On Right Strongly McCoy Rings

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Dedicated to T.Y.Lam for his seventieth birthday.

Abstract

In this paper we introduce and investigate right strongly McCoy rings, that is, rings for which every right module has the McCoy property. We show, in particular, that a von Neuman regular ring or a Frobenius ring is right McCoy if and only if it is right strongly McCoy. We also give characterizations of domains and semiprime Goldie rings which are right strongly McCoy.

Introduction

All rings considered in this paper are associative with unity. For any subset S of a right R-module M, $\mathrm{ann}_R(S)$ will denote the annihilator of S, i.e., $\mathrm{ann}_R(S) = \{r \in R \mid Sr = 0\}$.

McCoy observed that if R is a commutative ring then, for any polynomial $f(x) \in R[x]$ with $\mathrm{ann}_{R[x]}(f(x)) \neq 0$, one always has $\mathrm{ann}_R(f(x)) \neq 0$. Following a suggestion of T.Y. Lam, P. Nielsen defined in [8] a ring to be right McCoy if it satisfies the above property. The notion of left McCoy ring is defined similarly and a ring is McCoy if it is left and right McCoy. Recall that a ring is semicommutative (resp. reversible) if for any $a, b \in R$

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such that $ab = 0$ we also have $aRb = 0$ (resp. $ba = 0$). In the paper just mentioned, P. Nielsen gave an example of a semicommutative ring that is not McCoy. This answered a question prompted by the facts that, on one hand, if $R[x]$ is semicommutative then R is McCoy (cf. [4]) and, on the other hand, semicommutativity of R does not imply semicommutativity of $R[x]$ (cf. [3]). In [8], P. Nielsen also showed that any reversible ring is McCoy. In Proposition 1.1 we will give a short proof of this fact. In [1], V. Camillo and P. Nielsen studied the McCoy conditions and some of their generalizations in connection with other ring properties such as duo, quasi-duo, symmetric, etc.

Following a definition given in [2], we say that a right R-module M is McCoy if $\mathrm{ann}_R(f(x)) \neq 0$, for any $f(x) \in M[x]$ such that $\mathrm{ann}_{R[x]}(f(x)) \neq 0$. We define a ring R to be right strongly McCoy if every right R-module is McCoy.

The aim of the paper is two fold: to investigate the behaviour of the right strongly McCoy property under various ring extensions and to determine classes of rings in which being right strongly McCoy is equivalent to being right McCoy.

In the first section we give basic properties and construct some examples of McCoy modules and right strongly McCoy rings. It appears that, contrary to the McCoy property, not every commutative ring is right strongly McCoy. We also observe that right duo semiprime rings are right strongly McCoy.

Section 2 begins by showing that a domain is right strongly McCoy if and only if it is a right Ore domain (Theorem 2.1). Since a domain is obviously a McCoy ring, this offers a wide range of examples of right McCoy rings that are not right strongly McCoy. This result gives also a negative answer to a question posed in [2]. In Theorems 2.3, 2.4 and Corollary 2.5 we show that the behaviour of the right strongly McCoy property is very nice with respect to right Ore localizations, formation of corner rings, and direct products of rings. Theorem 2.6 states, in particular, that a semiprime right Goldie ring is right McCoy ring if and only if it is strongly McCoy. After giving some more properties of McCoy modules in Proposition 2.8, we prove in Theorem 2.11 that for von Neuman regular rings and FGF rings (i.e., rings such that finitely generated modules can be embedded in a free module) the notions of right McCoy and right strongly McCoy coincide. As an application we obtain that any group algebra over a commutative domain of an abelian group is always a right strongly McCoy ring. The paper ends with some examples and comments.

1 Preliminaries

We begin this section with a short proof of Theorem 2 of [8]. Recall that a ring R is reversible if $ab = 0$ implies $ba = 0$, for any $a, b \in R$.

Proposition 1.1. *Every reversible ring is right McCoy.*

Proof. Suppose R is a reversible ring. Let $f(x) = a_n x^n + \ldots + a_0$ be an element of $R[x]$ such that $\mathrm{ann}_{R[x]}(f(x)) \neq 0$ and $0 \neq g(x) = b_m x^m + \ldots + b_0 \in \mathrm{ann}_{R[x]}(f(x))$ be of minimal degree m. We claim that $m = 0$, i.e., R is right McCoy. Assume $m \geq 1$. Since $f(x)g(x) = 0$, we have $a_n b_m = 0$. Thus, as R is reversible, $\deg(g(x)a_n) < \deg(g(x))$. The equality $f(x)(g(x)a_n) = 0$ and the choice of m imply that $g(x)a_n = 0$. Thus also $a_n g(x) = 0$, as R is reversible. Then $0 = f(x)g(x) = (a_n x^n + (a_{n-1}x^{n-1} + \ldots + a_0))g(x)$ yields that $a_{n-1}b_m = 0$ and as above we get $g(x)a_{n-1} = 0$. Continuing in this way we obtain $a_i g(x) = 0$, for all $0 \leq i \leq n$. This implies that $f(x)b_m = 0$ and contradicts the assumption that $\deg(g(x)) > 0$. □

Let M be a right R-module. Then the induced right $R[x]$-module $M \otimes_R R[x]$ will be denoted by $M[x]$. Elements from $M[x]$ can be seen as polynomials in x with coefficients in M with natural additive and right $R[x]$-module structures. For this reason we will call elements of $M[x]$ polynomials and present them in the form $\sum_i m_i x^i$, where $m_i \in M$.

