three conditions are equivalent[2]:

(A) *R is not a simple ring;*
(B) *There exists a (nonconstant) monic semi-invariant polynomial in R;*
(C) *D is a quasi-algebraic derivation, i.e. there exists an operator equation $b_nD^n + \cdots + b_1D + b_0D_{c,\theta} = 0$, where $c, b_i \in K$, $b_n \neq 0$, θ is an endomorphism of K, and $D_{c,\theta}$ denotes the θ-inner derivation sending any $x \in K$ to $cx - \theta(x)c$.*

Here, (B)$\Longleftrightarrow$(C) is not difficult, but the equivalence of (B), (C) with (A) is much harder, and Lemonnier's rather elaborate proof of this equivalence (using "deviations" and "codeviations") seemed to be not well-understood. In our view, it seemed that there should exist a way of directly constructing a nonconstant invariant polynomial from a nonconstant semi-invariant polynomial, thereby yielding a natural proof for (B)$\Longrightarrow$(A). Indeed, it was this viewpoint which prompted the present investigation.

Now we can summarize the results obtained in this paper. We shall assume, throughout §§2-4, that K is a division ring, but S is not assumed to be an automorphism unless stated otherwise. In §2, we prove various identities on $p(t)t - tp(t)$ for any monic semi-invariant polynomial $p(t)$, and recapture Lemonnier's result that, when $p(t)$ is of minimal degree[3], $p(t)t$ has the form $(t+c)p(t)+d$ for suitable scalars c, d, determined explicitly by $p(t)$. The element d thus provides a measure of how far $p(t)$ is from being invariant; further, it has the remarkable property that, for any $a \in K$, $S^{n+1}(a)d = da$, where $n = deg\ p(t)$. From this result, it follows that, *if S is not an automorphism, then $p(t)$ is already invariant, and if S is an automorphism of infinite inner order*[4], *then, in fact, any semi-invariant polynomial is invariant.* These results are contained in (2.5) and (2.6).

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[2]Lemonnier's results hold, more generally, for any simple ring K.

[3]Of course, constant polynomials are both invariant and semi-invariant. Thus, whenever we speak of invariant or semi-invariant polynomials of minimal degree, it will always be understood that we are considering only *nonconstant* polynomials.

[4]The *inner order* of an automorphism S is the order of the image of S in the group of automorphisms of K modulo the normal subgroup of inner automorphisms.

In §3, we study the powers of a minimal monic semi-invariant polynomial $p(t)$, and obtain a criterion for $p(t)^r$ to be invariant (Theorem 3.1). This gives us a handle on the set of *invariant* polynomials contained in the left ideal $R \cdot p(t)$. More formally, let $J = \{h(t) \in R : \ h(t)R \subseteq R \cdot p(t)\}$ be the "bound" of $R \cdot p(t)$: this is the largest 2-sided ideal of R contained in $R \cdot p(t)$ (see [J_2: p.39]). In (3.8), we show that this bound J is given by $R \cdot p(t)^s$ for some integer $s \leq deg\, p(t)$. In the case when S is not an inner automorphism, we show further that $p(t)^s$ is, in fact, an invariant polynomial *of minimal degree in* R (Theorem 3.10). In particular, this provides a strong form of the implication (B)$\Longrightarrow$(A) above, in the mean time generalizing it to the case when S is any endomorphism. Here, the conclusion that $p(t)^s$ is invariant of minimal degree is quite significant, because then $p(t)^s$ can be used to describe the center of R as well as the set of all the 2-sided ideals in R, via the results of Cauchon [Ca].

In §4, continuing to assume that K is a division ring, we study the bound J of $R \cdot p(t)$ when $p(t)$ is a semi-invariant polynomial *not necessarily of minimal degree*. If $S(K)$ has finite right codimension m in K and $deg\, p(t) = n$, we show that J has the form $R \cdot f(t)$ where $f(t)$ is an invariant polynomial of degree $\leq n(1 + m + \cdots + m^{n-1})$ (Theorem 4.2). This part of our paper is closely related to the work of Carcanague [C], although Carcanague's results do not seem to apply to non-minimal semi-invariant polynomials. Also, our results in §4 are independent of those in the earlier sections; in particular, in the case when K is a division ring with $[K : S(K)]_r < \infty$, the argument in §4 gives a fairly quick direct proof for the fact that the existence of a nonconstant semi-invariant polynomial implies the existence of a nonconstant invariant polynomial.

Although we work under the assumption that K is a division ring in §§2-4, many of the proofs can be extended to the case when K is a simple ring. However, some extra considerations are needed in handling the non-monic polynomials, and generally we need to assume that S is an automorphism. The details of this generalization are presented in the last section (§5) of this paper.

§2. Structure of Semi-Invariant Polynomials

Throughout §§2-4 of this paper, we let K be a division ring, S be an endomorphism of K *which may not be an automorphism*, and we let R denote the skew polynomial ring $K[t, S, D]$. We begin this section by proving some basic identities concerning semi-invariant polynomials. We shall denote a typical semi-invariant polynomial in R by $p(t) = \sum_{i=0}^{n} a_i t^i$. Here, n denotes the degree of $p(t)$, and we shall assume, after a scaling, that $p(t)$ is monic (i.e. $a_n = 1$). Under this assumption, the condition that $p(t)$ is semi-invariant can be expressed in the form

$$p(t)a = S^n(a)p(t) \quad \forall\, a \in K. \tag{2.1}$$

Given such a polynomial $p(t)$, we shall associate to it the polynomial

$$
\begin{aligned}
q(t) &= p(t)t - tp(t) \\
&= \sum_{i=0}^{n} a_i t^{i+1} - \sum_{i=0}^{n} (S(a_i)t + D(a_i))t^i \qquad (2.2) \\
&= (a_{n-1} - S(a_{n-1}))t^n + \cdots + (a_0 - S(a_0) - D(a_1))t - D(a_0).
\end{aligned}
$$

In the next two Propositions, we shall establish some interesting properties of $q(t)$, in the general (K, S, D)-setting.

