ON UJ-RINGS

M. TAMER KOŞAN, ANDRÉ LEROY, JERZY MATCZUK

Abstract. UJ-rings are studied, i.e. ring in which all units can be presented in a form $1+x$, for some $x \in J(R)$. The behavior of UJ-rings under various algebraic construction is investigated. In particular, it is shown that the problem of lifting the UJ property from a ring R to the polynomial ring $R[x]$ is equivalent to the Köthe's problem for $\mathbb{F}_2$-algebras.

Introduction

Throughout the paper all rings considered are associative and unital, except Section 2 where nil rings naturally appear. For a ring R, the Jacobson radical, the group of units and the set of all nilpotent elements of R are denoted by $J(R)$, $U(R)$ and $N(R)$, respectively.

The influence on the structure of rings of properties defined elementwise is intensively studied in the literature. For example, clean rings and their generalizations, rings with special types of units, generalizations of commutative rings have been investigated in relation to various global ring properties.

Let us notice that $1 + J(R)$ is always contained in $U(R)$, the aim of the paper is to investigate rings in which the equality $U(R) = 1 + J(R)$ holds. A ring R with this property will be called a UJ-ring, we will alternatively say that R has the UJ property. We will mainly focus on the behavior of UJ property under some classical ring constructions.

Recall that UU-rings, defined as rings with $U(R) = 1 + N(R)$ (i.e. rings with unipotent units) were introduced by Călugăreanu [1] and studied in details by Danchev and Lam in [4]. Of course when R is a UJ-ring with nil Jacobson radical, then R is a UU-ring.

Section 1 provides examples and gives some characterizations and basic properties of UJ-rings.

The behavior of UJ property under some classical ring constructions is studied in Section 2. In particular, it is proved (cf. Proposition 2.5) that if the polynomial ring $R[x]$ has the UJ property then R is a UJ-ring and the Jacobson radical $J(R)$ is nil. Moreover, as Theorem 2.6 shows, the converse of the above statement is equivalent to the Köthe's problem for $\mathbb{F}_2$-algebras. Theorem 2.8 offers a description of Morita contexts which are UJ-rings.

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The last section is devoted to study of some relations between UJ-rings and clean rings. In particular some characterizations of clean UJ-rings are presented.

1. Preliminaries

A ring R is said to be a UJ-ring if $1+J(R)=U(R)$. Since units lift modulo the Jacobson radical, R is a UJ-ring if and only if the factor ring $R/J(R)$ is a ring with trivial units, i.e. $U(R/J(R))=\{1\}$.

Recall that $J(R)$ is the largest ideal of R consisting of quasi-regular elements of R, i.e. invertible elements in the circle monoid $(R,\circ)$ (see [9, Exercises for §4]). In the following lemma $\mathcal{C}(R)$ denotes the set of all quasi-regular elements of R. $(\mathcal{C}(R),\circ)$ is a group isomorphic to $U(R)$ and the isomorphism is given by $\mathcal{C}(R)\ni x\leftrightarrow 1-x\in U(R)$. Therefore R is a UJ-ring if and only if $\mathcal{C}(R)$ is an ideal of R. This description can be used as a definition of UJ-rings for rings without unity.

In the following lemma we collect other characterizations of UJ-rings.

Lemma 1.1. *For a ring R, the following conditions are equivalent:*

(1) *$U(R)=1+J(R)$, i.e. R is a UJ-ring;*
(2) *$U(R/J(R))=\{1\}$;*
(3) *$\mathcal{C}(R)$ is an ideal of R (then $\mathcal{C}(R)=J(R)$);*
(4) *$rb-cr\in J(R)$, for any $r\in R$ and $b,c\in\mathcal{C}(R)$;*
(5) *$ru-vr\in J(R)$, for any $u,v\in U(R)$ and $r\in R$;*
(6) *$U(R)+U(R)\subseteq J(R)$ (then U(R)+U(R)=J(R)).*

Proof. The equivalence of $(1)-(3)$ was already observed. The implication $(3)\Rightarrow(4)$ is trivial.

