QUASI-DUO SKEW POLYNOMIAL RINGS

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Abstract

A characterization of right (left) quasi-duo skew polynomial rings of endomorphism type and skew Laurent polynomial rings are given. In particular, it is shown that (1) the polynomial ring $R[x]$ is right quasi-duo iff $R[x]$ is commutative modulo its Jacobson radical iff $R[x]$ is left quasi-duo, (2) the skew Laurent polynomial ring is right quasi-duo iff it is left quasi-duo. These extend some known results concerning a description of quasi-duo polynomial rings and give a partial answer to the question posed by Lam and Dugas whether right quasi-duo rings are left quasi-duo.

INTRODUCTION

An associative ring R with unity is called *right (left) quasi-duo* if every maximal right (left) ideal of R is two-sided or, equivalently, every right (left) primitive homomorphic image of R is a division ring. There are many open problems in the area (Cf. [5]). One of the most interesting is Problem 7.7[5]: "does there exist a right quasi-duo ring that is not left quasi-duo ? ". If it were so, there would exist right quasi-duo non-division rings, which are left primitive. Constructing such an example might be a quite challenging task as it is even not easy to construct left but not right primitive rings. The known examples of such sort are based on skew polynomial rings of endomorphism type. In that

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context it is interesting to determine when such skew polynomial rings are right (left) quasi-duo. This problem is not new and was studied by several authors. In [13] it was proved that if R is Jacobson semisimple and the polynomial ring $R[x]$ is right quasi-duo, then R is commutative and it was expected that the same holds when $R[x]$ is Jacobson semisimple.

In this article we obtain a complete description of right (left) quasi-duo skew polynomial rings of endomorphism type (Cf. Theorem 3.4) and skew Laurent polynomial rings (Cf. Theorem 4.2). The descriptions are left-right symmetric but do not imply that $R[x;\tau]$ is right quasi-duo iff it is left quasi-duo in general. However the descriptions do imply that in order to get a skew polynomial ring which is quasi-duo on one side only, one has to begin with a coefficient ring which already has this property. Thus skew polynomial rings of endomorphism type themselves do not help to construct an example of a ring which is right but not left quasi-duo.

Theorem 3.4 implies that the ordinary polynomial ring $R[x]$ is right (left) quasi-duo if and only if it is commutative modulo its Jacobson radical. This in particular shows that $R[x]$ is right quasi-duo if and only if it is left quasi-duo. The same result holds also for skew Laurent polynomial rings.

The article is organized as follows. In the first section we show that the class of right quasi-duo rings is hereditary on certain subrings and closed under passing to some overings. The most important for our further studies are the results which say that the quasi-duo property is preserved when passing from Laurent skew polynomial rings to skew polynomial rings and from a ring with an injective endomorphism to its Cohn-Jordan extension.

In Section 2 we obtain, based on a result of Bedi and Ram [1], a detailed description of the Jacobson radical of right (left) quasi-duo skew polynomial rings of automorphism type, which will be needed later on.

The characterization of right (left) quasi-duo skew polynomial rings of endomorphism type is given in Section 3.

The last section contains applications and examples. In particular a characterization of quasi-duo skew Laurent polynomial rings and several examples delimiting the obtained results are presented.

All rings considered in this article are associative with unity.

We frequently consider the endomorphism and automorphism cases separately. Thus we will use different letters for denoting those maps, namely σ and τ will stand for an automorphism and an endomorphism of a ring R, respectively.

We will say that a subset B of R is *τ-stable* if $\tau(B) \subseteq B$ and $\tau^{-1}(B) \subseteq B$, where $\tau^{-1}(B)$ denotes the preimage of B in R. Notice that if I is an ideal of R such that $\tau(I) \subseteq I$, then τ induces an endomorphism of the factor ring R/I and I is τ-stable iff the induced endomorphism is injective. When $\tau = \sigma$ is an automorphism of R, then I is σ-stable if $\sigma(I) = I$.

The skew polynomial ring and skew Laurent polynomial ring are denoted by $R[x;\tau]$ and $R[x,x^{-1};\sigma]$, respectively. The coefficients from R are written on the left of the indeterminate x.

The Jacobson radical of a ring R will be denoted by $J(R)$. It is clear from the definition, that a ring R is right (left) quasi-duo if and only if $R/J(R)$ is right (left) quasi-duo and that $R/J(R)$ is a reduced ring in case it is right (left) quasi-duo. We will use these properties many times in the article.

1 QUASI-DUO PROPERTY OF RINGS AND THEIR SUBRINGS

We start with the following simple observation.

Lemma 1.1. *If R is a unital subring of a right quasi-duo ring T, then every maximal right ideal I of R such that $IT \neq T$ is two-sided.*

Proof. If $T \neq IT$, then IT is contained in a maximal right ideal M of T which is a two-sided ideal, as T is a right quasi-duo ring. Clearly $I \subseteq M \cap R$ and maximality of I yields that $I = M \cap R$, as otherwise $M \cap R = R$ and $M = T$ would follow. □

A subring S of a ring T is called [4] a *corner subring* of T if S is a ring with a unity, possibly different from that of T, and if there exist an additive subgroup C of T such that $T = S \oplus C$ and $SC, CS \subseteq C$. The subgroup C is called a *complement* of S. We say that S is a *left corner* of T if the complement C satisfies $SC \subseteq C$ only.

The most classical example of a corner subring is a subring S of the form eTe, where $e \in T$ is an idempotent. Another example is the following. Let H be a monoid with an identity e. If $T = \bigoplus_{g\in H} R_g$ is a H-graded ring, then R_e is a corner of T.

