Ore Extensions Satisfying a Polynomial Identity

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Abstract

Necessary and sufficient conditions for an Ore extension $S = R[x; \sigma, \delta]$ to be a PI ring are given in the case $\sigma$ is an injective endomorphism of a semiprime ring $R$ satisfying the ACC on annihilators. Also, for an arbitrary endomorphism $\tau$ of $R$, a characterization of Ore extensions $R[x; \tau]$ which are PI rings is given, provided the coefficient ring $R$ is noetherian.

Introduction

The aim of the paper is to give necessary and sufficient conditions for an Ore extension $R[x; \sigma, \delta]$ to satisfy a polynomial identity. One of the special feature is that we do not assume that $\sigma$ is an automorphism.

Clearly if $R[x; \sigma, \delta]$ satisfies a polynomial identity, then $R$ has to be a PI ring as well. Henceforth we will always assume that $R$ is a PI ring.

In [14], Pascaud and Valette showed that when $R$ is semiprime and $\sigma$ is an automorphism of $R$, then the Ore extension $R[x; \sigma]$ satisfies a polynomial identity if and only if $\sigma$ is of finite order on the center of $R$. We shall obtain similar results even when $\sigma$ is not an automorphism.

The PI property of general Ore extensions $R[x; \sigma, \delta]$ was studied by Cauchon in his thesis [4] in the case when the base ring $R$ is simple and $\sigma$ is

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an automorphism of $R$. In particular, Cauchon remarked that nonconstant central polynomials of $R[x;\sigma,\delta]$ appears naturally in this context.

On the other hand, the case of a noetherian base ring was considered by Damiano and Shapiro in [6]. They proved, in particular, that the Ore extension $R[x;\sigma]$ over a noetherian PI ring $R$ satisfies a polynomial identity if and only if the automorphism $\sigma$ is of finite order on the center of $R/B$, where $B$ denotes the prime radical of $R$. This result will be generalized to arbitrary endomorphism of $R$ in the last section.

Let us also mention that, mainly for Ore extensions coming from quantum groups, some authors are also interested in the relations between the PI degree of a ring $R$ and the PI degree of Ore extensions over $R$ (Cf. e.g. [3],[8]).

A somewhat related work is that of Bergen and Grzeszczuk (Cf. [1]), where a characterization of smash products $R\#U(L)$ satisfying polynomial identity is given, where $R$ is a semiprime algebra of characteristic 0 acted by a Lie color algebra $L$.

In Section 1, we recall some classical results and develop tools that will play an essential role in later sections.

In Section 2, we analyse the special case when $R$ is a prime ring. The main results being Theorem 2.7, Theorem 2.10 and Theorem 2.12. We prove, in particular, that for a prime PI ring $R$ and an injective endomorphism $\sigma$ of $R$, $R[x;\sigma,\delta]$ is PI if and only if there exists a nonconstant polynomial in the center of $R[x;\sigma,\delta]$ with regular leading coefficient if and only if the center $Z(R)$ of $R$ has finite uniform dimension over $Z(R)^{\sigma,\delta} := \{z \in Z(R) \mid \sigma(z) = z, \delta(z) = 0\}$.

In Section 3, we extend the results from the previous section to the case of a semiprime coefficient ring $R$ satisfying the ACC on annihilators. We first recall the results of Cauchon and Robson related to the action of an injective endomorphism $\sigma$ and a $\sigma$-derivation $\delta$ on a semisimple ring. Then we study the case of Ore extensions of endomorphism type ($\delta = 0$) showing in particular (Proposition 3.2) that the above mentioned result of Pascaud and Valette can be generalized to the case when $\sigma$ is an injective endomorphism of $R$. The general Ore extension $R[x;\sigma,\delta]$ is then analysed, first in the case of a semisimple base ring (Cf. Corollary 3.5). Then it is shown, in Theorem 3.7, that the necessary and sufficient conditions for $R[x;\sigma,\delta]$ to be a PI ring obtained in Section 2 are also valid under the assumption that $R$ is a semiprime PI ring with the ACC on annihilators.