Proposition 2.3. (1) $q(t)a - S^{n+1}(a)q(t) = (S^n D(a) - DS^n(a))p(t) \ \ \forall a \in K$;
(2) $S^n D - DS^n = D_{c,S}S^n$, *where* $c := a_{n-1} - S(a_{n-1})$.

Proof. For any $a \in K$, we have by (2.1):

$$
\begin{aligned}
q(t)a &= (p(t)t - tp(t))a \\
&= p(t)(S(a)t + D(a)) - tS^n(a)p(t) \\
&= S^{n+1}(a)p(t)t + S^n D(a)p(t) - (S^{n+1}(a)t + DS^n(a))p(t) \\
&= S^{n+1}(a)q(t) + (S^n D(a) - DS^n(a))p(t).
\end{aligned}
$$

Transposition of $S^{n+1}(a)q(t)$ now yields (1). For (2), we simply compare the (left) coefficients of t^n on the two sides of (1). On the RHS of (1), the coefficient is $S^n D(a) - DS^n(a)$ since $p(t)$ is monic; on the LHS of (1), the coefficient is $cS^n(a) - S^{n+1}(a)c = D_{c,S}S^n(a)$ in view of (2.2). Since this holds for all $a \in K$, we have proved (2). **QED**

Remark. In the case when $p(t)$ is an invariant polynomial, (2) above was first proved by Leroy, Tignol and van Praag [L_1: Lemme 2.2]. In his thesis, Lemonnier proved (2) for all monic semi-invariant polynomials $p(t)$, but assumed that S is an automorphism of K [Le: Ch.I, p.42]. Our proof above works for any endomorphism S and any monic semi-invariant polynomial $p(t)$. (The assumption that K be a division ring is not needed.) Note that, in the special case when $S = I$, Prop. 2.3 boils down to the fact that $q(t) = p(t)t - tp(t)$ commutes with scalars (*cf.* [L_3: Lemme 1.7], and [LL: (3.12)]).

Proposition 2.4. *In the above notations, we have for every* $a \in K$*:*

$$[q(t) - cp(t)]a = S^{n+1}(a)[q(t) - cp(t)].$$

In particular, the polynomial $q(t) - cp(t)$ *is semi-invariant.*

Proof. Substituting (2) into (1) in Proposition 2.3, we have

$$q(t)a = S^{n+1}(a)q(t) + (cS^n(a) - S^{n+1}(a)c)p(t).$$

Transposing the term $cS^n(a)p(t) = cp(t)a$ yields the desired result. **QED**

Now we come to the main result of this section. Again, in the special case when S is an automorphism of K, part (1) below was first proved by Lemonnier [Le: Ch. I, p.60]. However, Lemonnier's proof of (1) depended heavily on using the notion of quasi-algebraic derivations. Our approach does not involve this notion, and (1) below is deduced easily from (2.4).

Theorem 2.5. *Let* $p(t) = \sum_{i=0}^{n} a_i t^i$ *be a monic semi-invariant polynomial of the smallest degree* $n \geq 1$. *Then*
(1) $p(t)t = (t+c)p(t) + d$, *where* $c := a_{n-1} - S(a_{n-1})$, *and* $d := -(D(a_0) + ca_0)$;
(2) *For any* $a \in K$, $S^{n+1}(a)d = da$;
(3) $p(t)$ *is invariant iff* $d = 0$;
(4) $p(t)$ *is invariant unless* S^{n+1} *is an inner automorphism of* K. *In particular, if* S *is not an automorphism, then* $p(t)$ *is an invariant polynomial.*

Proof. Since $q(t) = p(t)t - tp(t) = ct^n + \cdots$, the polynomial $q(t) - cp(t)$ has degree $< n$. But by (2.4), it is semi-invariant. By the minimality of n, we must have $q(t) - cp(t) = e$ for some $e \in K$. Comparing the constant terms on both sides, we see that $e = -D(a_0) - ca_0 = d$. This proves (1). Replacing $q(t) - cp(t)$ in (2.4) by d, we get the equation in (2). If $d = 0$, then $p(t)t = (t+c)p(t)$; this together with the semi-invariance of $p(t)$ imply that $p(t)$ is invariant. Conversely, if $p(t)$ is invariant, then $p(t)t = (t+c')p(t)$ for some $c' \in K$, and $(t+c)p(t) + d = (t+c')p(t)$ clearly implies that $d = 0$. This proves (3). Finally, if S^{n+1} is not an inner automorphism of K, then (2) implies that d must be zero, and (3) implies that $p(t)$ is invariant. **QED**

In view of the above Theorem, for a monic semi-invariant polynomial $p(t)$ of minimal degree, the element $d = -(D(a_0) + ca_0) \in K$ provides a measure of the deviation of $p(t)$ from being invariant. From part (4) of the Theorem, we can also deduce the following result about semi-invariant polynomials which are <u>not</u> of minimal degree:

Corollary 2.6. *Let* $p'(t)$ *be a monic semi-invariant polynomial of non-minimal degree* n'. *Assume that* S *is an automorphism whose inner order is* $> n'$. *Then* $p'(t)$ *is invariant. In particular, if* S *is an automorphism of infinite inner order, then any semi-invariant polynomial is invariant.*