Definition 1.2. (1) A right R-module M is called McCoy if for any polynomial $m(x) \in M[x]$ with $\mathrm{ann}_{R[x]}(m(x)) \neq 0$, we have $\mathrm{ann}_R(m(x)) \neq 0$.
(2) We say that a ring R is right strongly McCoy if every right R-module is McCoy.

Similarly one can define left strongly McCoy rings.

Observe that every submodule of a McCoy module is McCoy and in the definition of right strongly McCoy rings it is enough to consider finitely generated modules only. Notice also that every right R-module with nonzero annihilator is McCoy. The above observations directly give the following:

Proposition 1.3. *A ring R is right strongly McCoy if and only if every faithful finitely generated right R-module is McCoy.*

Henceforth we will use the characterization of right strongly McCoy rings given in the above proposition.

Remark 1.4. Let M be a right R-module such that any finite subset S of M has nonzero annihilator in R. Then M is McCoy. In particular, singular R-modules, torsion modules over semiprime Goldie rings or over rings with Goldie dimension one are examples of McCoy modules.

The following proposition offers another class of McCoy modules.

Proposition 1.5. *Suppose that R is a commutative ring. Then every cyclic right R-module is McCoy.*

Proof. Let $M = cR$ be a cyclic faithful R-module and $m(x) \in M[x]$ be such that $m(x)g(x) = 0$, for some $0 \neq g(x) \in R[x]$. By assumption there exists $f(x) \in R[x]$ such that $m(x) = cf(x)$. Since M is faithful and R is commutative, we get $f(x)g(x) = 0$. R is a McCoy ring, so $f(x)r = 0$, for certain $0 \neq r \in R$. Then also $m(x)r = 0$. □

Since a right duo ring is right McCoy (cf. Theorem 8.2 in [1]), the same argument as in the above proof also shows that cyclic modules over right duo rings are right McCoy. The following example offers a module M generated by two elements over a commutative ring R, which is not McCoy. Thus the ring R is right McCoy but not right strongly McCoy.

Example 1.6. Let K be a field and V a K-linear vector space with basis v_1, v_2, v_3. Let a_0, a_1 be endomorphisms of V defined by setting:

$$a_0(v_1) = v_2, \quad v_2, v_3 \in \ker a_0 \quad \text{and} \quad a_1(v_3) = -v_2, \quad v_1, v_2 \in \ker a_1.$$

Then $a_0^2 = a_1^2 = a_0a_1 = a_1a_0 = 0$. Let $R = K[a_0, a_1]$. Then V has natural structure of an R-module such that $0 = v_1a_1 = v_3a_0 = v_3a_1 + v_1a_0$. This means that $(v_1x+v_3)(a_1x+a_0) = 0$. Since R is a three dimensional K-algebra with basis $1, a_0, a_1$, it can be directly checked that $\operatorname{ann}_R(v_1x + v_3) = 0$. This means that V is not McCoy as an R-module.

Let M be a right R-module. For formulating the next lemma we set $D(M) = \{m \in M \mid \operatorname{ann}_R(m) \neq 0\}$ and define $T(M) \subseteq D(M)$ as the set of all elements $m \in M$ such that $\operatorname{ann}_R(m)$ contains a regular element. Recall that $T(M)$ is always a submodule of M, called the torsion part of M, provided the set of all regular elements of R satisfies the right Ore condition. When R is a domain, then clearly $D(M) = T(M)$.

The following lemma offers another sufficient condition for an R-module M to be McCoy.

Lemma 1.7. *Suppose $T(M) = D(M)$ is an R-submodule of M. Then M is a McCoy module.*

Proof. Let $v(x) = v_nx^n + \ldots + v_0 \in M[x]$ and $g(x) = g_mx^m + \ldots + g_0 \in R[x]$, with $g_m \neq 0$, be such that $v(x)g(x) = 0$. Then $v_n \in D(M) = T(M)$. Thus, as $T(M)$ is a submodule of M, there exists a regular element $u \in R$ such that $v_ng_{m-1}u = 0$. Therefore, making use of the equality $v_ng_{m-1} + v_{n-1}g_m = 0$, we get $v_{n-1}g_m u = 0$. This means that $v_n, v_{n-1} \in D(M)$. Notice that for any finite subset S of $T(M)$, there exists a regular element $u \in R$ such that $Su = 0$. Thus, continuing the above procedure we easily see that there exists a regular element $u \in R$ such that $v(x)u = 0$, i.e., M is McCoy. □

We close this section by presenting two classes of right strongly McCoy rings. Before doing so we need the following definition. We say that a right R-module M is semicommutative if $mr = 0$ implies $mRr = 0$, for any $m \in M$ and $r \in R$, i.e., $\mathrm{ann}_R(m)$ is a two-sided ideal of R, for any $m \in M$.