The last section is devoted to the study of the PI property of the Ore extension $R[x;\sigma]$ where $\sigma$ is an arbitrary endomorphism of $R$ and the coefficient ring $R$ is noetherian. In particular, we give in Theorem 4.7, a necessary and sufficient conditionn for $R[x;\sigma]$ to satisfy a polynomial identity.
1 Preliminaries

Throughout the paper $R$ stands for an associative ring with unity and $Z(R)$ for its center. For any multiplicatively closed subset $S$ of $Z(R)$, $RS^{-1}$ denotes the localization of $R$ with respect to the set of all regular elements from $S$. In particular, $RZ(R)^{-1}$ is the localization of $R$ with respect to the Ore set of all central regular elements of $R$.

For a right $R$-module $M$, $\text{udim}_R(M)$ denotes its uniform dimension.

In the following proposition we gather classical results which are consequences of a generalized Posner’s Theorem and the theorem of Kaplansky (Cf. Rowen’s book [15]).

**Proposition 1.1.** For a semiprime PI ring $R$, the following conditions are equivalent:

1. $R$ is a left (right) Goldie ring;
2. $R$ satisfies the ACC condition on left (right) annihilators;
3. $R$ has finitely many minimal prime ideals;
4. $R$ possesses a semisimple classical left (right) quotient ring $Q(R)$ which is equal to the central localization $RZ(R)^{-1}$.

If $R$ satisfies one of the above equivalent conditions and $\bigoplus_{i=1}^{n} B_i$ is a decomposition of the semisimple ring $Q(R)$ into simple components, then $\dim_{Z(B_i)} B_i$ is finite, for every $1 \leq i \leq n$. In particular, $\text{udim}_{Z(Q(R))}(Q(R)) < \infty$.

We will use frequently the above proposition without referring to it.

The following observation is probably well-known but we could not find it in the literature:

**Proposition 1.2.** Let $B = \bigcup_{i=0}^{\infty} A_i$ be a filtered ring and $\text{gr}(B) = \bigoplus_{i=0}^{\infty} A_i/A_{i-1}$ denote its associated graded ring. If $B$ satisfies a polynomial identity then $\text{gr}(B)$ also satisfies a polynomial identity.

**Proof.** Let $G(B)$ denote the Rees ring of $B$, that is $G(B)$ is a subring of the polynomial ring $B[x]$ consisting of all polynomials $\sum a_i x^i \in B[x]$ such that $a_i \in A_i$, for any $i \geq 0$. Then $G(B)$ is a PI ring as a subring of the PI ring $B[x]$. It is known that the ring $G(B)/xG(B)$ is isomorphic to $\text{gr}(B)$ and the thesis follows. \qed

An Ore extension of a ring $R$ is denoted by $R[x;\sigma,\delta]$, where $\sigma$ is an endomorphism of $R$ and $\delta$ is a $\sigma$-derivation, i.e. $\delta : R \to R$ is an additive map
such that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$, for all $a, b \in R$. Recall that elements of $R[x; \sigma, \delta]$ are polynomials in $x$ with coefficients written on the left. Multiplication in $R[x; \sigma, \delta]$ is given by the multiplication in $R$ and the condition $xa = \sigma(a)x + \delta(a)$, for all $a \in R$.

The Ore extension $R[x; \sigma, \delta]$ has a natural filtration given by the degree and the associated graded ring is isomorphic to $R[x; \sigma]$. Therefore, by the above proposition, we have:

**Corollary 1.3.** If $R[x; \sigma, \delta]$ is a PI ring, then $R[x; \sigma]$ also satisfies a polynomial identity.

**Lemma 1.4.** Let $\sigma$ be an endomorphism of $R$. Then:

1. Suppose that $\sigma$ is injective and there exists $n \geq 1$ such that $\sigma^n|_{Z(R)}$ is an automorphism of $Z(R)$. Then $\sigma|_{Z(R)}$ is an automorphism of $Z(R)$.