Setting $b=1+u, c=1+v$, for $u,v\in U(R)$, and applying (4) we get (5).

Taking $r=1$ in (5) we get $u-v\in J(R)$, for any $u,v\in U(R)$ and $U(R)+U(R)\subseteq J(R)$ follows. Notice that every $a\in J(R)$ can be written as a sum of two units: $a=1+(a-1)$, so (6) holds.

Finally, using (6) we get $U(R)-1\subseteq J(R)$, i.e. (1) holds. □

Let us now mention a few basic examples of UJ-rings.

Examples 1.2. (1) Any ring with trivial units is UJ. In particular, the class of UJ-rings contains: all Boolean rings, all free, both commutative and noncommutative, algebras over the field $\mathbb{F}_2$.

(2) Any local ring R with a maximal ideal M such that $R/M=\mathbb{F}_2$. In particular the rings $\mathbb{Z}/2^n\mathbb{Z}$, $\mathbb{Z}_{(2\mathbb{Z})}$ and $R=\mathbb{F}_2[[x]]$ are UJ.

(3) If R is a UJ-ring, then ring $T_n(R)$ of n by n upper triangular matrices over R and $R[x]/(x^n)$ are UJ-rings.

In the following proposition, we collect some basic properties of UJ-rings.

Proposition 1.3. *Let R be a UJ-ring. Then:*

(1) *$2\in J(R)$;*

(2) *If R is a division ring, then $R = \mathbb{F}_2$;*
(3) *$R/J(R)$ is reduced (i.e. it has no nonzero nilpotent elements) and hence abelian (i.e. every idempotent is central);*
(4) *If $x, y \in R$ are such that $xy \in J(R)$, then $yx \in J(R)$ and $xRy, yRx \subseteq J(R)$;*
(5) *Let $I \subseteq J(R)$ be an ideal of R. Then R is a UJ-ring if and only if R/I is a UJ-ring;*
(6) *R is Dedekind finite;*
(7) *The ring $\prod_{i\in I} R_i$ is UJ if and only rings R_i are UJ, for all $i \in I$.*

Proof. Statements (1) and (2),(3) are direct consequences of Lemma 1.1(6) and Lemma 1.1(2), respectively. (4) follows from (3).

If $I \subseteq J(R)$, then $(R/I)/J(R/I) \simeq R/J(R)$. This gives (5).

By (3) $R/J(R)$ is reduced, so it is Dedekind finite. Let $a, b \in R$ be such that $ab = 1$. Then, as $R/J(R)$ is Dedekind finite, we get $ba - 1 \in J(R)$. Thus the idempotent ba is invertible, so $ba = 1$ and (6) follows.

The last statement is a consequence of the facts that $J(\prod_{i\in I} R_i) = \prod_{i\in I} J(R_i)$ and $U(\prod_{i\in I} R_i) = \prod_{i\in I} U(R_i)$ □

Statements (5), (3), (2) of the above proposition give immediately the following characterization of semilocal UJ-rings.

Proposition 1.4. *A semilocal ring R is UJ if and only if $R/J(R) \simeq \mathbb{F}_2 \times \ldots \times \mathbb{F}_2$.*

In particular we have

Corollary 1.5. *The ring $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is UJ if and only if n is a power of 2.*

Let us finish this section with the following:

Remark 1.6. A ring R is a UJ-ring with nil Jacobson radical if and only if R is a UU-ring and $N(R)$ is an ideal of R.

The following example of Bergman (see [4, Example 2.5]) shows that UU-rings with nil Jacobson radical do not have to be UJ.

Example 1.7. Let R be the $\mathbb{F}_2$-algebra generated by x, y with the only relation $x^2 = 0$. Then $U(R) = 1 + \mathbb{F}_2 x + xRx$, so R is a UU-ring. Moreover $J(R) = 0$, so R is not a UJ-ring.

2. UJ property under algebraic constructions

The main purpose of this section is to clarify the connection between Köthe's problem and UJ property of rings. Later on we present necessary and sufficient conditions for a Morita context to be a UJ-ring.