A natural example of a left corner is the base ring R of a skew polynomial ring $R[x;\tau,\delta]$, where τ and δ stand for an endomorphism and a τ-derivation of R, respectively. Clearly $R[x;\tau,\delta] = R\oplus C$, where $C = R[x;\tau,\delta]x$ and $CR \not\subseteq C$ if $\delta \neq 0$.

Theorem 1.2. *Let S be a left corner of a right quasi-duo ring T. Then S is also right quasi-duo.*

Proof. Let us fix a complement C of S in T and define $R = S + (1-e)\mathbb{Z} \subseteq T$, where e stands for the unity of S. Since S is a homomorphic image of R, it is enough to show that R is right quasi-duo. Let I be a maximal right ideal of R. If $IT \neq T$, then, by Lemma 1.1, I is a two-sided ideal of R . Suppose $IT = T$. Notice that e is a central idempotent of R, $eR = S$ and $SC \subseteq C$. Using the above we obtain $e \in eIT \cap S = (Ie(S \oplus C)) \cap S \subseteq ((I \cap S) \oplus SC) \cap S \subseteq I$, i.e. $e \in I$. This yields that I is a two-sided ideal of R, as it is the preimage of an ideal of the commutative ring R/eR. □

In the case S is a corner of T, the above proposition was proved by Lam and Dugas ([5], Theorem 3.8) by completely different arguments. From Theorem 1.2 one immediately obtains the following corollary. The first statement was proved, using long arguments, in [7].

Corollary 1.3. *1. If R is a right quasi-duo ring, then eRe is also a right quasi-duo ring, for every nonzero idempotent e of R.*

2. Suppose H is a monoid with an identity e. If $T = \bigoplus_{g\in H} R_g$ is a H-graded ring which is right quasi-duo, then $R = R_e$ is a right quasi-duo ring.

3. Let τ be an endomorphism and δ a τ-derivation of a ring R. If the skew polynomial ring $R[x;\tau,\delta]$ is right quasi-duo, then R is right quasi-duo.

Let us observe that for every positive integer n, the rings $S = R[x;\tau]$ and $T = R[x,x^{-1};\sigma]$ are naturally graded by the cyclic group G of order n with the e-components equal to $S_0 = R[x^n] \subseteq S$ and $T_0 = R[x^n,x^{-n}] \subseteq T$ which are isomorphic to $R[x;\tau^n]$ and $R[x,x^{-1},\sigma^n]$, respectively. Hence Corollary 1.3 implies the following:

Corollary 1.4. *1. If $R[x;\tau]$ is right quasi-duo, then R and $R[x;\tau^n]$ are right quasi-duo, for every $n \geq 1$.*

2. If $R[x,x^{-1};\sigma]$ is right quasi-duo, then R and $R[x,x^{-1};\sigma^n]$ are right quasi-duo, for every $n \geq 1$.

If R is an Ore domain with a division ring of factions Q, then Q is a quasi-duo ring though R need not be quasi-duo. The following theorem gives necessary and sufficient conditions for a ring to be right quasi-duo in terms of its localization with respect to a multiplicatively closed set generated by regular right normalizing elements. Recall that such sets are always right denominator sets and that an element $s \in R$ is *right normalizing* if $Rs \subseteq sR$, i.e. the right ideal sR is two-sided.

Theorem 1.5. *Let S be the multiplicatively closed set generated by the set $X \subseteq R$ of regular right normalizing elements of R such that the quotient ring RS^{-1} is right quasi-duo. Then the following conditions are equivalent:*

1. R is a right quasi-duo ring;

2. R/xR is a right quasi-duo ring, for every $x \in X$.

Proof. The implication (1) $\Rightarrow$ (2) is clear as the class of right quasi-duo rings is homomorphically closed.

(2) $\Rightarrow$ (1) Suppose R/xR is right quasi-duo, for any $x \in X$. Let M be a maximal right ideal of R.

If $M \cap S = \emptyset$, then MS^{-1} is a proper right ideal of RS^{-1} and Lemma 1.1 yields that M is a two-sided ideal of R in this case.

Assume $M \cap S \neq \emptyset$. Then, there are $k \geq 1$ and $x_1, \ldots, x_k \in X$, such that the element $y = x_1 \cdot \ldots \cdot x_k \in M$. Since x_iR, for any $1 \leq i \leq k$, is a two-sided ideal of R, we obtain that the product of ideals $x_1Rx_2R \ldots x_kR \subseteq yR$ is contained in the annihilator P of the simple right R-module R/M, which is a primitive ideal. Therefore there exists $1 \leq i \leq k$, such that $x = x_i \in P \subseteq M$. This and the assumption imply that the canonical image of M in R/xR is a two-sided maximal ideal of R/xR and so is M in R.

The above shows that every maximal right ideal of R is two-sided, i.e. the statement (1) holds. $\square$

In the following example we construct a domain R which is not right-quasi duo but its localization by a central element is a duo ring.

Example 1.6. Let p be an odd prime number, $A \subseteq \mathbb{Q}$ denote the localization of $\mathbb{Z}$ with respect to the maximal ideal $p\mathbb{Z}$ and let $\mathbb{H}$ stand for the Hamilton quaternion algebra over $\mathbb{Q}$ and $R = A[i, j, k] \subseteq \mathbb{H}$. Then R is not right quasi-duo, as R can be homomorphically mapped onto 2×2 matrices over the field $\mathbb{Z}_p$. Let $S = \{p^k \mid k \geq 1\}$. Then S is a central multiplicatively closed set of R such that the localization $RS^{-1} = \mathbb{H}$ is a duo ring.

Notice that $R[x, x^{-1}; \sigma]$ can be considered both as a localization of $R[x, \sigma]$ and of $R[x, \sigma^{-1}]$ with respect the multiplicatively closed set generated by x. Thus, applying Corollary 1.4(2) and Theorem 1.5 to one of the above mentioned rings and $X = \{x\}$ we get the following corollary.