2. Suppose that $R$ is a semiprime PI ring and $\sigma$ is injective when restricted to the center $Z(R)$, then $\sigma$ is injective on $R$.

3. Suppose that $R$ is simple finite dimensional over $Z(R)$ and $\sigma|_{Z(R)}$ is an automorphism of $Z(R)$ then $\sigma$ is an automorphism of $R$.

**Proof.** (1). Let $[a, b]$ denote the commutator of elements $a, b \in R$; i.e. $[a, b] = ab - ba$. Pick $n \geq 1$ such that $\sigma^n|_{Z(R)}$ is an automorphism of $Z(R)$.

Then, for any $r \in R$ and $z \in Z(R)$, we have

$$\sigma^{n-1}([\sigma(z), r]) = [\sigma^n(z), \sigma^{n-1}(r)] = 0.$$

Hence $[\sigma(z), r] \in \ker \sigma^{n-1} = 0$. This gives (1).

The statement (2) is clear, as any nonzero ideal of a semiprime PI ring intersects the center nontrivially.

(3). Since $R$ is simple, $Z(R)$ is a field. Let $R'$ be the left $Z(R)$-linear space $R$ with the action of $Z(R)$ twisted by $\sigma$, i.e. $z \cdot r = \sigma(z)r$, for $z \in Z(R)$, $r \in R'$. Now the thesis is a consequence of the fact that $\sigma : Z(R)R \rightarrow Z(R)R'$ is an injective homomorphism of $Z(R)$-linear spaces of the same finite dimension.

Braun and Hajarnavis showed in [2] that if $\sigma$ is an injective endomorphism of a prime noetherian PI ring $R$ such that $\sigma|_{Z(R)} = \text{id}_{Z(R)}$, then $\sigma$ is an automorphism of $R$. We will see in Example 2.2, that if $R$ is a prime PI ring (so it has the ACC on annihilators), then $\sigma$ does not have to be onto if $\sigma|_{Z(R)} = \text{id}_{Z(R)}$.

In the lemma below we quote some known results. The first statement is a special case of a result of Jategaonkar (Cf. Proposition 2.4 [7]). The second one is exactly Theorem 3.8 and Proposition 3.6 from [11], respectively.
Lemma 1.5. Let $R$ be a semiprime left Goldie ring. Suppose that the endomorphism $\sigma$ is injective. Then:

1. Let $C$ be the set of all regular elements of $R$. Then $\sigma(C) \subseteq C$;

2. $R[x;\sigma,\delta]$ is a semiprime left Goldie ring. When $R$ is prime, then $R[x;\sigma,\delta]$ is also a prime ring.

As we will see in the following result, the above lemma will enable us to reduce some of our considerations to the case when the coefficient ring $R$ is semisimple.

Proposition 1.6. Let $R$ be a semiprime left Goldie ring. Suppose that $\sigma$ is an injective endomorphism of $R$. Then $\sigma$ and $\delta$ can be uniquely extended to the classical ring of quotient $Q(R)$ of $R$, and the following conditions are equivalent:

1. $R[x;\sigma,\delta]$ is a PI ring.

2. $Q(R)[x;\sigma,\delta]$ is a PI ring.

Proof. By Lemma 1.5(1), $\sigma(C) \subseteq C$, where $C$ is the set of all regular elements of $R$. This means that $\sigma$ and $\delta$ can be uniquely extended to an injective endomorphism $\sigma$ and a $\sigma$-derivation $\delta$ of $Q(R) = C^{-1}R$ and we can consider the Ore extension $Q(R)[x;\sigma,\delta]$.

The implication $(2) \Rightarrow (1)$ is obvious.