It is known and easy to check (see [4]) that a subring of a UU-ring is always a UU-ring. We will see in the example below that the UJ property is not hereditary on subrings but anyway we have the following:

Proposition 2.1. *Let R be a UJ-ring and Z a subring of R such that $U(Z) = U(R) \cap Z$. Then Z is also a UJ-ring. In particular this applies to $Z = Z(R)$ the center of R.*

Proof. Since $U(Z) = U(R) \cap Z$, we also have $J(R) \cap Z \subseteq J(Z)$. Thus, using $U(R) = 1 + J(R)$ we get $1 + J(Z) \subseteq U(Z) = U(R) \cap Z = (1 + J(R)) \cap Z = 1 + (J(R) \cap Z) \subseteq 1 + J(Z)$ and $U(Z) = 1 + J(Z)$ follows. □

Example 2.2. Let $R = \mathbb{F}_2[[x]]$. Then R is a UJ-ring and its subring S generated by $1 + x$ and $(1 + x)^{-1} = \sum_{i=0}^{\infty} x^i$ is not a UJ-ring, as it is isomorphic to $\mathbb{F}_2[x, x^{-1}]$.

Now we will concentrate on the UJ property of polynomial rings. In this context let us notice that:

Lemma 2.3. *Let R be a ring with trivial units. Then $U(R[X]) = \{1\}$, where $R[X]$ denotes the polynomial ring in the set X of commuting indeterminates.*

Proof. Since being a unit in $R[X]$ is a local property, i.e. depends only on finitely many indeterminates, we may assume that X is a finite set.

By assumption $U(R) = \{1\}$, so R does not contain nontrivial nilpotent elements, i.e. it is a reduced ring. [7, Corollary 1.7] characterizes reduced rings as rings such that $U(R[x]) = U(R)$ and the thesis follows easily. □

Let us recall that a ring R is 2-primal if its prime radical $B(R)$ coincides with the set of all its nilpotent elements.

Proposition 2.4. *Let R be a 2-primal UU-ring. Then, for any set X of commuting indeterminates, the polynomial ring $R[X]$ is a UJ-ring.*

Proof. It is known that $B(R[X]) = B(R)[X]$ (cf.[9, Theorem 10.19]). Thus the assumptions imposed on R and Lemma 2.3 imply that the ring $R[X]/B(R[X]) \simeq (R/B(R))[X]$ has trivial units. Now, by Proposition 1.3(5), $R[X]$ is a UJ-ring. □

Proposition 2.5. *If the polynomial ring $R[x]$ is UJ, then R is a UJ-ring and $J(R)$ is a nil ideal of R.*

Proof. It is known that $J(R[x]) = I[x]$ for some nil ideal I of R. Thus, as $R[x]$ is UJ, we have $1 + J(R) \subseteq U(R[x]) = 1 + J(R[x]) = 1 + I[x]$. This implies that $J(R) = I$ is nil. Then, as $R[x]$ is a UJ-ring, $\{1\} = U(R[x]/J(R[x])) = U((R/J(R)[X])$. Hence also $U(R/J(R)) = \{1\}$, i.e. R is a UJ-ring. □

The above proposition shows that if the UJ property lifts from a ring R to the polynomial ring $R[x]$, then $J(R)$ has to be a nil ideal. The next theorem says that the problem of lifting the UJ property is equivalent to Köthe's problem for algebras over the field $\mathbb{F}_2$. Recall that Köethe's problem (formulated in 1930) asks whether a ring R has no nonzero nil one-sided ideals provided R has no nonzero nil ideals. It is known (see Theorem 6, [8]) that the problem has a positive solution if and only if it has positive solution for algebras over fields. There are many other problems in ring theory which are equivalent or related to it (see [12]), one more is indicated below.