Corollary 1.7. *If $R[x, x^{-1}; \sigma]$ is a right quasi-duo ring, then so are $R[x; \sigma]$ and $R[x; \sigma^{-1}]$.*

We will see in Example 4.5 that the reverse implication in the above corollary does not hold in general. However it will turn out that the polynomial ring $R[x]$ is right quasi-duo if and only if $R[x, x^{-1}]$ is right quasi-duo.

It is known that if the endomorphism τ of the ring R is injective, then there exists a universal over-ring $A(R, \tau)$ of R, called the *τ-Cohn-Jordan extension of R,* such that τ extends to an automorphism of $A(R, \tau)$ and $A(R, \tau) = \bigcup_{i=0}^{\infty} \tau^{-i}(R)$.

The following proposition will enable us to replace an injective endomorphism by an automorphism in some of our considerations.

Proposition 1.8. *Let τ be an injective endomorphism of R. Then:*

1. *If R is right (left) quasi-duo, then so is $A(R, \tau)$.*

2. If $R[x;\tau]$ is right (left) quasi-duo, then so is $A(R,\tau)[x;\tau]$.

Proof. (1) We present the proof for right quasi-duo property. Recall (Cf. [5]) that a ring R is right quasi-duo if and only if $Ra+Rb=R$ implies $aR+bR=R$, for any $a,b\in R$. We will use this characterization in the proof.

Let $a,b\in A(R,\tau)=A$ be such that $Aa+Ab=A$. Then, $1=za+wb$ for some $z,w\in A$ and by definition of A, one can pick $n\in\mathbb{N}$ such that $1=\tau^n(z)\tau^n(a)+\tau^n(w)\tau^n(b)$ is an equality in R. Hence, as R is right quasi-duo, we obtain $\tau^n(a)R+\tau^n(b)R=R$. Then also $\tau^n(a)A+\tau^n(b)A=A$ and $aA+bA=A$ follows as τ is an automorphism of A. This proves (1).

(2) The injective endomorphism τ of R can be extended to an endomorphism of $R[x;\tau]$ by setting $\tau(x)=x$. Then one can check that $A(R[x;\tau],\tau)=A(R,\tau)[x;\tau]$. Therefore, applying the statement (1) to the ring $R[x;\tau]$ we obtain (2). □

The following example shows that the converse of the statement (1) in the above lemma does not hold in general. However, we will see later, as a consequence of Theorem 3.4, that the converse implication in Proposition 1.8(2) holds.

Example 1.9. Let $\mathbb{H}$ denote the Hamilton quaternion algebra over the field $\mathbb{Q}$ of rationals and $A=\mathbb{H}(x_i\mid i\in\mathbb{Z})=\mathbb{H}\otimes_{\mathbb{Q}}K$, where $K=\mathbb{Q}(x_i\mid i\in\mathbb{Z}$ is the field of rational functions in the set $\{x_i\mid i\in\mathbb{Z}\}$ of indeterminates. Let σ denote the $\mathbb{Q}$-automorphism of A determined by conditions: $\sigma|_{\mathbb{H}}=\mathrm{id}_{\mathbb{H}}$ and $\sigma(x_i)=x_{i+1}$, for all $i\in\mathbb{Z}$. Define $R=\mathbb{H}(x_i\mid i\geq 1)[x_0]\subseteq A$. Then σ induces an endomorphism, also denoted by σ, of R such that $A=\bigcup_{i=0}^{\infty}\sigma^{-i}(R)$. This means that $A=A(R,\sigma)$. Clearly A is a division ring, so it is a quasi-duo ring and R is not right (left) quasi-duo, as it is a polynomial ring over a noncommutative division ring.

2 THE JACOBSON RADICAL OF QUASI-DUO SKEW POLYNOMIAL RINGS OF AUTOMORPHISM TYPE

In this short section we get precise description of the Jacobson radical of a skew polynomial ring of automorphism type $R[x;\sigma]$, in the case $R[x;\sigma]$ is right quasi-duo. For that we will need the following definition and result by Bedi and Ram.

An element $a\in R$ is called *σ-nilpotent* if for every $m\geq 1$ there exists $n\geq 1$ such that $a\sigma^m(a)\cdots\sigma^{mn}(a)=0$ and a subset B of R is called $\sigma-nil$ if every element of A is σ-nilpotent.

Theorem 2.1. *([1], Theorem 3.1) If σ be an automorphism of a ring R, then there exist σ-nil ideals $K\subseteq J(R)$ and I of R such that $J(R[x;\sigma])=(I\cap J(R))+I[x;\sigma]x$ and $J(R[x,x^{-1};\sigma])=K[x,x^{-1};\sigma]$.*

Lemma 2.2. *Suppose that $R[x;\sigma]$ is right quasi-duo. Let M be a maximal ideal of $R[x;\sigma]$ and $M_0 = M \cap R$. Then precisely one of the following conditions holds:*

1. *If there exist $n \geq 1$ such that $x^n \in M$, then $x \in M$, M_0 is a maximal ideal of R and $M = M_0 + R[x;\sigma]x$;*

2. *If for every $n \geq 1$, $x^n \notin M$, then M and M_0 are σ-stable, $M_0 = \{r \in R \mid \exists_{m\geq 0}\ rx^m \in M\} = \{r \in R \mid rx \in M\}$ and $A = R/M_0$ is a domain such that $A[x;\sigma]$ is right quasi-duo.*

Proof. The n-th power $(x)^n$ of the ideal generated by x is equal to $R[x;\sigma]x^n$. Thus if $x^n \in M$, then $x \in M$ and clearly $M = M_0 + R[x;\sigma]x$, i.e. (1) holds.