Let M be a semicommutative right R-module. The arguments used in the proof of Lemma 1 of [8] can also be applied to semicommutative modules. In particular, when $v(x)g(x) = 0$, where $v(x) = v_n x^n + \ldots + v_0 \in M[x]$ and $g(x) = g_m x^m + \ldots + g_0 \in R[x]$, then $v(x)g_0^n = 0$. This implies that every semicommutative module over a reduced ring R is McCoy. This observation gives immediately the following:

Proposition 1.8. *Every right duo semiprime ring is right strongly McCoy. In particular, if R is a strongly von Neumann regular ring or R is a commutative reduced ring, then R is right strongly McCoy.*

Proposition 1.9. *Let $n \geq 2$. Then the ring $T = R[y]/(y^n) = R < y \mid y^n = 0 >$ is right strongly McCoy if and only if R is right strongly McCoy.*

Proof. Suppose T is right strongly McCoy. Let M be a right R-module and $m(x) \in M[x]$ and $0 \neq w(x) \in R[x]$ be such that $m(x)w(x) = 0$. Let $\hat{M} = M \otimes_R T$. As T is a free left R-module, M is an R-submodule of $\hat{M}$ and $M[x] \subseteq \hat{M}[x]$. Thus, as $\hat{M}$ is a McCoy T-module, we can pick $0 \neq t = t_k y^k + \ldots + t_0 \in T$ such that $m(x)t = 0$, where $t_i \in R$, $0 \leq i \leq k \leq n-1$, $t_k \neq 0$. Then $m(x)t_k = 0$ as T is free as an R-module. This implies that M is a McCoy R-module and shows that R is a right strongly McCoy ring.

Suppose now that R is right strongly McCoy. Let M be a right T-module and $a(x) \in M[x]$ be such that $\mathrm{ann}_{T[x]}(a(x)) \neq 0$. Let $0 \neq g(x) = b_m x^m + \ldots + b_0 \in \mathrm{ann}_{T[x]}(a(x))$. Eventually multiplying $g(x)$ by a suitable power of y, we may additionally suppose that $g(x)y = 0$, which means that there exist $r_i \in R$, with $0 \leq i \leq m$, such that $b_i = r_i y^{n-1}$, i.e., $g(x) = h(x)y^{n-1}$, where $h(x) \in R[x]$. If $a(x)y^{n-1} = 0$, then we are done. Assume $a(x)y^{n-1} \neq 0$. Considering M as an R-module, we have in $M[x]_{R[x]}$: $(a(x)y^{n-1})h(x) = 0$. By assumption M is a McCoy R-module, hence there exists $0 \neq r \in R$ such that $a(x)(y^{n-1}r) = 0$, i.e., $0 \neq y^{n-1}r \in \mathrm{ann}_T(a(x))$. □

Remark 1.10. Let R be the ring from Example 1.6. Then:
1. R is a homomorphic image of the polynomial ring $K[x, y]$ which, by Proposition 1.8, is right strongly McCoy. Thus the class of right strongly

McCoy rings is not closed under homomorphic images.
2. By Proposition 1.9, the ring $T = K[x,y]/(x^2,y^2)$ is right strongly McCoy and R is isomorphic to a subring of T generated by $x + (x^2,y^2)$ and $xy + (x^2,y^2)$. Thus the class of right strongly McCoy rings is not closed under taking subrings.
3. Every proper homomorphic image of R is isomorphic either to K or to $K[y]/(y^2)$. Thus, by the above proposition, every proper homomorphic image of R is right strongly McCoy and R is not right strongly McCoy.

2 Properties of right strongly McCoy rings

We begin this section with a characterization of domains which are right strongly McCoy rings.

Theorem 2.1. *For a domain R the following conditions are equivalent:*

1. *Every right R-module is McCoy, i.e., R is right strongly McCoy;*
2. *Every right cyclic R module is McCoy;*
3. *R is a right Ore domain;*
4. *$T(M)$ is a submodule of M, for any right R-module M.*

Proof. The implication (1) $\Rightarrow$ (2) is a tautology.

(2) $\Rightarrow$ (3) Suppose now that R is a domain which does not satisfy the right Ore conditions. This means that we can pick nonzero elements $a, b \in R$ such that $aR \cap bR = 0$. Let U denote the right ideal $a^2R + b^2R + (ab + ba)R$ of R. First we will show that if $ar \in U$, for some $r \in R$, then $r \in aR$. Suppose $ar = a^2w_1 + b^2w_0 + (ab + ba)v$, for some $w_1, w_0, v \in R$. Then $ar = a(aw_1 + bv) + b(bw_0 + av)$. Using twice the fact that R is a domain with $aR \cap bR = 0$, we obtain consecutively $bw_0 + av = 0$ and $w_0 = v = 0$, i.e., $ar = a^2w_1$. This shows that indeed $r = aw_1 \in aR$. Since the definition of U is symmetric with respect to a and b, we also have that if $br \in U$, then $r \in bR$.

Now, let M denote the cyclic right R-module R/U and v_1, v_0 stand for canonical images of a and b in M, respectively. The above considerations show that $\mathrm{ann}_R(v_1) = aR$ and $\mathrm{ann}_R(v_0) = bR$. Therefore, the polynomial $v_1x + v_0 \in M[x]$ has zero annihilator in R. However one can easily see that $(v_1x + v_0)(ax + b) = 0$. This shows that the cyclic right R-module M is not a McCoy module.