$(1) \Rightarrow (2)$. Suppose $R[x;\sigma,\delta]$ satisfies a polynomial identity. Then, by Lemma 1.5(2), $R[x;\sigma,\delta]$ is a semiprime PI left Goldie ring. Thus, $Q(R)[x;\sigma,\delta]$ is a semisimple PI ring.

Clearly all elements from the set $C$ are invertible in $Q(R)[x;\sigma,\delta]$ and every element from $Q(R)[x;\sigma,\delta]$ can be presented in the form $c^{-1}p$ for some $c \in C$ and $p \in R[x;\sigma,\delta]$. This means that $C$ is a left Ore set in $R[x;\sigma,\delta]$ and $S^{-1}(R[x;\sigma,\delta]) = Q(R)[x;\sigma,\delta]$. This yields, in particular, that there exists a natural embedding of $Q(R)[x;\sigma,\delta]$ into the PI ring $Q(R[x;\sigma,\delta])$. This shows that $Q(R)[x;\sigma,\delta]$ satisfies a polynomial identity.

In the sequel we will need the following:

Lemma 1.7. Let $R$ be a ring and $S$ be a right Ore set of regular elements. If $M$ is a right $R$-module which is $S$-torsion free then

$$\text{udim}_R(M) = \text{udim}_{RS^{-1}}(M \otimes_R RS^{-1}).$$

In particular, if $R$ is a commutative integral domain and $M$ is a torsion free right $R$-module then $\text{udim}_R(M) = \text{dim}_K(M \otimes_R K)$, where $K$ is the field of fractions of $R$. 
The particular case comes from [10] Theorem 6.14 and the proof given there can be easily extended to get the first statement.

In the next lemma we will consider Ore extensions of the form $R[x; \phi]$, where $\phi$ denotes either an automorphism or a derivation of $R$. $R^\phi$ will denote a subring of constants, i.e. $R^\phi = \{ x \in R \mid \phi(x) = x \}$ when $\phi$ is an automorphism and $R^\phi = \{ x \in R \mid \phi(x) = 0 \}$, when $\phi$ is a derivation.

**Lemma 1.8.** Let $R$ be a ring and $Z = Z(R[x; \phi])$, where $\phi$ is either an automorphism or a derivation of $R$. Then:

1. $\text{udim}_{R^\phi}(R) \leq \text{udim}_Z(R[x; \phi])$.

2. Suppose that $R$ is a semiprime ring with the ACC on annihilators. If $R[x; \phi]$ is a PI ring, then:

   (a) $\text{udim}_{R^\phi}(R) < \infty$.

   (b) If every regular element of $Z(R)^\phi$ is regular in $Z(R)$, then:

   $Z(R)Z(R)^{-1} = Z(R)Z(R)^{\phi}^{-1}$ and $Q(R) = RZ(R)^{\phi}^{-1}$.

**Proof.** (1). For $f = \sum_{i=0}^{n} a_i x^i \in Z$, let $\phi(f)$ denote $\sum_{i=0}^{n} \phi(a_i)x^i$. Then, $0 = [x, f] = (\phi(f) - f)x$, if $\phi$ is an automorphism of $R$. If $\phi$ is a derivation of $R$, then $0 = [x, f] = \phi(f)$. The above yields that $Z \subseteq R^\phi[x]$.

For any element $r \in R^\phi$ we have $xr = rx$. Therefore, if $M$ is a right $R^\phi$-submodule of $R$, then $M[x]$ is a right $R^\phi[x]$-submodule of $R[x; \phi]$. In particular, $M[x]$ has also a structure of $Z$-module, as $Z \subseteq R^\phi[x]$. One can easily check that direct sums of $R^\phi$-submodules of $R$ lift to direct sums of $R^\phi[x]$-submodules of $R[x; \phi]$. Therefore $\text{udim}_{R^\phi}(R) \leq \text{udim}_{R^\phi[x]}(R[x; \phi]) \leq \text{udim}_Z(R[x; \phi])$.