Proof. Let $p(t)$ be a monic semi-invariant polynomial of minimal degree $n \geq 1$. By [LL: (2.11)], we know that there is a representation $p'(t) = \sum_{i=0}^{r} c_i p(t)^i$ where $c_r = 1$ and $rn = n'$. The proof of [LL: (2.11)] shows further that $c_i \neq 0$ only if $S^{n'-ni}$ is an inner automorphism of K. Thus, if we assume that S has inner order $> n'$, we must have $c_0 = c_1 = \cdots = c_{r-1} = 0$, and hence $p'(t) = p(t)^r$. Since $n' > n$, (2.5)(4) implies that $p(t)$ is invariant; therefore, $p'(t)$ is also invariant. **QED**

§3. Powers of a Minimal Semi-Invariant Polynomial

In this section, we continue to write $p(t) = \sum_{i=0}^{n} a_i t^i$ for a monic semi-invariant polynomial in $R = K[t, S, D]$ of degree $n \geq 1$, but in addition, we shall assume that $p(t)$ has been chosen such that n is minimal. Our main objective will be to study the powers of $p(t)$. For this purpose, the following notation will be useful. For any endomorphism θ of K, and any integer $r \geq 1$, let us write $T_{r,\theta}$ for the operator $\theta^{r-1} + \theta^{r-2} + \cdots + \theta + Id_K$ on K, i.e. $T_{r,\theta}(a) = \sum_{i=0}^{r-1} \theta^i(a)$ for every $a \in K$. Using this notation, we have the following natural extension of parts of (2.5).

Theorem 3.1. *Let $p(t)$, c and d be as in (2.5). Then, for any $r \geq 1$, we have*

$$p(t)^r t - tp(t)^r = T_{r,S^n}(c)p(t)^r + T_{r,S^n}(d)p(t)^{r-1}. \tag{3.2}$$

In particular, $p(t)^r$ is invariant iff $T_{r,S^n}(d) = 0$.

Proof. For $r = 1$, (3.2) is just (2.5)(1). Inductively, if (3.2) holds for some r, then

$$\begin{aligned} p(t)^{r+1}t &= p(t)(p(t)^r t) \\ &= p(t)[tp(t)^r + T_{r,S^n}(c)p(t)^r + T_{r,S^n}(d)p(t)^{r-1}] \\ &= [tp(t) + cp(t) + d]p(t)^r + S^n(T_{r,S^n}(c))p(t)^{r+1} + S^n(T_{r,S^n}(d))p(t)^r \\ &= tp(t)^{r+1} + [S^n(T_{r,S^n}(c)) + c]p(t)^{r+1} + [S^n(T_{r,S^n}(d)) + d]p(t)^r \\ &= tp(t)^{r+1} + T_{r+1,S^n}(c)p(t)^{r+1} + T_{r+1,S^n}(d)p(t)^r. \end{aligned}$$

This completes the inductive proof of (3.2). If $T_{r,S^n}(d) = 0$, then $p(t)^r t = (t + T_{r,S^n}(c))p(t)^r$, so $p(t)^r$ is indeed invariant. Conversely, if $p(t)^r$ is invariant, then $p(t)^r t = (t+e)p(t)^r$ for some $e \in K$, and we have $ep(t)^r = T_{r,S^n}(c)p(t)^r + T_{r,S^n}(d)p(t)^{r-1}$, which clearly implies that $T_{r,S^n}(d) = 0$. **QED**

Remark 3.3. In generalization of (2.5)(2), one can also show that, for any $a \in K$ and any $r \geq 1$, $T_{r,S^n}(d)a = S^{n+1}(a)T_{r,S^n}(d)$. However, we will not need this equation in the sequel, so we shall leave its proof as an exercise.

Proposition 3.4. *Keeping the above notations, we have $T_{j(n+1),S^n}(d) = jT_{n+1,S^n}(d) = jT_{n+1,S}(d)$ for any integer $j \geq 1$. In particular, if $charK = p > 0$, then $p(t)^{p(n+1)}$ is invariant.*

Proof. We may clearly assume that $d \neq 0$. Then, by (2.5)(2), $S^{n+1} = I_d$, where I_d denotes the inner automorphism on K associated with d ($I_d(a) = dad^{-1}$ $\forall a \in K$). Since I_d fixes d, we have $S^m(d) = S^{m'}(d)$ whenever $m \equiv m' (\mathrm{mod}\ (n+1))$. For any $j \geq 1$, we have then

$$\begin{aligned} T_{j(n+1),S^n}(d) &= \sum_{i=1}^{j(n+1)} (S^n)^{j(n+1)-i}(d) \\ &= \sum_{i=1}^{j(n+1)} S^i(d) \quad (\text{since } n(j(n+1)-i) \equiv i\ (\mathrm{mod}\ (n+1)) \ ) \\ &= j\left(S(d) + \cdots + S^{n+1}(d)\right) \\ &= jT_{n+1,S}(d). \end{aligned}$$

For $j = 1$, this gives $T_{n+1,S^n}(d) = T_{n+1,S}(d)$, so we have proved the first part of (3.4). Using this, the last part of (3.4) follows from (3.1). **QED**

As it turns out, some power of $p(t)$ will also be invariant in the case of characteristic zero. To treat the case of general characteristic, we shall now study the left ideal $R \cdot p(t)$ generated by $p(t)$ in $R = K[t, S, D]$, and try to calculate its bound $J = \{h(t) \in R : h(t)R \subseteq R \cdot p(t)\}$. The first main step toward this calculation is the following:

Proposition 3.5. $p(t)^n \in J$.