When R is a right Ore domain, then $T(M)$ is just the torsion submodule of R, i.e., (3) $\Rightarrow$ (4). By assumption, R is a domain. Thus $T(M) = D(M)$ and the implication (4) $\Rightarrow$ (1) is given by Lemma 1.7. $\square$

It is known (cf.[8]), that there exist rings which are left but not right McCoy. The above theorem shows that the notion of strongly McCoy rings is also not left-right symmetric, as there are many examples of domains which satisfy the Ore condition only on one side.

We have seen that not all commutative rings are strongly McCoy, but Theorem 2.1 implies that commutative domains are always strongly McCoy.

The above theorem gives immediately a negative answer to Question 1 of [2] whether a ring R has to be right duo, provided every cyclic right R-module is McCoy. By Theorem 2.1, any right Ore domain R which is not right duo is a good example. For example, one can take $R = K[x;\sigma]$ - the skew polynomial ring over a field K, where σ is a non-identity automorphism of K (cf.[6]).

Remark 2.2. Let $\mathcal{M}$, $\mathcal{CM}$, $\mathcal{SM}$ denote the classes of all right McCoy rings, all rings R such that every cyclic right R-module is McCoy and all right strongly McCoy rings, respectively. Then obviously $\mathcal{SM} \subseteq \mathcal{CM} \subseteq \mathcal{M}$. Proposition 1.5 and Example 1.6 show that there exist rings for which every cyclic right R-module is McCoy but the ring is not right strongly McCoy, i.e., $\mathcal{CM} \neq \mathcal{SM}$. If R be a domain which is not right Ore then R is a McCoy ring and Theorem 2.1 says that $R \notin \mathcal{CM}$. This means that the introduced three classes of rings are different from each other.

Theorem 2.3. *Let S be a right Ore set consisting of regular elements of a ring R. Then the localization RS^{-1} is a right strongly McCoy ring if and only if R is a right strongly McCoy ring.*

Proof. Using the facts that every right RS^{-1}-module is a right R-module and for every finite subset $A \subseteq RS^{-1}$ there exists $s \in S$ such that $As \subseteq R$ one can easily prove that if R is right strongly McCoy then so is RS^{-1}.

Suppose now that RS^{-1} is right strongly McCoy and let M be a right R-module. It is known that the kernel of the canonical map from M to its localization $MS^{-1} = M \otimes_R RS^{-1}$ is equal to the S-torsion submodule $T_S(M) = \{m \in M \mid ms = 0, \text{ for some } s \in S\}$. Thus we may consider the factor R-module $M/T_S(M)$ as an R-submodule of MS^{-1}. By assumption, MS^{-1} is a McCoy RS^{-1}-module and, as every nonzero right ideal of RS^{-1} has nonzero intersection with R, we deduce that $M/T_S(M)$ is a McCoy R-module.

Let $m(x) \in M[x]$ and $0 \neq g(x) \in R[x]$ be such that $m(x)g(x) = 0$. Let $\bar{m}(x)$ denote the natural image of $m(x)$ in $M[x]/(T_S(M)[x]) = (M/T_S(M)[x]$.

Since $M/T_S(M)$ is a McCoy R-module and $\bar{m}(x)g(x) = 0$, there exists $0 \neq r \in R$ such that $\bar{m}(x)r = 0$, that is, $m(x)r \in T_S(M)[x]$. Notice that for any finite subset A of $T_S(M)$, we can always find a regular element $q \in S$ such that $Aq = 0$. In particular, there exists a regular element $q \in R$ such that $m(x)rq = 0$ and $rq \neq 0$ as $r \neq 0$ and q is regular. This means that M is McCoy and hence R is right strongly McCoy. □

Notice that the above theorem offers a direct proof of the implication (3) ⇒ (1) in Theorem 2.1.

Theorem 2.4. *Let $e \notin \{0, 1\}$ be a central idempotent of a ring R. Then R is right strongly McCoy if and only if rings eR and $(1 - e)R$ are right strongly McCoy.*

Proof. Suppose R is a right strongly McCoy ring. Let M be an eR-module and $m(x) \in M[x]$ be such that $m(x)g(x) = 0$, for some $0 \neq g(x) \in eR[x]$. M has a natural structure of an R-module, given by $M(1 - e) = 0$. Thus we can consider the right R-module $N = M \oplus R(1 - e)$. By assumption, N is a McCoy module and the equality $(m(x) + (1 - e))g(x) = 0$ yields that there exists $0 \neq r \in R$ such that $(m(x) + (1 - e))r = 0$. This means that $r \in eR$ and $m(x)r = 0$ and shows that eR is right strongly McCoy. Similarly $(1 - e)R$ is right strongly McCoy.