Suppose $x \notin M$ and let $m \in M$. Then $xm = \sigma(m)x$ and $mx = x\sigma^{-1}(m)$ belong to M. Since M is prime ideal and x is a centralizing element, we obtain $\sigma(m), \sigma^{-1}(m) \in M$, for $m \in M$, i.e. M is σ-stable. A similar argument leads to $M_0 = \{r \in R \mid \exists_{m\geq 0}\ rx^m \in M\}$. Now $B = R/M_0$ embeds canonically into $R[x;\sigma]/M$ which is a division ring (as $R[x;\sigma]$ is right quasi-duo), so B is a domain and $B[x;\sigma]$ is right quasi-duo as a homomorphic image of $R[x;\sigma]$. Hence (2) holds. □

Let us remark that the ideal M_0 from Lemma 2.2(1) need not be maximal in R. For example, if K is a field, then the ideal $M = (tx-1)$ of $K[[t]][x]$ is maximal and $M_0 = 0$ is not maximal in the base ring $K[[t]]$.

Although only automorphisms are considered in this section, we will formulate the following definition in more general setting. For a ring R with an endomorphism τ we set

$$N(R) = \{a \in R \mid \exists_{n\geq 1}\ \ a\tau(a)\ldots\tau^n(a) = 0\}.$$

The definition of $N(R)$ depends on the choice of the endomorphism τ, we hope that the proper choice of τ will be clear from the context. Notice that the set $N(R)$ is τ-stable

Proposition 2.3. *Let σ be an automorphism of R and $R[x;\sigma]$ be right (left) quasi-duo ring. Then:*

1. *$N(R)$ is a two-sided, σ-stable ideal of R such that $J(R[x;\sigma]) = (N(R) \cap J(R)) + N(R)[x;\sigma]x$, the ring $R/N(R)$ does not contain nonzero σ-nilpotent elements and the ring $R/(N \cap J(R))$ is reduced. In particular, every nilpotent element of R is σ-nilpotent.*

2. *Let $\mathcal{A}$ denote the set of all maximal ideals M of $R[x;\sigma]$ such that $x \notin M$ and $\mathcal{B}$ the set of remaining maximal ideals of $R[x;\sigma]$. Then:*

   (a) $\bigcap_{M\in\mathcal{B}} M = J(R) + R[x;\sigma]x$;

(b) $N(R) = \bigcap_{M\in\mathcal{A}} M_0$*, where* $M_0 = M \cap R$ *and* $\bigcap_{M\in\mathcal{A}} M = N(R)[x;\sigma]$*;*

Proof. (1) Let I be the ideal of R described in Theorem 2.1. Clearly $I \subseteq N(R)$, as I is σ-nil. Since $R[x;\sigma]/J(R[x;\sigma])$ is a semiprimitive one-sided quasi-duo ring, it is reduced. If $a \in N(R)$, then $ax \in R[x;\sigma]$ is a nilpotent element and $N(R) \subseteq I$ follows, i.e. $I = N(R)$. Now it is easy to complete the proof of (1) using Theorem 2.1 and the fact that $R[x;\sigma]/J(R[x;\sigma])$ is a reduced ring.

(2)(a) If M_0 is a maximal ideal of R, then $M_0 + R[x;\sigma]x \in \mathcal{B}$. This implies, as R is a right quasi-duo, that $\bigcap_{M\in\mathcal{B}} M = J(R) + R[x;\sigma]x$, i.e. (2)(a) holds.

(b) Using Lemma 2.2 and the statement (a) we obtain

$$\begin{aligned} N(R) &= \{a \in R \mid ax \in J(R[x;\sigma])\} = \bigcap_{M\in\mathcal{A}\cup\mathcal{B}} \{a \in R \mid ax \in M\} \\ &= \bigcap_{M\in\mathcal{A}} \{a \in R \mid ax \in M\} = \bigcap_{M\in\mathcal{A}} M_0. \end{aligned}$$

This shows that the first equation from (b) holds. The second one is a consequence od (1) and (a) above. □

The above proposition offers precise description of $J(R[x;\sigma])$ provided $R[x;\sigma]$ is right (left) quasi-duo. Later on, as a side effect of the main Theorem 3.4, we will obtain a similar characterization of the Jacobson radical of a right (left) quasi-duo skew polynomial ring $R[x;\tau]$ of endomorphism type.

3 MAIN RESULTS

In this section we characterize skew polynomial rings which are right (left) quasi-duo. When τ is an automorphism, then the opposite ring to $R[x;\tau]$ is also a skew polynomial ring of automorphism type. This means that the characterizations of right and left quasi-duo properties of skew polynomial rings are left-right symmetric, in this case. We show in Theorem 3.4 that we also have such symmetry for arbitrary endomorphisms. This will be achieved by reducing the general case to the case of skew polynomial rings of automorphism type.

Recall that a ring R with an automorphism σ is called *σ-prime* if the product IJ of nonzero σ-stable ideals I, J of R is always nonzero and a σ-stable ideal P of R is σ-prime if the ring R/P is σ-prime. By $P_\sigma(R)$ we will denote *the σ-pseudoradical of* R, that is $P_\sigma(R)$ is the intersection of all non-zero σ-prime ideals of R (by definition, the empty intersection is equal to R).

The commutator $ab - ba$ of elements a and b from a ring R will be denoted by $[a, b]$.

In the sequel we will need the following two lemmas, the first one is a special case of Lemma 3.2(1) from [10].