Proof. By (3.1), we have $p(t)^r t \subseteq R \cdot p(t)^{r-1}$ for every $r \geq 1$. Thus, $p(t)^n t^i \in R \cdot p(t)^{n-i} \subseteq R \cdot p(t)$ for every $i \leq n-1$. From this, we see that $p(t)^n \cdot r(t) \in R \cdot p(t)$ for every polynomial $r(t)$ of degree $< n$. But for any $g(t) \in R$, we can write $g(t) = g'(t)p(t) + r(t)$, where $g'(t) \in R$ and $deg\ r(t) < n$. Thus, $p(t)^n g(t) = p(t)^n g'(t)p(t) + p(t)^n r(t) \in R \cdot p(t)$, and so $p(t)^n \in J$. **QED**

Theorem 3.6. *There exists a nonconstant semi-invariant polynomial iff there exists a nonconstant invariant polynomial.*

Proof. If $p(t)$ is any monic semi-invariant polynomial of minimal degree n, as above, then the bound J of $R \cdot p(t)$ is nonzero since it contains $p(t)^n$. Thus, $J = R \cdot f(t)$ for some polynomial $f(t) \neq 0$, and $f(t)$ is invariant since J is a two-sided ideal. **QED**

Remark 3.7. In view of the equivalence of the conditions (B) and (C) mentioned in the Introduction, (3.6) can also be paraphrased as follows: *R is nonsimple iff the S-derivation D is quasi-algebraic.* In particular, this affirms the conjecture made by two of the present authors in [LL: (3.10)].

We can now complete our computation of the bound J for the left ideal $R \cdot p(t)$.

Theorem 3.8. *$J = R \cdot p(t)^s$ for some $s \leq n$.*

Proof. Write $J = R \cdot f(t)$, where $f(t)$ is monic. If S is not an automorphism, $p(t)$ is already invariant (by (2.5)(4)), and we have $f(t) = p(t)$. Therefore, we may assume that S is an automorphism. In this case, it is known (*cf.* [LL: (2.9)]) that $f(t)$ can be written in the form $\sum_{j=0}^{s} d_j p(t)^j$, where $d_s = 1$. On the other hand, $p(t)^n \in J$ implies that $p(t)^n = h(t)f(t)$ for some $h(t) \in R$. Since $p(t)^n$ and $f(t)$ are both semi-invariant, so is $h(t)$ [LL: (2.7)(3)], and hence we can write $h(t) = \sum_{i=0}^{r} c_i p(t)^i$, where $c_r = 1$. Then,

$$p(t)^n = \sum_{i=0}^{r} c_i p(t)^i \sum_{j=0}^{s} d_j p(t)^j = \sum_{i,j} c_i S^{ni}(d_j) p(t)^{i+j}. \tag{3.9}$$

Let $l \leq r$ be the smallest integer such that $c_l \neq 0$, and $m \leq s$ be the smallest integer such that $d_m \neq 0$. If $m < s$, then $l + m < r + s = n$; isolating the term in (3.9) corresponding to $(i,j) = (l,m)$, we get $c_l S^{nl}(d_m) p(t)^{l+m} \in R \cdot p(t)^{l+m+1}$, and hence $c_l S^{nl}(d_m) = 0$, a contradiction. Thus, we must have $m = s$, i.e. $d_0 = \cdots = d_{s-1} = 0$, from which we conclude that $f(t) = p(t)^s$. **QED**

Our next result reveals some other remarkable properties of the generator $p(t)^s$ for the bound J of $R \cdot p(t)$.

Theorem 3.10. *For any integer $m \geq 0$, $p(t)^m$ is invariant iff $s|m$. If S is not an inner automorphism, then $p(t)^s$ is an invariant polynomial in R of minimal degree.*

Note that knowing $p(t)^s$ to be an invariant polynomial of minimal degree is quite significant since, with this knowledge, one can give fairly precise descriptions of the

center of R as well as the set of all invariant polynomials in R, by using the results of Cauchon [Ca: Th. 5.1.4, Th. 5.2.2].

To begin the proof of (3.10), let $d \in K$ be as defined in (2.5). If $d = 0$, then $p(t)$ is already invariant, so we have here $s = 1$, and both conclusions in (3.10) are obvious. (We do not need any assumptions on S in this case.) In the following, let us assume, therefore, that $d \neq 0$. By (2.5), S^{n+1} is the inner automorphism associated with d; in particular, S is an automorphism. For the first conclusion in the theorem, of course, any power $p(t)^{sh}$ is invariant; conversely, if $p(t)^m$ is invariant and $m = sh + e$ where $0 \leq e < s$, then from $p(t)^m = p(t)^{sh}p(t)^e$, we can see that $p(t)^e$ is invariant (*cf.* [LL: (2.7)]). If $e > 0$, then $p(t)^e \in J = R \cdot p(t)^s$, and this would contradict $e < s$. Thus, $e = 0$ and we have $s|m$. To prove the second part of (3.10), let k be the inner order of S, so that $k|(n+1)$. The crux of the proof is the following lemma.

Lemma 3.11. (1) *Any monic semi-invariant polynomial $g(t)$ has the form $p(t)^r + bp(t)^{r-k} + b'p(t)^{r-2k} + \cdots$, where $r \geq 1$, and $b, b', \cdots \in K$.*
(2) *Assume that $k > 1$ (i.e. S is not an inner automorphism). If the $g(t)$ above is invariant, then in fact $p(t)^r$ is invariant.*