Suppose both eR and $(1 - e)R$ are right strongly McCoy. Let M be an R-module and $a(x) \in M[x]$ be such that $\mathrm{ann}_{R[x]}(a(x)) \neq 0$. Assume $\mathrm{ann}_R(a(x)) = 0$. Then both $a(x)e$ and $a(x)(1 - e)$ are nonzero. Let $0 \neq g(x) \in \mathrm{ann}_{R[x]}(a(x))$. Eventually replacing e by $1 - e$, we may assume that $eg(x) \neq 0$. By assumption the eR-module Me is McCoy and the equality $(a(x)e)(er) = a(x)g(x)e = 0$ implies that there exists nonzero element $r \in eR \subseteq R$ such that $a(x)r = a(x)er = 0$. This contradicts the assumption that $\mathrm{ann}_R(a(x)) = 0$ and completes the proof. □

Remark that the ring R from Example 1.6 is a subdirect product of rings $R/(a_0)$ and $R/(a_1)$ which are isomorphic to $k[y]/(y^2)$. Thus Proposition 1.9 and Example 1.6 show that subdirect product of finite number of right strongly McCoy rings does not have to be right strongly McCoy. The following corollary shows that the class of right strongly McCoy rings is closed under products.

Corollary 2.5. *The ring $R = \prod_{i \in I} R_i$ is right strongly McCoy if and only if R_i is right strongly McCoy, for all $i \in I$.*

Proof. Let e_i, for $i \in I$, denote the unity of the ring R_i. Suppose R is right strongly McCoy. Then, by the above theorem, $R_i = Re_i$ is right strongly McCoy, for any $i \in I$.

Suppose R_i is right strongly McCoy, for each $i \in I$. Let M be an R-module and $m(x) \in M[x]$ be such that $m(x)g(x) = 0$, for some $0 \neq g(x) \in R[x]$. Then there exist idempotent $e = e_i$, for a suitable $i \in I$, such that $g(x)e \neq 0$. As the eR-module Me is McCoy, there is $0 \neq r \in eR \subseteq R$, such that $m(x)r = m(x)er = 0$. □

Now, with the help of obtained results we get the following theorem:

Theorem 2.6. *Let R be a semiprime right Goldie ring. Then the following conditions are equivalent:*

(1) every right R-module is McCoy, i.e., R is right strongly McCoy;

(2) every cyclic right R-module is McCoy;

(3) R is right McCoy;

(4) The classical right quotient ring Q of R is isomorphic to a finite product of division rings;

(5) For every minimal prime ideal P of R, the factor ring R/P is a domain.

(6) R is a reduced ring.

Proof. The implications (1) ⇒ (2) ⇒ (3) are tautologies and the equivalences (4) ⇔ (5) ⇔ (6) are known (cf. Propositions 11.22 and 12.7 of [5]).

(3) ⇒ (4) Let Q denote the classical right ring of quotients of R. Suppose $f(x)g(x) = 0$, for some $f(x), g(x) \in Q[x]$, where $g(x) \neq 0$. We can pick regular elements $t, q \in R$ such that $f(x)t, t^{-1}g(x)q \in R[x]$. Then the equation $(f(x)t)(t^{-1}g(x)q) = 0$ is an equation in $R[x]$ and $t^{-1}g(x)q \neq 0$. Thus, as R is a right McCoy ring we can find $0 \neq r \in R$ such that $f(x)tr = 0$ and $tr \neq 0$, as t is a regular element. This shows that Q is a right McCoy ring.

Notice that if a right McCoy ring R is a product of rings A and B, then the rings A, B have to be right McCoy (cf. Lemma 4.1 in [1]). Therefore the right McCoy semisimple artinian ring Q has to be a product of division rings as, for $n \geq 2$, the matrix ring $M_n(D)$ is never right McCoy (cf. Proposition 10.2 in [1]).

(4) ⇒ (1) Suppose the classical right quotient ring Q of R is isomorphic to a finite product of division rings. Then, by Corollary 2.5, Q is right strongly McCoy. Now, Theorem 2.3 completes the proof of (1). □

Corollary 13 of [3] says that in the above theorem one could add another equivalent statement that R is an Armendariz ring. Since a semiprime ring is reduced iff it is 2-primal, one could also add the statement that R is 2-primal.

It is known that if R is a right McCoy ring, then so is the polynomial ring $R[x]$. As observed in [2] the same argument shows that if a module M is right McCoy then $M[x]$ is a right McCoy $R[x]$-module. We do not know whether the right strongly McCoy property lifts from R to $R[x]$. The following proposition shows that this is true in two special cases.

Proposition 2.7. *Suppose that R is either a domain or a semiprime right Goldie ring. Then R is right strongly McCoy if and only if the polynomial ring $R[x]$ is right strongly McCoy.*

Proof. It is known and easy to check that a ring R is a right Ore domain iff $R[x]$ is a right Ore domain. Thus, in case R is a domain, the result is a direct consequence of Theorem 2.1. Suppose now that R is semiprime right Goldie. Then $R[x]$ is also semiprime right Goldie and clearly R is reduced iff $R[x]$ is reduced. Hence, by Theorem 2.6, R is right strongly McCoy iff $R[x]$ is right strongly McCoy. □

The following proposition leads to two interesting families of rings for which the right McCoy property is equivalent to the right strongly McCoy property.