Lemma 3.1. *Let R be a domain. Suppose that $R[x;\sigma]$ contains a maximal ideal M such that $M \cap R = 0$. Then $P_\sigma(R) \neq 0$.*

Lemma 3.2. *Let τ be an endomorphism of a ring R and $U \subseteq V$ be ideals of R. If $\tau(V) \subseteq V$, then $U + V[x;\tau]x$ is a two-sided ideal of $R[x;\tau]$ and the ring $R[x;\tau]/(U + V[x;\tau]x)$ is right (left) quasi-duo if and only if R/U and $R[x;\tau]/V[x;\tau] \simeq (R/V)[x;\tau]$ are right (left) quasi-duo rings.*

Proof. It is clear that $U + V[x;\tau]x$ is a two-sided ideal of $R[x;\tau]$ and that τ induces an endomorphism, also denoted by τ, of the ring R/V. Notice that $(U+R[x;\tau]x)\cap V[x;\tau] = U+V[x;\tau]x$. This implies that $R[x;\tau]$ is a subdirect product of rings R/U and $(R/V)[x;\tau]$. Now it is easy to complete the proof with the help of Corollary 3.6(2) [5], which states that a finite subdirect product of right (left) quasi-duo rings is also such. $\square$

The following theorem is the key ingredient for our description of one-sided quasi-duo skew polynomial rings.

Theorem 3.3. *Let R be a domain with an automorphism σ. If $R[x;\sigma]$ is right quasi-duo, then R is commutative and $\sigma = \mathrm{id}_R$.*

Proof. Suppose that $R[x;\sigma]$ is right quasi-duo. Let $\mathcal{A}$ denote the set of all maximal ideals M of $R[x;\sigma]$ such that $x \notin M$. Since R is a domain and $R[x;\sigma]$ is right quasi-duo, Proposition 2.3 implies that:
(i) $0 = J(R[x;\sigma]) = \bigcap_{M\in\mathcal{A}} M$, i.e. $R[x;\sigma]$ is a subdirect product of division rings $R[x;\sigma]/M$, where M ranges over $\mathcal{A}$.
(ii) $M_0 = M \cap R$ is a σ-stable ideal of R, for any $M \in \mathcal{A}$.

Let $M \in \mathcal{A}$ and $\pi\colon R[x;\sigma] \to R[x;\sigma]/M = D$ be an epimorphism onto a division ring D. We claim that D is commutative. Since M_0 is a σ-stable ideal of R, π induces an epimorphism $f\colon (R/M_0)[x;\sigma] \to D$ such that $\ker f \cap (R/M_0) = 0$. Moreover $x \notin \ker f$, as $x \notin M$. Since D is a division ring, the above yields that R/M_0 is a domain, $\ker f$ is a maximal ideal of $(R/M_0)[x;\sigma]$ not containing x. Thus, eventually replacing R by R/M_0 we may assume that $M \in \mathcal{A}$ has zero intersection with the base ring R. Then, Lemma 3.1, shows that $P_\sigma(R) \neq 0$.

Take $0 \neq a \in P_\sigma(R)$ and consider a maximal right ideal W of $R[x;\sigma]$ containing $ax + 1$. By assumption, W is a two-sided ideal. Clearly $x \notin W$ and, by Lemma 2.2, $W_0 = W \cap R$ is a (completely) prime, σ-stable ideal of R. Thus, if W_0 would be nonzero it would contain $P_\sigma(R)$ and $W = R[x;\sigma]$ would follow. Therefore $W_0 = 0$.

Observe that $[a, ax+1] = (a^2 - a\sigma(a))x \in W$, as W is a two-sided ideal of $R[x;\sigma]$. Since x is a normalizing element of $R[x;\sigma]$ and W is a prime ideal, we obtain $a(a-\sigma(a)) \in W \cap R = 0$. This proves that $a(a-\sigma(a)) = 0$, for any $a \in P_\sigma(R)$ and the fact that R is a domain implies that σ is identity on $P_\sigma(R)$.

Then for arbitrary $r \in R$, we have $ar \in P_\sigma(R)$ and $0 = \sigma(ar) - ar = a(\sigma(r) - r)$ and $\sigma = \mathrm{id}_R$ follows.

Let $b \in R$. Then, as $\sigma = \mathrm{id}_R$, we have $[b, a]x = [b, ax+1] \in W$ and, similarly as above, we get $[b, a] = 0$ for all $b \in R$. This means that $0 = [b, ar] = a[b, r]$, for all $b, r \in R$, and yields commutativity of the domain R. This shows that D is commutative as a homomorphic image of a commutative ring $R[x; \sigma] = R[x]$ and completes the proof of the theorem. $\square$

For the endomorphism τ of a ring R let us set $\mathcal{K} = \bigcup_{i=1}^{\infty} \ker \tau^i$. Then, as $\mathcal{K}$ is τ-stable, τ induces an injective endomorphism, also denoted by τ, of the ring $\bar{R} = R/\mathcal{K}$. In particular, we can consider the τ-Cohn-Jordan extension $A(\bar{R}, \tau)$ of $\bar{R}$.

Keeping the above notation we can formulate the following theorem which characterizes one-sided quasi-duo skew polynomial rings of endomorphism type.

Theorem 3.4. *For a ring R with an endomorphism τ, the following conditions are equivalent:*

1. *$R[x; \tau]$ is a right (left) quasi-duo ring;*
2. *R and $\bar{R}[x; \tau]$ are right (left) quasi-duo rings;*
3. *R and $A(\bar{R}, \tau)[x; \tau]$ are right (left) quasi-duo rings;*
4. *The following conditions hold:*
   (a) *R is right (left) quasi-duo and $J(R[x; \tau]) = (J(R) \cap N(R)) + N(R)[x; \tau]$;*
   (b) *$N(R)$ is a τ-stable ideal of R, the factor ring $R/N(R)$ is commutative and the endomorphism τ induces identity on $R/N(R)$.*

Proof. It is clear that (1) implies (2), as the rings in (2) are homomorphic images of $R[x; \tau]$.

The implication (2) $\Rightarrow$ (3) is given by Proposition 1.8(2) applied to the ring $\bar{R}$.