Proof. Part (1) here is a special case of [LL: (2.11)(2)], but for the convenience of the reader we shall include a proof here. By [LL: (2.9)], we know that $g(t)$ has the form $\sum_{i=0}^{r} b_i p(t)^i$, with $b_r = 1$, and $deg\ g(t) = nr$ for some r. The semi-invariance condition $g(t)a = S^{nr}(a)g(t)$ now becomes

$$\sum S^{nr}(a)b_i p(t)^i = \sum b_i p(t)^i a = \sum b_i S^{ni}(a)p(t)^i,$$

i.e. $S^{nr}(a)b_i = b_i S^{ni}(a)$ for $0 \leq i \leq r$. Replacing a by $S^{-ni}(a)$, this amounts to $S^{n(r-i)}(a)b_i = b_i a$ (for all $a \in K$). Therefore, whenever $b_i \neq 0$, we have $S^{n(r-i)} = I_{b_i}$, and so $k|n(r-i)$. Since $k|(n+1)$, this implies that $k|(r-i)$, so i must have the form $r, r-k, r-2k, \cdots$, proving (1). For (2), we use the invariance condition $g(t)t = (t+e)g(t)$ (for some $e \in K$), that is:

$$[p(t)^r + bp(t)^{r-k} + \cdots]t = (t+e)[p(t)^r + bp(t)^{r-k} + \cdots].$$

Transposing the term $tp(t)^r$ to the LHS, we get

$$p(t)^r t - tp(t)^r = ep(t)^r - bp(t)^{r-k}t + tbp(t)^{r-k} + \cdots .$$

By (3.2), the LHS is $T_{r,S^n}(c)p(t)^r + T_{r,S^n}(d)p(t)^{r-1}$. Comparing the coefficients of t^{nr}, we see that $e = T_{r,S^n}(c)$. Cancelling the term $ep(t)^r$, we get

$$T_{r,S^n}(d)p(t)^{r-1} = -bp(t)^{r-k}t + tbp(t)^{r-k} + \cdots . \tag{3.12}$$

Since we assume $d \neq 0$, $p(t)$ is not invariant, and so $n = deg\ p(t) \geq 2$ (see [LL: (2.6)]). In view of this, and the fact that $k \geq 2$, the degree of the RHS of (3.12) is at most

$$n(r-k) + 1 = nr - nk + 1 \leq nr - 2n + 1 < n(r-1).$$

Therefore, the leading coefficient $T_{r,S^n}(d)$ of the term $t^{n(r-1)}$ in the LHS of (3.12) must be zero. It follows then from (3.1) that $p(t)^r$ is invariant. **QED**

Using the above Lemma, we shall now give the conclusion of the proof of (3.10). Let $g(t)$ be any monic invariant polynomial, written as in (1) in the Lemma. Then, by (2) above, $p(t)^r$ is invariant, and so $p(t)^r \in J = R \cdot p(t)^s$. From this, we have clearly $deg\ g(t) = deg\ p(t)^r \geq deg\ p(t)^s$. **QED**

Combining (3.10) with (3.4), we get the following extra information in characteristic p (no assumptions on S are necessary). The easy proof is left to the reader.

Corollary 3.13. *Suppose char* $K = p > 0$. *Then, in the notations of* (3.10), $s|p(n+1)$. *In particular:* (1) *If* $n+1$ *is prime, then* $p(t)^p$ *is invariant;*
(2) *If* $n < p$, *then* $s|(n+1)$. *If, in addition,* $n+1$ *is prime, then* $p(t)$ *is invariant.*

It remains now to treat the case when S is an inner automorphism. If $S = I_\alpha$ ($\alpha \in K\backslash\{0\}$), then, as is well-known, $K[t, S, D] = K[t', I, D']$ for $t' = \alpha^{-1}t$ and $D' = \alpha^{-1}D$. Therefore, after a change of variable, we may assume that $S = I$. In this case, it is known that invariant polynomials in $K[t, S, D]$ are just the central polynomials ([A_2: Remark 1, p.95], [LL: (2.4)(2)]). The relation between invariant and semi-invariant polynomials is given by the following result, in supplement to (3.10).

Theorem 3.14. *Let* $R = K[t, I, D]$, *and let* $p(t) = \sum_{i=0}^{n} a_i t^i$ *be a monic semi-invariant polynomial of minimal degree* $n \geq 1$ *in* R.
(1) *If char* $K = 0$, *then* $n = 1$ *and* $p(t)$ *is invariant. (In this case,* $p(t)$ *exists iff* D *is an inner derivation.)*
(2) *Let char* $K = p > 0$. *Then either* $p(t)$ *is already invariant, or else* $p(t)^m$ *is invariant iff* m *is a multiple of* p.

Proof. (1) The fact that $p(t)$ must be linear (under the assumption *char* $K = 0$) follows from [LL: (3.11)(3)]. Given this, it is easy to see that $p(t)$ exists iff D is inner (see, *e.g.* [LL: Ex. (2.6)]). (In particular, this recovers the well-known result that, in the case of characteristic 0, any algebraic derivation on K is inner.)

(2) Let c, $d \in K$ be as in (2.5). Then we have $T_{m,S^n}(d) = \sum_{i=0}^{m-1}(S^n)^i(d) = md$. Assume that $p(t)$ is not invariant, so $d \neq 0$. Then $md = 0$ iff $p|m$, so by (3.1), $p(t)^m$ is invariant iff $p|m$. **QED**

Example 3.15. (*cf.* [LL: Ex. 2.5(b)]) Let K be a division ring of characteristic 2, with $S = I$, and D a non-inner derivation with $D^2 = 0$. Let a be a central element with $D(a) \neq 0$. Then, as is easily seen, $p(t) = t^2+a$ is not invariant, but it is a semi-invariant polynomial in $R = K[t, S, D]$ of minimal degree. By direct computation, $p(t)^2 = t^4+a^2$, and $p(t)^2 t = tp(t)^2$, so $p(t)^2$ is a central polynomial, as predicted by (3.14) (or (3.13)). Also, it can be checked that $R \cdot p(t)^2$ is the bound of $R \cdot p(t)$. However, contrary to the last conclusion of (3.10), $p(t)^2$ is not an invariant polynomial of minimal degree; in fact, an invariant polynomial of minimal degree is given here by t^2. Of course, (3.10) does not apply to this example since $S = I$ here. This example serves to show that the hypothesis that S not be an inner automorphism is essential for the second part of (3.10).