Theorem 2.6. *The following conditions are equivalent:*

(1) *For any UJ-ring R with nil Jacobson radical, the polynomial ring $R[x]$ is also UJ;*

(2) *For any nil* $\mathbb{F}_2$*-algebra* A*,* $J(A[x]) = A[x]$*;*
(3) *For any nil* $\mathbb{F}_2$*-algebra* A *and* $n \geq 1$ *the matrix algebra* $M_n(A)$ *is nil;*
(4) *Köthe's problem has a positive solution in the class of* $\mathbb{F}_2$*-algebras.*

Proof. The equivalence of statements (2)-(4) is a well known result of Krempa ([8]).

(1) $\Rightarrow$ (2) Assume that (1) holds and A is a nil $\mathbb{F}_2$-algebra. Let A^* be the $\mathbb{F}_2$-algebra obtained from A by adjoining unity with the help of $\mathbb{F}_2$. Note that $A^* = A \cup (1 + A)$, $J(A^*) = A$ and $A^*/J(A^*) = \mathbb{F}_2$. In particular, A^* is a UJ-ring and, by (1), $A^*[x]$ also has the UJ property. Consequently, $U(A^*[x]/J(A^*[x])) = \{1\}$ follows. Let N be the ideal of A^* such that $J(A^*[x]) = N[x]$. As $U((A^*/N)[x]) = \{1\}$, we get A^*/N is reduced. This yields $A = N$, i.e $J(A[x]) = A[x]$.

(2) $\Rightarrow$ (1) Let R be a ring as in (1). By Proposition 1.3(1), $2 \in J(R)$. Thus, as $J(R)$ is nil, $(2R)[x]$ is a nilpotent ideal of $R[x]$. Therefore, in virtue of Proposition 1.3(5), to show that $R[x]$ has the UJ property, it is enough to prove that $R[x]/(2R[x]) \simeq (R/2R)[x]$ is a UJ-ring. Thus, eventually replacing R by $R/2R$, we may assume, that $2 = 0$ in R, i.e. R is an algebra over the field $\mathbb{F}_2$. Then, the property (2) gives $J(R[x]) = J(R)[x]$ (because $J(R)$ is nil), and so $R[x]/J(R[x]) \simeq (R/J(R))[x]$. Since R is UJ, we get $U(R/J(R)) = \{1\}$ and Lemma 2.3 implies that $U((R/J(R))[x]) = \{1\}$. This proves that $R[x]$ is UJ, as desired. □

Let us observe that whenever $n > 1$, the matrix ring $M_n(R)$ does not have the UJ property. Indeed, the ring $M_n(R)/J(M_n(R)) \simeq M_n(R/J(R))$ is not reduced when $n > 1$, so $M_n(R)$ can not be UJ, as observed in Proposition 1.3(3).

Proposition 2.7. *Let* R *be a ring with an idempotent* $e \in R$*. The following conditions are equivalent:*

(1) R *is a* UJ*-ring;*
(2) eRe *and* $(1-e)R(1-e)$ *are* UJ*-rings, and* $eR(1-e), (1-e)Re \subseteq J(R)$*.*

Proof. Suppose R is a UJ-ring. Then, taking $x = e$ and $y = 1 - e$ in Proposition 1.3(4), we obtain $eR(1-e), (1-e)Re \subseteq J(R)$. Recall that $J(eRe) = J(R) \cap eRe$, thus the natural homomorphism form eRe into $R/J(R)$ induces an embedding of $eRe/J(eRe)$ into $R/J(R)$. Moreover, by Proposition 1.3(3), $\bar{e} = e + J(R)$ is a central idempotent of $\bar{R} = R/J(R)$. Thus $\{\bar{1}\} = U(\bar{R}) = U(\bar{e}\bar{R}) \times U((\bar{1} - \bar{e})\bar{R})$, so the ring $eRe/J(eRe) \simeq \bar{e}R$ has trivial units, i.e. eRe is a UJ-ring. Similarly, $(1-e)R(1-e)$ is a UJ-ring.

Suppose (2) holds. Making use of Pierce decomposition of R with respect to e and the assumption that $eR(1-e), (1-e)Re \subseteq J(R)$, it is clear that $R/J(R) \simeq eRe/J(eRe) \times (1-e)R(1-e)/J((1-e)R(1-e))$ and $U(R/J(R)) = \{\bar{1}\}$ follows as both eRe and $(1-e)R(1-e)$ are UJ-rings. □

The above proposition can be extended to Morita context but instead of using Proposition 1.3 we will use the description of N-radicals (Jacobson radical is such) of Morita contexts given in [6, Theorem 2.7].