Proposition 2.8. *For any index set I we have the following:*

1. *Suppose M is a McCoy right R-module. Then the direct sum $\bigoplus_{i\in I} M_i$, with $M_i = M$ for all $i \in I$, is a McCoy R-module;*
2. *Suppose R is a right artinian ring and M is a McCoy right R-module. Then the direct product $\prod_{i\in I} M_i$, with $M_i = M$ for all $i \in I$, is a McCoy R-module;*
3. *If I is a directed set and $\{M_i : i \in I\}$ is a direct system, then the direct limit $\varinjlim M_i$ is a McCoy R-module, provided M_i is a McCoy module, for every $i \in I$.*

Proof. Let M_R be a McCoy right R-module. We claim that for any finite set $\{m_1(x), \ldots, m_t(x)\} \subset M[x]$ we have,

$$\bigcap_{i=1}^{t} \mathrm{ann}_{R[x]} m_i(x) \neq 0 \quad \text{if and only if} \quad \bigcap_{i=1}^{t} \mathrm{ann}_R m_i(x) \neq 0.$$

The condition is obviously sufficient. Let $0 \neq g(x) \in \bigcap_{i=1}^{t} \mathrm{ann}_{R[x]} m_i(x)$ and let $k \geq 0$ denote the maximum of $\deg_x m_i(x)$ for $1 \leq i \leq t$. Let $f(x) = m_1(x) + m_2(x)x^k + \ldots + m_t(x)x^{(t-1)k} \in M[x]$. Then $f(x)g(x) = 0$

and the McCoy property of M implies that there exists $0 \neq r \in R$ such that $f(x)r = 0$, i.e., $m_i(x)r = 0$, for all $1 \leq i \leq t$. This proves the claim.

(1) Let $m(x) = (m_i(x))_{i\in I} \in (\bigoplus_{i\in I} M_i)[x] = \bigoplus_{i\in I}(M_i[x])$ be such that $m(x)g(x) = 0$ for some $0 \neq g(x) \in R[x]$. This means that $m_i(x)g(x) = 0$, for all $i \in I$. Let F be a finite subset of I such that $m_i(x) = 0$, for all $i \in I \setminus F$. The claim proved above shows that there exists $0 \neq r \in ann_R\{m_i(x) \mid i \in F\}$. Then clearly $m(x)r = 0$, as desired.

(2) Let $m(x) = (m_i(x))_{i\in I} \in (\prod_{i\in I} M_i)[x] \subseteq \prod_{i\in I}(M_i[x])$ be such that $m(x)g(x) = 0$ for some $0 \neq g(x) \in R[x]$. This means that $m_i(x)g(x) = 0$, for all $i \in I$. Therefore, by the first part of the proof, $\mathrm{ann}_R(S_F) \neq 0$, for any finite subset F of I, where $S_F = \{m_i(x) \mid i \in F\}$. Since R is right artinian we can choose a finite subset $K \subseteq I$ such that $\mathrm{ann}_R(S_K)$ is minimal amongst annihilators of S_F, where F ranges over all finite subsets of I. For $i \in I$, let us define the set $K_i = K \cup \{i\}$. Clearly $\mathrm{ann}_R(S_{K_i}) \subseteq \mathrm{ann}_R(S_K)$ and minimality of $\mathrm{ann}_R(S_K)$ forces $\mathrm{ann}_R(S_{K_i}) = \mathrm{ann}_R(S_K)$, for any $i \in I$. The above shows that there exists a nonzero element $r \in \mathrm{ann}_R(S_K)$ and then $m(x)r = 0$, as desired.

(3) Let $m(x) := \sum_{j=0}^{n} m_j x^j \in (\varinjlim M_i)[x]$ and $f(x) \in R[x]$ be such that $m(x)f(x) = 0$. One can choose $k \in I$ such that the elements $m_j \in \varinjlim M_i$, $0 \leq j \leq n$, are represented by elements in M_k. Since M_k is a McCoy module there exists $r \in R$ such that $m(x)r = 0$. This shows that M is a McCoy module. □

Corollary 2.9. *Let M be a right module over a right McCoy ring R. Then M is a McCoy module if one of the following properties holds:*

- *(a) M is a submodule of a free R-module;*
- *(b) M is a projective R-module;*
- *(c) R, as a right module, is a cogenerator and M is finitely cogenerated right R-module;*
- *(d) M is a flat R-module;*

Proof. Suppose (a) holds. Then Proposition 2.8(1) and the fact that a submodule of a McCoy module is McCoy imply that M is McCoy. In particular, if (b) holds, i.e., M is projective, it has to be McCoy.

The assumptions imposed in (c) imply that M is isomorphic to a submodule of a finitely generated free right R-module (cf. Propositions 19.1 and 19.6 of [5]). This implies that M is McCoy.

Suppose (d) holds. It is known that any flat module is a direct limit of finitely generated free modules (cf. Theorem 4.34 in [5]). Now the result is a consequence of Proposition 2.8(3). □

Theorem 2.10. *Let R be a right artinian ring and M a right R-module which is a cogenerator. If M is McCoy then R is a right strongly McCoy ring.*

Proof. Since M is a cogenerator, any R-module can be embedded in a direct product of copies of M. This fact and Proposition 2.8(2) yield that all right R-modules are McCoy, i.e., R is strongly right McCoy. □

Let us recall that a ring R is a right FGF ring if every finitely generated right R-module can be embedded in a free right R-module. In particular, quasi-Frobenius rings are right FGF rings. Let us mention that the FGF conjecture asks if every right FGF ring is quasi-Frobenius.