§4. Semi-Invariant Polynomials Not Necessarily of Minimal Degree

Continuing to assume that K is a division ring, , we shall give in this section some results on $R \cdot p(t)$ where $p(t)$ is a monic semi-invariant polynomial, not necessarily of minimal degree. In particular, what can we say about the bound $J = \{h(t) \in R : h(t)R \subseteq R \cdot p(t)\}$? We begin with an example which shows that J may not be generated by a power of $p(t)$.

Example 4.1. Let K be a division ring with S an automorphism of order 2, and let $D = 0$. Let a be a central element with $S(a) \neq a$, and let $p(t) = t^2 + a$. Then (*cf.* [LL: Ex. 2.5(a)]) $p(t)$ is semi-invariant, but not invariant. We leave it to the reader to check that $p(t)^2 = t^4 + 2at^2 + a^2$ is also not invariant[5]. On the other hand, one can check that $f(t) = (t^2 + S(a))(t^2 + a) = t^4 + (a + S(a))t^2 + S(a)a$ is invariant, and in fact that $J = R \cdot f(t)$. Therefore, in this example, we have $p(t), p(t)^2 \notin J$, and J is not generated by a power of $p(t)$. Here, (3.5) and (3.8) fail to apply since t is an invariant (and hence semi-invariant) polynomial of degree lower than that of $p(t)$.

In the case when $S(K)$ has finite right codimension in K, we can indeed get some results on the bound of $R \cdot p(t)$ (not assuming $p(t)$ to be of minimal degree) which would "explain" the example above. This is given in the following:

Theorem 4.2. *Let K be a division ring such that the right dimension $[K : S(K)]_r = m < \infty$. Let $p(t)$ be any monic semi-invariant polynomial in $R = K[t, S, D]$ of degree $n \geq 1$. Then the bound of $R \cdot p(t)$ has the form $R \cdot f(t)$ where $f(t)$ is an invariant polynomial of degree $\leq n(1 + m + \cdots + m^{n-1})$. In particular, if S is an automorphism of K, we have deg $f(t) \leq n^2$.*

In what follows, we shall present a proof of Theorem 4.2. The idea of our proof is close to that of the proof of Carcanague's Theorem 4 in [C]. Given the monic semi-invariant polynomial $p(t) \in R$, we shall try to exploit the fact that $V := R/R \cdot p(t)$ has the structure of an (R, K)-bimodule. Here, the right K-structure on V is well-defined since $p(t) \cdot K \subseteq K \cdot p(t)$ implies that $R \cdot p(t)$ is a right K-subspace of R. (In [C], Carcanague considered V, instead, as a right module over its full endomorphism ring $End_R V$. Also, Carcanague assumed $p(t)$ to be irreducible, while we assume $p(t)$ to be semi-invariant.) We shall deduce Theorem 4.2 from the following general fact about bimodules.

Proposition 4.3. *Let R be a ring containing a division subring K. Let $V = {}_R V_K$ be a nonzero (R, K)-bimodule with the property that $N := [V : K]_r < \infty$ and $n := [V : K]_l < \infty$. Then there is a 2-sided ideal $J \subsetneq R$ such that $[R/J : K]_l \leq nN$. In particular, if $[R : K]_l > nN$, then $J \neq 0$ and R is not a simple ring.*

Proof. Consider the ring homomorphism $\phi : R \longrightarrow End(V_K)$ given by the the left action of R on V_K. Via ϕ, we can view $E := End(V_K)$ as a left R-module, in particular a left K-vector space. We fix a basis $\{e_1, \cdots, e_N\}$ for V_K and identify E

[5] If $p(t)$ were invariant, it must commute with t since $D = 0$; but a direct calculation shows that this is not the case as long as $S(a) \neq a$.

with the matrix ring $\mathbb{M}_N(K)$ in the usual way: any $\lambda \in E$ is identified with a matrix (λ_{ij}) where $\lambda(e_j) = \sum_{i=1}^N e_i \lambda_{ij}$. The crucial step is to compute the dimension of E over K with respect to the left action of K defined via ϕ above[6]. Letting C_k be the left ideal of $\mathbb{M}_N(K) = E$ consisting of matrices whose nonzero entries occur only on the k-th column, we have a direct decomposition $E = C_1 \oplus \cdots \oplus C_N$. Since $E \cdot C_k \subseteq C_k$, these C_k's are left K-subspaces of E and so $[E : K]_l = \sum_{k=1}^N [C_k : K]_l$. We claim that, *for any k, C_k is isomorphic to V as left K-spaces.* In fact, we have a map $\sigma_k : C_k \longrightarrow V$ sending any $M \in C_k$ with k-th column $(x_1, \cdots, x_N)^T$ to the vector $v = \sum_{j=1}^N e_j x_j \in V$. For $a \in K$, let $\phi(a) = (a_{ij}) \in E$. The action of a on M gives a matrix in C_k with k-th column $(y_1, \cdots, y_N)^T = (a_{ij}) \cdot (x_1, \cdots, x_N)^T$. On the other hand, the left action of a on $v \in V$ is

$$a \cdot v = a \cdot \sum_j e_j x_j = \sum_j \sum_i e_i a_{ij} x_j = \sum_i e_i \sum_j a_{ij} x_j = \sum_i e_i y_i.$$

Thus, σ_k is an isomorphism of left K-spaces[7]. From this, we conclude that

$$[E : K]_l = \sum_{k=1}^N [C_k : K]_l = \sum_{k=1}^N [V : K]_l = nN. \tag{4.4}$$