(3) $\Rightarrow$ (4) Case 1, when $\tau = \sigma$ is an automorphism of R, i.e. $R = \bar{R} = A(R, \tau)$. Suppose that R and $R[x; \sigma]$ are right quasi-duo. Proposition 2.3 shows that the statement (4)(a) holds, in this case.

Let $\mathcal{A}$ denote the set of all maximal ideals of $R[x; \sigma]$ such that $x \notin M$. Proposition 2.3 together with Lemma 2.2(2) yield that $N(R)$ is a σ-stable ideal of R and the ring $(R/N(R))[x; \sigma] \simeq R[x; \sigma]/(N(R)[x; \sigma])$ is a subdirect product of domains $(R/M_0)[x, \sigma] \simeq R[x; \sigma]/(M[x; \sigma])$, where M ranges over $\mathcal{A}$ and $M_0 = M \cap R$. Now the statement (4)(b) is a consequence of Theorem 3.3. Therefore Case 1 holds for right quasi-duo property of $R[x; \sigma]$. Notice that we also have proved that the ring $R/N(R)$ is reduced.

Making use of the isomorphism $(R[x;\sigma])^{op} \simeq R^{op}[x;\sigma^{-1}]$, where T^{op} denotes the ring opposite to a ring T, one can easily get the left version of Case 1.

Case 2, when τ is an arbitrary endomorphism of R. Recall that for a ring B with an endomorphism τ, $N(B) = \{b \in B \mid \exists_{n\geq 1} b\tau(b)\ldots\tau^n(b) = 0\}$. Let $A = A(\bar{R},\tau)$, where $\bar{R} = R/\mathcal{K}$. It is easy to check that $\mathcal{K} \subseteq N(R)$, $N(\bar{R}) = N(A)\cap\bar{R}$ and $N(\bar{R}) = N(R)/\mathcal{K}$. This implies that $N(R)$ is the kernel of the composition of natural homomorphisms $R \to \bar{R} \to \bar{R}/N(\bar{R}) \hookrightarrow A/N(A)$. The above and Case 1 applied to the ring A imply that $N(R)$ is a τ-stable ideal of R, $R/N(R)$ is a commutative reduced ring as $A/N(A)$ is such and τ induces identity on $R/N(R)$, as τ induces identity on $A/N(A)$. This means that the statement (4)(b) holds and also proves that $J(R[x;\tau]) \subseteq N(R)[x;\tau]$, since the factor ring $R[x;\tau]/(N(R)[x;\tau]) \simeq (R/N(R))[x]$ is semiprimitive as $R/N(R)$ is reduced. In fact, the last inclusion together with the fact that $J(R[x;\tau]) \cap R \subseteq J(R)$ imply that $J(R[x;\tau]) \subseteq (J(R) \cap N(R)) + N(R)[x;\tau]x$.

For every $\bar{r} \in N(\bar{R})$, $\bar{r}x$ is a nilpotent element of $\bar{R}[x;\tau] = (R/\mathcal{K})[x;\tau]$. Hence, since $A[x;\tau]$ is a right (left) quasi-duo ring, $N(\bar{R})[x;\tau]x \subseteq J(A[x;\tau])$. Thus for every $f(x) \in N(\bar{R})[x;\tau]x$ there exists $g(x) \in A[x;\tau]$ such that $(1 - f(x))(1+g(x)) = 1$. On the other hand $(1-f(x))(1+f(x)+(f(x))^2+\cdots) = 1$ in the power series ring $A[[x;\tau]]$. Consequently $g(x) = f(x) + (f(x))^2 + \cdots \in N(\bar{R})[x;\tau]$, so $(N(\bar{R})[x;\tau]x) \subseteq J(\bar{R}[x;\tau])$. Therefore, as $\mathcal{K}[x;\tau]x$ is a nil ideal of $R[x;\tau]$ and $N(\bar{R}) = N(R)/\mathcal{K}$, we get that $N(R)[x;\tau]x \subseteq J(R[x;\tau])$. Now, with the help of this inclusion, one can check that every element of the ideal $J = (J(R) \cap N(R)) + N(R)[x;\tau]x$ is quasi regular in $R[x;\tau]$. This together with the above proved inclusion $J(R[x;\tau]) \subseteq J$ show that $J(R[x;\tau]) = (J(R)\cap N(R)) + N(R)[x;\tau]$ and completes the proof of the implication (3) $\Rightarrow$ (4).

The implication (4) $\Rightarrow$ (1) is a direct consequence of Lemma 3.2 with $U = J(R) \cap N(R)$ and $V = N(R)$. $\square$

Remark 3.5. 1. Statements (2) and (3) of the above theorem are of rather technical nature, but are important steps for obtaining the equivalence of (1) and (4). When τ is an automorphism of R, then both (2) and (3) boil down to the statement (1), as $A(\bar{R};\tau) = R = \bar{R}$ in this case.
2. Note that $R[x;\tau]$ is a commutative ring if and only if R is commutative and $\tau = \mathrm{id}_R$. Thus the condition (4)(b) can be equivalently stated as "$N(R)$ is a τ-stable ideal and the ring $R[x;\tau]/(N(R)[x;\tau]) \simeq (R/N(R))[x;\tau]$ is commutative". One can reformulate some other results in the article in a similar way.
3. Observe that it is shown in the proof of Theorem 3.4 that if $R[x;\tau]$ is right (left) quasi-duo, then the commutative ring $R/N(R)$ has to be reduced.
4. Theorem 3.4 shows that the use of skew polynomial rings of endomorphism type can not be effective in constructing a ring which is quasi-duo only on one side, as for that aim it would be necessary to construct the base ring with the same properties first.