(2)(a). Suppose that $R[x, \phi]$ is a PI ring and the coefficient ring $R$ is semiprime with the ACC condition on annihilators. Then $R$ also satisfies a polynomial identity, so $R$ is a semiprime PI left Goldie ring. Therefore, by Lemma 1.5, $R[x, \phi]$ is a semiprime Goldie PI ring with a semisimple quotient ring $Q(R[x, \phi]) = R[x, \phi]Z^{-1}$.

Making use of the statement (1), Lemma 1.7 and Proposition 1.1, we obtain:

$$\text{udim}_{R^\phi}(R) \leq \text{udim}_Z(R[x; \phi]) = \text{udim}_{Z^{-1}}(Q(R[x; \phi])) < \infty.$$  

This gives the statement (a).

(b). Suppose that every regular element of $Z(R)^\phi$ is regular in $Z(R)$. That is $B = Z(R)Z(R)^{\phi}^{-1}$ and $Z(R)^{\phi}(Z(R)^{\phi})^{-1}$ means localizations with respect the same Ore set of all regular elements of $Z(R)^\phi$.

Notice that, in order to prove the statement (b), it is enough to show that $Z(R)Z(R)^{-1} = B$, as $Q(R) = RZ(R)^{-1}$. Since the ACC on annihilators is a
hereditary condition on subrings and \(Z(R)\) is a reduced ring (i.e. \(Z(R)\) does not contain nontrivial nilpotent elements), \(Z(R)^\sigma\) is a commutative reduced ring with the ACC on annihilators. Therefore, its classical quotient ring \(A = Z(R)^\sigma(Z(R)^\sigma)^{-1} \subseteq B\) is a finite product of fields, say \(A = \bigoplus_{i=1}^n K_i\), where \(K_i = e_i A\) for suitable primitive orthogonal idempotents, \(1 \leq i \leq n\).

Recall that \(\sigma\) is either an automorphism or a derivation of \(R\). Hence \(Z(R)\) is stable by \(\phi\) and we can consider the Ore extension \(Z(R)[x; \phi]\) and, since \(R[x; \phi]\) is a PI ring, \(Z(R)[x; \phi]\) is also a PI ring. Now, we can apply the statement \((2)(a)\) to \(Z(R)\) and get \(\text{udim}_{Z(R)^\phi}(Z(R)) < \infty\). Consequently, by Lemma 1.7, we get \(\text{udim}_A(B) < \infty\). This implies that, for any \(1 \leq i \leq n\), \(e_i B\) is a finite dimensional algebra over the field \(K_i = e_i A\). Therefore every regular element in \(e_i B\) is invertible in \(e_i B\). This, in turn, implies that every regular element of \(B = \bigoplus_{i=1}^n e_i B\) is invertible in \(B = Z(R)(Z(R)^\phi)^{-1}\). This shows that \(B\) is equal to \(Z(R)Z(R)^{-1}\) and completes the proof of the lemma.

Remark 1.9. Let us observe that if \(R\) is a prime ring, then the assumption of the statement \((2b)\) from the above proposition is always satisfied.

## 2 Prime Coefficient Ring

In this section \(R\) will stand for a prime PI ring. We first continue to gather some information on the behaviour of \(\sigma\) on the center.

### Proposition 2.1

Let \(R\) be a prime PI ring and \(\sigma\) an endomorphism of \(R\) such that \(\sigma^n\) is an automorphism of the center \(Z(R)\) of \(R\). Then:

1. \(\sigma\) extends uniquely to an automorphism of the localization \(RZ(R)^{-1}\).

2. Suppose additionally that \(\sigma^n|_{Z(R)} = \text{id}_{Z(R)}\). Then there is \(0 \neq u \in Z(R)\) such that \(\sigma(u) = u\), \(\sigma\) is an automorphism of \(RS^{-1}\) and \(\sigma^n\) is an inner automorphism of \(RS^{-1}\), where \(S = \{u^k \mid k \geq 0\}\).