Let $J := ker(\phi) \subsetneq R$. Since R/J embeds into E as a left K-space, we have $[R/J : K]_l \leq [E : K]_l = nN$. **QED**

Proof of Theorem 4.2. Using the notations in (4.2), let $V = R/R \cdot p(t)$, where R is now $K[t, S, D]$. As we have observed in the paragraph preceding the statement of (4.3), we have $V = {}_R V_K$, with $[V : K]_l = n = deg\ p(t)$. We claim that

$$N := [V : K]_r = 1 + m + m^2 + \cdots + m^{n-1}, \quad where\ m = [K : S(K)]_r < \infty. \tag{4.5}$$

Assuming this claim, R will have a 2-sided ideal $J \subsetneq R$ with $[R/J : K]_l \leq nN = n(1 + m + \cdots + m^{n-1})$ by Proposition 4.3. Clearly, $J \neq 0$ since $[R : K]_l = \infty$. Moreover, this J is obtained as the kernel of the ring homomorphism $\phi : R \longrightarrow End((R/R \cdot p(t))_K)$, which consists of polynomials $h(t) \in R$ such that $h(t)R \subseteq R \cdot p(t)$. Thus, J is precisely the bound of $R \cdot p(t)$. Writing $J = R \cdot f(t)$, we have then $f(t) \neq 0$, and $deg\ f(t) = [R/J : K]_l \leq n(1 + m + \cdots + m^{n-1})$, as desired.

We now prove the claim (4.5) by explicitly constructing a right K-basis on V_K. The construction here is similar to that in [Co: pp. 57-58] (except that Cohn assumed $p(t)$ to be invariant). Write $K = \oplus_{i=1}^m z_i S(K)$. Then, by iteration, we get

$$\begin{aligned} K &= \bigoplus_{i=1}^m z_i \left(\bigoplus_{j=1}^m S(z_j) S^2(K) \right) = \bigoplus_{i,j} z_i S(z_j) S^2(K), \\ K &= \bigoplus_{i,j,k} z_i S(z_j) S^2(z_k) S^3(K), \ \cdots, \ etc. \end{aligned}$$

[6]This action is not to be confused with the usual action $a \cdot (\lambda_{ij}) = (a\lambda_{ij})$ of K on $\mathbb{M}_N(K) = E$. In fact, with respect to this action, E has left K-dimension N^2, but with respect to the K-action defined by ϕ, E has left K-dimension nN, as we shall show in (4.4).

[7]An explicit left K-basis on E can be given as follows. Let $\{e_l^* : 1 \leq l \leq n\}$ be a *left* K-basis on V, and let $\lambda(k, l) \in E$ be the endomorphism of V_K sending e_k to e_l^*, and sending all other e_j's to zero. Then $\{\lambda(k, l) : 1 \leq k \leq N,\ 1 \leq l \leq n\}$ gives a left K-basis on E.

The claim (4.5) will follow if we can show that

$$\{1,\ z_i\bar{t},\ z_iS(z_j)\bar{t}^2,\ \cdots,\ z_{i_1}S(z_{i_2})\cdots S^{n-2}(z_{i_{n-1}})\bar{t}^{n-1}\}$$

form a basis of V_K. To see this, let $V_k = K\cdot\bar{1} + K\cdot\bar{t} + \cdots + K\cdot\bar{t}^{k-1}$ $(1 \le k \le n)$. Then $0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V$ is a *right* K-filtration. It is therefore sufficient to show that, for any $k < n$, the images of $\{z_{i_1}S(z_{i_2})\cdots S^{k-1}(z_{i_k})\bar{t}^k\}$ form a basis of the *right* K-space V_{k+1}/V_k. This is done by an easy computation. To illustrate the ideas used, let us do it explicitly for $k = 2$. On V_3, we have for any $a \in K$:

$$\begin{aligned} at^2 &= \sum_{i,j} z_iS(z_j)S^2(a_{ij})t^2 \quad (\text{for unique } a_{ij} \in K) \\ &= \sum_{i,j} z_iS(z_j)(t^2a_{ij} + \text{an element in } (K + K\cdot t)) \\ &\equiv \sum_{i,j} z_iS(z_j)t^2\cdot a_{ij} \pmod{V_2}. \end{aligned}$$

This shows that, modulo V_2, the elements $\{z_iS(z_j)\bar{t}^2\}$ form a right K-basis for V_3. Obviously, the same argument works for V_{k+1}/V_k. **QED**

§5. The Case When K is a Simple Ring

In this section, we shall indicate how some of the proofs in §§2-3 can be modified to give similar results in the case when K is a simple ring. Thus, we shall assume now that K *is a simple ring* rather than a division ring. In order to get reasonable generalizations of our results, however, we shall need to assume that S *is an automorphism of* K.

Because of the possible existence of zero-divisors, we may no longer have $\deg fg = \deg f + \deg g$ for $f, g \in K[t, S, D]$. Also, K may have left-invertible elements which are not right-invertible. Therefore, great caution must be exercised in dealing with $R := K[t, S, D]$ when K is only a simple ring. For instance, note that, with respect to the definitions given earlier, an invariant polynomial $f(t)$ need no longer be semi-invariant (unless we are given that the leading coefficient of $f(t)$ is, say, right-invertible).