Theorem 2.11. *If a ring R is either von Neumann regular or FGF, then R is right McCoy if and only if it is right strongly McCoy.*

Proof. Of course, every right strongly McCoy ring is right McCoy.

It is well-known that over a regular ring, any right R-module is flat (cf. Theorem 4.21 in [5]). Thus, Corollary 2.9 (d) implies that a von Neumann regular right McCoy ring R is right strongly McCoy.

In the case of FGF rings, the proof is given by Corollary 2.9 (a). □

Any group algebra of a finite group over a field is Frobenius. Thus the above theorem gives immediately the following corollary:

Corollary 2.12. *Let KG be a group algebra, where K is a field and G is a finite group. Then KG is right strongly McCoy iff KG is right McCoy.*

Example 1.6 shows that, contrary to the right McCoy property, not every commutative ring is right strongly McCoy. Proposition 2.1 implies that every commutative domain is strongly McCoy. The following theorem offers another class of commutative rings which are right strongly McCoy.

Theorem 2.13. *Let D be a commutative domain and G an abelian group. Then the group ring DG is right strongly McCoy.*

Proof. Notice that, for any polynomial $g(x) \in DG[x]$, there exists a finite subgroup H of G such that $g(x) \in DH[x]$. This means that we can assume in the proof that the group G is finitely generated. Then, as G is abelian, we can write $DG = (DF)H$ where H is the torsion part of G and F is free abelian group of finite rank. DF is commutative domain and the set S of all nonzero elements of DF is an Ore set of regular elements of DG. By Theorem 2.3, the ring DG is right strongly McCoy iff its localization $(DG)S^{-1}$ is right strongly McCoy. However the ring $(DG)S^{-1}$ is isomorphic to $((DF)S^{-1})H$ and Corollary 2.12 says that this ring is right strongly McCoy. □

For our next example we need the following technical lemma

Lemma 2.14. *Let $R = K\langle a, b\rangle$ be the free algebra over a field K. Suppose $f(x), g(x) \in R[x]$ are such that $g(x) \neq 0$ and $f(x)g(x) \in bR[x]$. Then there are $h(x) \in bR[x]$ and $c(x) \in K[x] \subseteq R[x]$ such that $f(x) = h(x) + c(x)$. Moreover, if $c(x) \neq 0$, then $g(x) \in bR[x]$.*

Proof. Let S denote the multiplicative semigroup generated by $\{a, b, x\} \subseteq R[x]$. Every element of S can be uniquely presented in the form $x^n w$, where $n \geq 0$ and w is a word, possibly empty, in alphabet $\{a, b\}$. We can introduce the lexicographical order in S by setting $1 < x < a < b$. Then S is an ordered semigroup.

Let $h(x)$ be the sum of all terms of $f(x)$ belonging to $bR[x]$ and $c(x) = f(x) - h(x)$. If $c(x) = 0$ we are done.

Suppose $c(x) \neq 0$. Since $(f(x) - h(x))g(x) \in bR[x]$, we have also $c(x)g(x) \in bR[x]$. In particular, as $g(x) \neq 0$, $0 \neq vq \in bR[x]$, where v and q denote leading terms of $c(x)$ and $g(x)$ respectively. This implies that $v \in K[x]$, as otherwise v would be of the form $kx^n a...$, for suitable $k \in K$, $n \geq 0$ and vq would belong to $aR[x]$, which is impossible. Then the condition $0 \neq vq \in bR[x]$ implies that $q \in bR[x]$. By the choice, every nonzero term of $c(x)$ is smaller than v. Therefore $c(x) \in K[x]$. Notice that also $c(x)(g(x) - q) \in bR[x]$ and the degree of $g(x) - q$ is smaller than that of $g(x)$ and a simple inductive argument shows that $g(x) \in bR[x]$ provided $c(x) \neq 0$. □

The McCoy property of modules is hereditary on submodules. The following example shows that this property does not lifts from essential submodules, i.e., an R-module N can have an essential submodule which is McCoy but N itself is not McCoy.

Example 2.15. Let $R = K\langle a, b\rangle$. Then R is not right Ore domain. Let $N = v_1 R + v_0 R$ be the R-submodule of a module M defined in the proof of implication (2) ⇒ (3) of Theorem 2.1. Then the action of R on N is given by $v_0 \cdot a = v_0 a$, $v_0 \cdot b = 0$ and $v_1 \cdot a = 0$, $v_1 \cdot b = -v_0 a$ and, by the proof of Theorem 2.1, N is not right McCoy as a right R-module.

Set $L = v_0 R$. Then it is clear that L is an essential submodule of N. We have seen in the proof of Theorem 2.1 that $\mathrm{ann}_{R[x]}(v_0) = bR[x]$. This together with Lemma 2.14 imply that L is a McCoy R-module

In order to prove the last result, we need a definition:

Let M be a right R-module. For $f(x) = \sum_{i=0}^{n} a_i x \in M[x]$ and $r \in R$ we define evaluation $f(r) = \sum_{i=0}^{n} a_i r^i \in M$.