**Proof.** (1) By Lemma 1.4, \(\sigma\) is injective and \(\sigma(Z(R)) = Z(R)\). Thus \(\sigma\) has a unique extension to the localization \(RZ(R)^{-1}\) which, by Posner’s Theorem, is a simple, finite dimensional algebra over the center \(Z(R)Z(R)^{-1}\). Now the statement \((1)\) is a direct consequence of Lemma 1.4(3).

(2) By \((1)\), \(\sigma\) is an automorphism of \(RZ(R)^{-1}\) and the theorem of Skolem-Noether implies that \(\sigma^n\) is an inner automorphism of \(RZ(R)^{-1}\). Therefore, one can choose a regular element \(a \in R\) with the inverse \(bv^{-1} \in RZ(R)^{-1}\) such that \(\sigma^n(r) = arbv^{-1}\) for all \(r \in R\). Let \(u = v\sigma(v) \ldots \sigma^{n-1}(v)\). Then \(\sigma(u) = u\), the element \(bu^{-1}\) has the inverse \(a\sigma(v) \ldots \sigma^{n-1}(v)\) and also determines \(\sigma^n\). This yields the thesis. \(\square\)
In the sequel we will say that an automorphism $\sigma$ of a ring $R$ is of finite inner order if $\sigma^n$ is an inner automorphism of $R$, for some $n \geq 1$.

At this point we make a small digression not directly related to the main theme of the paper. It is known (Cf. [9]) that for any ring $R$ with a fixed injective endomorphism $\sigma$ there exists a universal over-ring $A$ of $R$, called a Jordan extension of $R$, such that $\sigma$ extends to an automorphism of $A$ and $A = \bigcup_{i=0}^{\infty} \sigma^{-i}(R)$. In this case we will write $R \subseteq_{\sigma} A$.

It is easy to check that if $\sigma$ becomes an inner automorphism of $A$, then $R = A$. Also, if $R$ is a prime PI ring, then $A$ is prime PI as well.

Recall that an automorphism of a prime PI ring $R$ is $X$-inner if and only if it becomes inner when extended to the classical quotient ring $Q(R) = RZ(R)^{-1}$.

Suppose that $R$ is a prime PI ring and $\sigma$ is an endomorphism of $R$ such that $\sigma^n|_{Z(R)} = \text{id}|_{Z(R)}$. Then, by Proposition 2.1, $R \subseteq_{\sigma} A \subseteq RS^{-1}$, where $S$ consists of powers of a single central element. Moreover $\sigma$ is an $X$-inner automorphism of $A$. The following example shows, that all inclusions $R \subseteq_{\sigma} A \subseteq RS^{-1}$ can be strict. This example will be used again later in the paper.

**Example 2.2.** Let $R = \begin{bmatrix} \mathbb{Z} + x\mathbb{Q}[x] & \mathbb{Z} + x\mathbb{Q}[x] \\ x\mathbb{Q}[x] & \mathbb{Z} + x\mathbb{Q}[x] \end{bmatrix}$ be a subring of $M_2(\mathbb{Q}[x])$ and $\sigma$ denote the inner automorphism of $M_2(\mathbb{Q}[x])$ adjoint to the element $u = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then $\sigma\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 2b \\ \frac{1}{2}c & d \end{bmatrix}$. This means that the restriction of $\sigma$ to $R$ is an injective endomorphism of $R$ which is not onto. One can check that $R \subseteq_{\sigma} A = \begin{bmatrix} \mathbb{Z} + x\mathbb{Q}[x] & \mathbb{Z}[\frac{1}{2}] + x\mathbb{Q}[x] \\ x\mathbb{Q}[x] & \mathbb{Z} + x\mathbb{Q}[x] \end{bmatrix}$ and $\sigma$ becomes an inner automorphism on the localization $RS^{-1}$, where $S$ denotes the multiplicatively closed set generated by 2.

**Definition 2.3.** Let $R$ be a ring, $\sigma$ an endomorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. We say that the center of the Ore extension $R[x; \sigma, \delta]$ is nontrivial if it contains a nonconstant polynomial.