For ease of reference, let us first recall some known facts ([Ca: p.5.3], [L_1: (2.1)]):

Lemma 5.1. (1) *Let I be a (K,K)-submodule of R. If I contains a polynomial $f(t)$ of degree m, then it contains a monic polynomial $g(t)$ of degree m;*
(2) *Let $e \in K\backslash\{0\}$ and ϕ be an automorphism of K. If $ed = \phi(d)e$ for all $d \in K$, then e is a unit of K.*

Proof. (1) Say $f(t) = at^m + \cdots$. Since $a \neq 0$ and S is an automorphism of K, there exists an equation $\sum b_iaS^m(c_i) = 1$ in K. Then $g(t) := \sum b_if(t)c_i \in I$ is monic of degree m. (2) Since ϕ is an automorphism, the hypothesis implies that $Ke = eK$. Therefore $Ke = eK = K$ since K is a simple ring. **QED**

Proposition 5.2. *The following statements about* $R = K[t, S, D]$ *are equivalent:*
(1) R *is not simple*;
(2) R *has a nonconstant monic invariant polynomial*;
(3) R *has a nonzero invariant polynomial which is not left-invertible.*

Proof. (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1) are clear. For (1) $\Longrightarrow$ (2), let I be an ideal of R other than $\{0\}$ and R. Let $f(t)$ be a nonzero polynomial in I of the least degree. By (5.1)(1), we may assume f is monic. By the usual division algorithm argument, we have $I = R \cdot f$, so f is invariant. **QED**

Remark 5.3. *In general, an invariant polynomial may very well have a left inverse.* For instance, let K be a simple ring with two elements a, b such that $ab \neq 1 = ba$. Then $b[(ab-1)t + a] = (bab - b)t + 1 = 1$. Thus, $(ab-1)t + a$ is a linear polynomial with a left inverse b, hence necessarily invariant.

In the present context (K a simple ring and S an automorphism of K), let us indicate how to modify our earlier arguments to give a proof for the fact that, if there exists a nonconstant monic semi-invariant polynomial in R, then there exists a nonconstant monic *invariant* polynomial in R. We proceed as follows. Let $\mathcal{F}$ be the set of all nonconstant monic semi-invariant polynomials. Assuming $\mathcal{F}$ is nonempty, let $p(t) \in \mathcal{F}$ be of the least degree, say n. We can define $q(t)$ as in §2, and (2.3), (2.4) remain valid as before. Using our earlier notations, we may assume that the semi-invariant polynomial $q(t) - cp(t)$ is not zero, for otherwise $p(t)$ is already invariant. Suppose $q(t) - cp(t)$ has degree $m(< n)$ and leading coefficient e. The following lemma gives the extra step needed to carry through our program:

Lemma 5.4. *The element* e *is a unit in* K.

Proof. Comparing the leading terms in the equation in (2.4), we have $eS^m(a) = S^{n+1}(a)e$ for all $a \in K$. Replacing a by $S^{-m}(d)$, we get $ed = S^{n-m+1}(d)e$ for any $d \in K$. Therefore, (5.1)(2) gives the desired conclusion. **QED**

Since $q(t) - cp(t)$ is semi-invariant, we see easily that $e^{-1}(q(t) - cp(t))$ is also semi-invariant. But the latter is monic and has degree $m < n$. By the minimal choice of n, we conclude that $m = 0$, $q(t) - cp(t) = e$ (and as before, $e = d := -(D(a_0) + ca_0)$). This enables us to recapture all of the earlier results in §§2-3 (in the case when S is an automorphism). The point is, that we try to stick to monic polynomials only in all our considerations. (Thus, for example, in restating Cor. 2.6, we just take the $p'(t)$ there to be of non-minimal degree in the family $\mathcal{F}$.) The couple of results needed from [LL] (for instance, [LL: (2.7), (2.9), (2.11)]) can easily be checked to be valid over simple rings when we consider only *monic* polynomials, and Lemma 5.1(2) will guarantee that certain nonzero elements arising in our considerations are actually units. Thus, in the case when K is a simple ring and S is an automorphism of K, we not only get back Lemonnier's result that $R = K[t, S, D]$ is non-simple iff D is quasi-algebraic, but we also have all the explicit results (in §3) about the bound of $R \cdot p(t)$, where $p(t)$ is a (monic) polynomial of the least degree in $\mathcal{F}$.

To complete our results, we shall record a couple of other closely related conditions equivalent to the existence of a nonconstant monic semi-invariant polynomial. This result is the analogue of Prop. 5.2 for semi-invariant polynomials. Note that, by our results above, the conditions in (5.5) below are all equivalent to those in (5.2).

Proposition 5.5 *The following statements about* $R = K[t, S, D]$ *are equivalent:*
(1) R *has a nonconstant monic semi-invariant polynomial;*
(2) R *has a nonzero semi-invariant polynomial which is not left-invertible;*
(3) R *has a nonzero semi-invariant polynomial* $f(t)$ *for which there is no* $b \in K$ *such that* $bf(t) = 1$.
In fact, if (3) *holds, then* R *has a nonconstant monic semi-invariant polynomial* $g(t)$ *of degree* $\leq \deg f(t)$.

Proof. Since (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are tautologies, it is sufficient to prove the last sentence of the Proposition. Fix the given polynomial $f(t)$ in (3), and let I be the (K,K)-submodule $K \cdot f(t) \subseteq R$. Among all nonzero polynomials in I, pick $g(t)$ to be of the least degree, say m. By (5.1)(1), we may assume that $g(t)$ is monic. For any $a \in K$, if $g_a(t) := g(t)a - S^m(a)g(t) \neq 0$, then $g_a(t) \in I$ and $\deg g_a(t) < \deg g(t)$, contradicting the minimal choice of m. Thus, $g(t)a = S^m(a)g(t)$ for every $a \in K$, so $g(t)$ is semi-invariant. But by assumption, $I = K \cdot f(t)$ does not contain 1; hence we must have $m \geq 1$, and $g(t)$ is the nonconstant monic semi-invariant polynomial we want. **QED**

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