4 APPLICATIONS AND EXAMPLES

It is known ([12], [1]) that if either R is one sided noetherian or the automorphism σ is of locally finite order, i.e. when for any $a \in R$ there is $n \geq 1$ such that $\sigma^n(a) = a$, then $J(R[x;\sigma]) = I[x;\sigma]$, for some nil ideal I of R. Since the Jacobson radical of quasi-duo rings contains all nilpotent elements, Theorem 3.4 gives immediately the following theorem.

Theorem 4.1. *Suppose that σ is an automorphism of R and either R is one-sided noetherian or σ is of locally finite order. Then $R[x;\sigma]$ is right quasi-duo if and only if $J(R[x;\sigma]) = \mathcal{N}(R)[x;\sigma]$, where $\mathcal{N}(R)$ is the nil radical of R, $R/\mathcal{N}(R)$ is commutative and the automorphism of $R/\mathcal{N}(R)$ induced by σ is equal to $\mathrm{id}_{R/\mathcal{N}(R)}$.*

The following theorem characterizes quasi-duo skew Laurent polynomial ring.

Theorem 4.2. *The skew Laurent polynomial ring $R[x,x^{-1};\sigma]$ is right (left) quasi-duo if and only if $J(R[x,x^{-1};\sigma]) = N(R)[x,x^{-1};\sigma]$, $R/N(R)$ is commutative and the automorphism of $R/N(R)$ induced by σ is equal $id_{R/N(R)}$ if and only if $R[x,x^{-1};\sigma]/J(R[x,x^{-1};\sigma])$ is commutative.*

Proof. Let $T = R[x,x^{-1};\sigma]$. We will prove the only nontrivial implication, that is, we will show that if T is one-sided quasi-duo, then $J(T)$, $N(R)$ and σ satisfy conditions described in the theorem. To this end suppose T is right quasi-duo. By Theorem 2.1, $J(R[x,x^{-1};\sigma]) = K[x,x^{-1};\sigma]$ for a suitable σ-nil ideal K of R. Clearly $K \subseteq N(R)$. Since $N(R)x$ consists of nilpotent elements and $T/J(T)$ is reduced, $N(R) \subseteq J(T)$ and $J(T) = N(R)[x,x^{-1};\sigma]$ follows, in this case. By Corollary 1.7, $R[x;\sigma]$ is right quasi-duo and Theorem 3.4 shows that $R/N(R)$ is commutative and the automorphism of $R/N(R)$ induced by σ is equal $id_{R/N(R)}$. □

It is known [3] that for every ring R, $J(R[x,x^{-1}]) \cap R[x] = J(R[x])$. Thus, Theorems 4.1 and 4.2 give the following:

Corollary 4.3. *$R[x]$ is right (left) quasi-duo if and only if $R[x,x^{-1}]$ is right (left) quasi-duo if and only if $J(R[x]) = \mathcal{N}(R)[x]$ and the factor ring $R/\mathcal{N}(R)$ is commutative, where $\mathcal{N}(R)$ denote the nil radical of R.*

Remark 4.4. If the polynomial ring $R[x,y]$ in two commuting indeterminates x,y is right-quasi-duo, then $R[x]$ is also right quasi-duo as a homomorphic image of $R[x,y]$. However the converse does not hold. Indeed, in [11] a nil ring C was constructed such that $C[x]$ is Jacobson radical but not nil. Let R be the ring obtained from C by adjoining unity by the ring $\mathbb{Z}$ of integers.

Then $J(R[x]) = C[x]$ and $R[x]/J(R[x]) \simeq \mathbb{Z}[x]$, so $R[x]$ is quasi-duo. However $R[x]/\mathcal{N}(R[x])$ is not commutative as otherwise C would be contained in $\mathcal{N}(R[x])$, which is impossible. Hence Corollary 4.3 shows that the ring $R[x,y] \simeq R[x][y]$ is not right (left) quasi-duo.

The aim of the following example is twofold. Firstly, it shows that there exists a ring R with an automorphism σ such that both $R[x;\sigma]$ and $R[x;\sigma^{-1}]$ are right quasi-duo but $R[x,x^{-1};\sigma]$ is not, i.e. the converse of Corollary 1.7 does not hold. Secondly, contrary to the cases of polynomial and skew Laurent polynomial rings, there exist right quasi-duo skew polynomial ring $R[x;\sigma]$ such that $R[x;\sigma]/J(R[x;\sigma])$ is not commutative.

Example 4.5. Let Δ be a right quasi-duo ring which is Jacobson semisimple (for example one can take Δ to be a division ring) and $P = \prod_{i\in\mathbb{Z}} \Delta_i$ be the direct product of $\Delta_i = \Delta$. We can identify the center F of Δ with the subring of P consisting with constant sequences $(f)_{i\in\mathbb{Z}}$, $f \in F$. Let $R = F + I \subseteq P$, where $I = \bigoplus_{i\in\mathbb{Z}} \Delta_i$. Let σ be the right shifting automorphism of R given by $\sigma((a_i)_{i\in\mathbb{Z}}) = (a_{i+1})_{i\in\mathbb{Z}}$.

One can easily check R is right quasi-duo. Observe also that $I[x;\sigma]x$ and $I[x;\sigma^{-1}]x$ are nil ideals of $R[x;\sigma]$ and $R[x;\sigma^{-1}]$, respectively, and the factor rings $R[x;\sigma]/(I[x;\sigma])$ and $R[x;\sigma^{-1}]/I[x;\sigma^{-1}]$ are isomorphic to the commutative polynomial ring $F[x]$. Now Lemma 3.2 yields that both $R[x;\sigma]$ and $R[x;\sigma^{-1}]$ are right quasi-duo. Notice that $N(R) = I$ and, as $J(\Delta) = 0$, the ring R is Jacobson semisimple. Thus, by Theorem 2.1, $J(R[x,x^{-1};\sigma]) = 0$ and $J(R[x;\sigma]) = I[x;\sigma]x$. Now, Theorem 4.2 implies that $R[x,x^{-1};\sigma]$ is not right quasi-duo. Notice also that if Δ is not commutative then $R[x;\sigma]/J(R[x;\sigma])$ is not commutative as well.