**Lemma 2.4.** Let $\sigma$ be an injective endomorphism of a prime ring $R$. Then:

1. If $R[x; \sigma, \delta]$ is a PI ring, then the center $Z(R[x; \sigma, \delta])$ is nontrivial.

2. Let $f = ax^n + \ldots \in Z(R[x; \sigma, \delta])$ be a polynomial of degree $n \geq 1$. Then $a$ is a regular element of $R$, $\sigma^n|_{Z(R)} = \text{id}|_{Z(R)}$ and $\sigma|_{Z(R)}$ is an automorphism of $Z(R)$.

**Proof.** (1). Suppose that $R[x; \sigma, \delta]$ is a PI ring. Then Lemma 1.5 and Proposition 1.1 imply that $R[x; \sigma, \delta]$ is a prime PI ring. Thus every essential one-sided ideal contains a nonzero central element. Since $\sigma$ is injective, the element $x$ is
regular in $R[x; \sigma, \delta]$. Therefore $R[x; \sigma, \delta]x$ contains a nonzero central element $f = ax^n + a_{n-1}x^{n-1} + \ldots + a_1x$, where $a \neq 0$ and $n \geq 1$ and (1) follows.

(2) Let $f = ax^n + \ldots \in Z(R[x; \sigma, \delta])$ be such that $a \neq 0$ and $n \geq 1$. Making use of $xf = fx$ and $rf = fr$, for any $r \in R$, we obtain $\sigma(a) = a$ and $ra = a\sigma^n(r)$, for any $r \in R$.

We claim that $a$ is a regular element in $R$. Indeed, if $b \in R$ is such that $ba = 0$, then $bRa = ba\sigma^n(R) = 0$. Hence $b = 0$, as $R$ is a prime ring. Thus $a$ is left regular. If $ab = 0$, then $0 = \sigma^n(a)\sigma^n(b) = a\sigma^n(b) = ba$. Since $a$ is left regular, $b = 0$ follows.

Now, for any $z \in Z(R)$ we have $az = za = a\sigma^n(z)$. Thus $a(z - \sigma^n(z)) = 0$ and $\sigma^n(z) = z$, for any $z \in Z(R)$, follows. The last assertion of (2) is then a consequence of Lemma 1.4(1).

**Proposition 2.5.** Let $R$ be a prime PI ring and $\sigma$ an injective endomorphism of $R$. The following conditions are equivalent:

1. $R[x; \sigma]$ is a PI ring.

2. $\sigma|_{Z(R)}$ is an automorphism of $Z(R)$ of finite order.

3. There exists $0 \neq u \in Z(R)$ such that $\sigma(u) = u$ and $\sigma$ is an automorphism of finite inner order of the localization $RS^{-1}$, where $S$ denotes the set of all powers of $u$.

**Proof.** The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are given by Lemma 2.4 and Proposition 2.1, respectively.

(3) $\Rightarrow$ (1). Suppose that (3) holds and let $\sigma^n$, $n \geq 1$, be an inner automorphism of $RS^{-1}$. By the choice of $S$, the set $S$ is central in $R[x; \sigma]$ and $(R[x; \sigma])S^{-1} = RS^{-1}[x; \sigma]$. Since $\sigma^n$ is an inner automorphism of $RS^{-1}$, the subring $RS^{-1}[x^n] \subseteq RS^{-1}[x; \sigma]$ is isomorphic to the usual polynomial ring $RS^{-1}[y]$, so it satisfies a polynomial identity, as $RS^{-1}$ is a PI ring. Now, the fact that $RS^{-1}[x; \sigma]$ is a finitely generated free module over the PI subring $RS^{-1}[x^n]$ implies that $R[x; \sigma]$ satisfies a polynomial identity. \qed

In case $\sigma$ is an automorphism of $R$, the equivalence (1) $\Leftrightarrow$ (2) in the above proposition was also obtained by Pascaud and Valette (Cf. [14]) using another approach.