Let us recall that a quadruple (R, V, W, S) is a Morita context where R, S are rings, V, W are $(R-S)$ and $(S-R)$ bimodules, respectively and the products $\phi : V \otimes_S W \to R$

and $\psi : W \otimes_R V \to S$ are given such that matrices $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ form an associative ring with natural matrix operations defined with the help of ϕ and ψ.

Theorem 2.8. *Let (R, V, W, S) be a Morita context and $T := \begin{pmatrix} R & V \\ W & S \end{pmatrix}$. The following conditions are equivalent:*

1. *T is a UJ-ring;*
2. *R, S are UJ-rings and $VW \subseteq J(R)$, $WV \subseteq J(S)$;*
3. *R, S are UJ-rings and $T/J(T) \simeq R/J(R) \oplus S/J(S)$.*

Proof. By [6, Theorem 3.18.14], we have $J(T) = \begin{pmatrix} J(R) & B \\ C & J(S) \end{pmatrix}$, where $B = \{v \in V \mid Wv \subseteq J(S)\} = \{v \in V \mid vW \subseteq J(R)\}$ and $C = \{w \in W \mid wV \subseteq J(S)\} = \{w \in W \mid Vw \subseteq J(R)\}$.

(1) $\Rightarrow$ (2) Suppose that T is a UJ-ring. Then $T/J(T)$ does not possess nonzero nilpotent elements. This forces $\begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix} \subseteq J(T)$, i.e. $B = V$, $C = W$, $VW \subseteq J(R)$, $WV \subseteq J(S)$ and $T/J(T) \simeq R/J(R) \oplus S/J(R)$. Thus, by (3) and (7) of Proposition 1.3, R, S are UJ-rings, i.e. (2) holds.

Implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are consequences of [6, Theorem 3.18.14] and Lemma 1.1, respectively. □

3. Clean rings and UJ property

Recall that an element $r \in R$ is clean (J-clean) provided there exist an idempotent $e \in R$ and an element $t \in U(R)$ ($t \in J(R)$) such that $r = e + t$. A ring R is clean (J-clean) if every element of R has such clean (J-clean) decomposition. It is known that every J-clean ring is clean (in fact if $-r = e + j$ is a J-clean decomposition of $-r \in R$, then $r = (1 - e) + (-1 - j)$ is a clean decomposition of r).

Proposition 3.1. *For a ring R, the following conditions are equivalent:*

1. *R is a UJ-ring.*
2. *All clean elements of R are J-clean.*

Proof. (1) $\Rightarrow$ (2) Assume that $r \in R$ is a clean element and $r = e + u$ is its clean decomposition. As R is a UJ-ring, $2 \in J(R)$ and $u = 1 + j$ for some $j \in J(R)$. Then $2e + j \in J(R)$ and $r = e + 1 + j = (1 - e) + (2e + j)$ is a J-clean decomposition of r, i.e. (2) holds.

(2) $\Rightarrow$ (1) Let $u \in U(R)$. Then u is a clean element and, by the hypothesis, u is J-clean. Let $u = e + j$ be a J-clean decomposition of u. Since $1 = eu^{-1} + ju^{-1}$, we obtain that $eu^{-1} = 1 - ju^{-1}$ is a unit of R. Hence $e = 1$. This means that $u = 1 + j$ and $U(R) = 1 + J(R)$ follows. □

Theorem 3.2. *For a ring R, the following conditions are equivalent:*

1. *R is a clean UJ-ring;*

(2) *$R/J(R)$ is a Boolean ring and idempotents lift modulo $J(R)$;*
(3) *R is a J-clean, UJ-ring;*
(4) *R is a J-clean ring.*

Proof. (1) $\Rightarrow$ (2) The imposed assumptions imply that $R/J(R)$ is a clean ring such that $U(R) = \{1\}$. In particular, $2 = 0$ in $(R/J(R))$ and every element $r \in R/J(R)$ is of the form $r = e_r + 1$, for a suitable idempotent e_r. Hence $r^2 = r$, i.e. $R/J(R)$ is Boolean. It is known (cf.[11, Lemma 17]), that idempotents lift modulo every ideal I of a clean ring R, so (2) follows.