The following example shows that it is possible that $R[x;\sigma]$ is right quasi-duo whereas $R[x;\sigma^{-1}]$ is not.

Example 4.6. Let K be a field, $R = K\langle x_i \mid i \in \mathbb{Z},\ x_i x_j = 0, \text{ for } i < j\rangle$ and I denote the ideal of R generated by all products $x_k x_l$, where $k, l \in \mathbb{Z}$ and $k > l$. Let σ be the K-automorphism of R given by $\sigma(x_i) = x_{i+1}$, for all $i \in \mathbb{Z}$.

Note that $R[x;\sigma]/I[x;\sigma]$ is isomorphic to $P[x;\sigma]$, where $P = \prod_{i\in\mathbb{Z}} \Delta_i$ and $\Delta_i = K[x_i]$, for $i \in \mathbb{Z}$. The same arguments as in Example 4.5 (with F replaced by $\prod_{i\in\mathbb{Z}} K$), show that this factor ring is right quasi-duo. Thus, for proving that $R[x;\sigma]$ is right quasi-duo, it is enough to show that $I[x;\sigma] \subseteq J(R[x;\sigma])$. To this end, observe that for any monomial $m \in I$ we have $mR[x;\sigma]m = 0$ in $R[x;\sigma]$. This implies $mR[x;\sigma] \subseteq J(R[x;\sigma])$, for all monomials $m \in I$ and hence $I \subseteq J(R[x;\sigma])$ follows.

Finally the ring $R[x;\sigma^{-1}]$ is not right quasi-duo as otherwise, by Theorem 3.4, N would be an ideal of R and R/N would be commutative, where $N = N(R)$ is taken with respect to σ^{-1}. However $x_1 x_0^2 \dots x_{-n+1}^2 x_{-n} \neq 0$, for any $n \geq 1$, i.e. $x_1 x_0 \notin N$ and clearly $0 = x_0 x_1 \in N$.

Corollary 1.7 shows that for the ring R from the above example the ring $R[x,x^{-1};\sigma]$ is also not right quasi-duo.

Note that in Examples 4.5 and 4.6 the ring R contains maximal ideals with infinite orbits under the action of σ. We close the article with a result showing that this is one of the reasons for which $R[x,x^{-1};\sigma]$ is not quasi-duo.

Theorem 4.7. *Let σ be an automorphism of R. For any maximal ideal M of R we have:*

1. *Suppose the ring $R[x;\sigma]$ is right quasi-duo. Then either $\sigma(M)=M$ or M has infinite orbit under the action of σ. Moreover, if $\sigma(M)=M$, then the ring R/M is a field and σ induces identity on R/M.*

2. *Suppose the ring $R[x,x^{-1};\sigma]$ is right quasi-duo. Then $\sigma(M)=M$, the ring R/M is a field and σ induces identity on R/M.*

Proof. Let M be a maximal ideal of R. Notice that R/M is a division ring, as $R[x;\sigma]$ is right quasi-duo.

(1)(a) Suppose that M has finite orbit $\{M,\sigma(M),\ldots,\sigma^n(M)\}$ under the action of σ and let $I=M\cap\sigma(M)\cap\cdots\cap\sigma^n(M)$. Then, σ induces an automorphism of $\bar{R}=R/I$, which is also denoted by σ. Notice that $\bar{R}$ is isomorphic to the direct product of division rings $R/\sigma^k(M)$, $0\leq k\leq n$, and σ induces its isomorphism permuting the components. It is clear that the set of σ-nilpotent elements of $\bar{R}$ forms an ideal if and only if $\sigma(M)=M$, so the first part of (1) is a consequence of Proposition 2.3.

Suppose now that $\sigma(M)=M$. Then σ induces an automorphism of the division ring R/M and $R/M[x;\sigma]$ is right quasi-duo. Now the remaining part of (1) is a consequence of Theorem 3.3.

(2) Since $R[x,x^{-1};\sigma]$ is right quasi-duo, there exists a maximal ideal of $R[x,x^{-1};\sigma]$, say W, such that $MR[x,x^{-1};\sigma]\subseteq W$. Notice that maximality of M implies that $W\cap R=M$. We also have $M+xMx^{-1}=M+\sigma(M)$ and $M+\sigma^{-1}(M)=M+x^{-1}Mx$ are contained in $W\cap R=M$. This shows that $\sigma(M)=M$.

By Corollary 1.7, the ring $R[x;\sigma]$ is also a right quasi-duo and with the help of the statement (1) one can easily complete the proof of the theorem. □

Concluding, in this article we got a complete and, as it seems, quite satisfactory characterization of quasi-duo skew polynomial rings of endomorphism type. Our results show that it is hard to expect that such rings might be helpful in constructing right but not left quasi-duo rings. It seems that more appropriate for that aim could be skew polynomial rings $R[x;\tau,\delta]$, where δ denotes a skew τ-derivation of R. The situation reminds a bit the case of duo property of rings (recall that a ring R is called right (left) duo if every right (left) ideal of R is two-sided). It is known (Cf.[8]) that $R[x;\tau]$ is right duo iff it is left duo, whereas there exists (Cf.[9]) a skew polynomial ring $R[x;\tau,\delta]$ such that it is right but not left duo. Note however that the examples constructed in [9] are left and right quasi-duo. In that context it would be interesting to characterize right and left quasi-duo property of skew polynomial rings $R[x;\tau,\delta]$.

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