The following technical lemma collects basic properties of evaluation as defined here:

Lemma 2.16. *Let R, C and M denote a ring, its center and a right R-module respectively. For $f(x) = \sum_{i=0}^{n} a_i x^i \in M[x]$, $g(x) \in R[x]$, $r \in R$, and $c \in C$, we have:*

(a) There exists a polynomial $q(x) \in M[x]$ such that $f(x) = q(x)(x-r) + f(r)$,

(b) $f(r) = 0$ if and only if $f(x) \in M[x](x-r)$,

(c) $(f(x)g(x))(c) = f(c)g(c)$,

(d) If $c_1, \dots c_{n+1} \in C$ are such that $f(c_j) = 0$, for $1 \le j \le n+1$, and $\mathrm{ann}_M(\prod_{1\le i<j\le n+1}(c_i - c_j)) = 0$ then $f = 0$.

Proof. (a) We proceed by induction on n. If $n = 0$ or $n = 1$, the result is clear. Suppose $n \ge 2$, we have $f(x) = a_n x^{n-1}(x-r) + h(x)$ where $h(x) := (a_n r + a_{n-1})x^{n-1} + \sum_{i=0}^{n-2} a_i x^i$. By the induction hypothesis we can write $h(x) = p(x)(x-r) + h(r)$, for a suitable $p(x) \in M[x]$. Since $h(r) = f(r)$ we obtain $f(x) = (a_{n-1}x^{n-1} + p(x))(x-r) + f(r)$.

(b) The proof, as easy, is left to the reader.

(c) We write $f(x) = q(x)(x-c) + f(c)$ and $g(x) = p(x)(x-c) + g(c)$ and, since $x - c$ is central, we have $f(x)g(x) = (f(x)p(x) + q(x)g(c))(x-c) + f(c)g(c)$.

(d) We have $f(x) = q_1(x)(x-c_1)$ for some $q_1(x) \in M[x]$. This gives $0 = f(c_2) = q_1(c_2)(c_2 - c_1)$. The assumption on annihilators implies $q_1(c_2) = 0$ and hence there exists $q_2(x) \in M[x]$ such that $q_1(x) = q_2(x)(x-c_2)$. We then have $f(x) = q_2(x)(x-c_2)(x-c_1)$. Continuing this process leads to $f(x) \in M[x](x-c_{n+1})\dots(x-c_2)(x-c_1)$. This is impossible unless $f(x) = 0$. □

Recall that the right R module M is semicommutative if for any element $m \in M$, $\mathrm{ann}_R(m)$ is a two-sided ideal of R. In the case of rings, it is well-known that if $R[x]$ is semicommutative then R is McCoy. Notice that this is untrue in the case of modules. Indeed, if R is commutative then every R-module is semicommutative, however, Example 1.6 shows that there are modules over commutative rings that are not McCoy. It is known (cf.[3]) that, in general, the semicommutativity property does not lift from a ring

R to the polynomial ring $R[x]$, but it does when R is an algebra over the field $\mathbb{Q}$ of rational numbers (Theorem 8.4 in [1]). The following proposition is inspired by this theorem.

Proposition 2.17. *Let R be a $\mathbb{Q}$ algebra and M a semicommutative right R-module. Then $M[x]$ is a semicommutative $R[x]$-module.*

Proof. Assume $m(x) \in M[x]$ and $0 \neq g(x) \in R[x]$ are such that $m(x)g(x) = 0$. Therefore, by Lemma 2.16, we have $m(c)g(c) = 0$, for any $c \in \mathbb{Q}$. The semicommutativity of M implies that $m(c)Rg(c) = 0$. This implies that, for any $r \in R$ and any $c \in \mathbb{Q}$, we have $(m(x)rg(x))(c) = m(c)rg(c) = 0$. Thus, Lemma 2.16(d) gives $m(x)rg(x) = 0$, for every $r \in R$. This yields that $\mathrm{ann}_{R[x]}(m(x))$ is a two-sided ideal of $R[x]$, i.e., $M[x]$ is semicommutative $R[x]$ module. □

References

[1] Camillo V., Nielsen Pace P., McCoy rings and zero-divisors, J. Pure Appl. Algebra, 212 (2008), 599-615.

[2] Cui J., Chen J., On McCoy modules, Bull. Korean Math.Soc. 48 (2011), 23-33.

[3] Huh C., Lee Y., Smoktunowicz A., Armendariz rings and semicommutative rings, Comm. Algebra 30 (2002), 751-761.

[4] Hirano Y., On annihilator ideals of a polynomial ring over a noncommutative ring, J. Pure Appl. Algebra 168 (2002), 45-52.

[5] Lam T. Y., Lectures on modules and rings, Graduate texts in mathematics 189, Springer-Verlag New York, (1999).

[6] Marks G., Duo rings and Ore extensions, J. Algebra 280 (2004,) 463-471.

[7] McCoy N.H., Remarks on divisors of zero, Amer. Math. Monthly 49 (1942), 286-2295.

[8] Nielsen Pace P., Semi-commutativity and the McCoy condition, J. Algebra 298 (2006), 134-141.