(2) $\Rightarrow$ (3) Suppose (2) holds and let $a \in R$. Then $a + J(R) \in R/J(R)$ is an idempotent. Hence there exists an idempotent $e \in R$ such that $a - e \in J(R)$, i.e. a is a J-clean element. This shows that R is J-clean. If $u \in U(R)$, then $u + J(R)$ is a unit in a Boolean ring $R/J(R)$. Thus $u - 1 \in J(R)$, so R is a UJ-ring.

(3) $\Rightarrow$ (4) This is a tautology.

(4) $\Rightarrow$ (1) This implication is given by Proposition 3.1. □

It is known (cf. [5, Theorem 5.9]) that a ring R is uniquely nil clean if and only if $R/J(R)$ is Boolean, $J(R)$ is nil and idempotents lift uniquely modulo $J(R)$. In particular, the class of uniquely nil clean ring is contained in the class of UJ-rings. However, slightly bigger class of conjugate nil clean rings is not included in UJ-rings, as the ring $M_2(\mathbb{F}_2)$ is conjugate nil clean (see [10, Corollary 2.4]) but it is not a UJ-ring. Assuming additionally in Theorem 3.2 that $J(R)$ is a nil ideal, we get:

Theorem 3.3. *For a ring R, the following conditions are equivalent:*
(1) *R is a clean UJ-ring with nil Jacobson radical $J(R)$;*
(2) *$R/J(R)$ is a Boolean ring and $J(R)$ is nil;*
(3) *R is a nil clean UJ-ring;*
(4) *R is a conjugate nil clean UJ-ring;*
(5) *R is a conjugate nil clean ring and $N(R)$ is an ideal of R;*
(6) *$R/J(R)$ is a Boolean ring and R is a UU-ring.*

Proof. By [5, Corollary 3.17], R is a nil clean ring if and only if $R/J(R)$ is nil clean and $J(R)$ is nil. In particular, when R is a UJ-ring, then R is nil clean if and only if R is J-clean and $J(R)$ is a nil ideal of R. Now the equivalence of statements $(1) - (3)$ is given by Theorem 3.2 and the fact that idempotents lift modulo nil ideals.

The implication (4) $\Rightarrow$ (3) is a tautology.

Statement (2) implies that R is a UJ-ring, thus the implication (2) $\Rightarrow$ (4) is a consequence of [10, Corollary 2.16] and the fact that Boolean rings are conjugate nil clean.

If R is nil clean, then $J(R)$ is nil. Therefore, the equivalence (4) $\Leftrightarrow$ (5) is a consequence of Remark 1.6.

Finally, one can easily check that both (2) and (6) are equivalent to $R/J(R)$ is Boolean and $J(R) = N(R)$. □

Comparing Theorems 3.2 and 3.3, let us observe that the class of J-clean rings having nil Jacobson radical is equal to the class of UJ nil clean rings but it is strictly contained in the class of all nil clean rings, as the the ring $M_2(\mathbb{F}_2)$ is nil clean however it is not a UJ-ring.

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DEPARTMENT OF MATHEMATICS, GEBZE TECHNICAL UNIVERSITY, 41400 GEBZE/KOCAELI, TURKEY

E-mail address: MTKOSAN@GTU.EDU.TR; TKOSAN@GMAIL.COM

DEPARTMENT OF MATHEMATICS, UNIVERSITÉ D'ARTOIS, RUE JEAN SOUVRAZ, 62307 LENS, FRANCE

E-mail address: ANDRE.LEROY@UNIV-ARTOIS.FR

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WARSAW, UL. BANACHA, 2, WARSZAWA, POLAND

E-mail address: JMATCZUK@MIMUW.EDU.PL