Before stating the next results we need to recall some definitions (Cf.[12]):

**Definition 2.6.** Let $R$ be a ring, $\sigma$ an endomorphism and $\delta$ a $\sigma$-derivation of $R$, respectively. We say that:
1. $\delta$ is quasi algebraic if there exists $n \geq 1$ and elements $b, a_1, \ldots, a_n \in R$, with $a_n \neq 0$, such that $\sum_{i=1}^n a_i \delta^i = \delta b, \sigma^n$ where $\delta b, \sigma^n$ denotes the inner $\sigma^n$-derivation adjoint to the element $b$, that is $\delta b, \sigma^n(r) = br - \sigma^n(r)b$, for any $r \in R$.

2. A polynomial $p \in R[x; \sigma, \delta]$ is right semi-invariant if for any element $a \in R$ there exists $b \in R$ such that $pa = bp$.

**Theorem 2.7.** Suppose $\sigma$ is an injective endomorphism of a prime PI ring $R$. Let $Q(R) = RZ(R)^{-1}$ denote the classical quotient ring of $R$. The following conditions are equivalent:

1. $R[x; \sigma, \delta]$ is a PI ring.
2. There exists a nonconstant central polynomial in $R[x; \sigma, \delta]$ with a regular leading coefficient.
3. The center of $R[x; \sigma, \delta]$ is nontrivial.
4. The center of $Q(R)[x; \sigma, \delta]$ is nontrivial.
5. There exists a nonconstant central polynomial in $Q(R)[x; \sigma, \delta]$ with invertible leading coefficient.
6. $Q(R)[x; \sigma, \delta]$ is a PI ring.
7. $\sigma$ is an automorphism of $Q(R)$ of finite inner order and $\delta$ is a quasi algebraic $\sigma$-derivation of $Q(R)$.
8. $\sigma$ is an automorphism of $Q(R)$ of finite inner order and $Q(R)[x; \sigma, \delta]$ contains a monic nonconstant semi-invariant polynomial.

**Proof.** The implication (1) $\Rightarrow$ (2) is given by Lemma 2.4. The implication (2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4). Suppose that the center of $R[x; \sigma, \delta]$ is nontrivial. Then, by Lemma 2.4(2), $\sigma(Z(R)) = Z(R)$. This means that we can extend uniquely $\sigma$ and $\delta$ to an endomorphism and a $\sigma$-derivation of $Q(R) = RZ(R)^{-1}$, respectively, and consider the over-ring $Q(R)[x; \sigma, \delta]$ of $R[x; \sigma, \delta]$. It is standard to check that $Z(R[x; \sigma, \delta]) \subseteq Z(Q(R)[x; \sigma, \delta])$. This shows that the center of $Q(R)[x; \sigma, \delta]$ is nontrivial.

The implication (4) $\Rightarrow$ (5) is given by Lemma 2.4(2) applied to the ring $Q(R)$ and the fact that every regular element of $Q(R)$ is invertible.

(5) $\Rightarrow$ (6). Let $f \in Q(R)[x; \sigma, \delta]$ be a central polynomial of degree $n \geq 1$ with an invertible leading coefficient. Then the subring $Q(R)[f] \subseteq Q(R)[x; \sigma, \delta]$
is isomorphic to the usual polynomial ring $Q(R)[y]$ in one indeterminate $y$, so the ring $Q(R)[f]$ satisfies a polynomial identity. Notice also that, due to Lemma 2.4(2) and Proposition 2.1, $\sigma$ is an automorphism of $Q(R)$. Now, the fact that $Q(R)[x; \sigma, \delta]$ is a free module over $Q(R)[f]$ with basis $1, x, \ldots, x^{n-1}$ implies that $Q(R)[x; \sigma, \delta]$ is a PI ring.

The implication (6) $\Rightarrow$ (1) is obvious.

The above shows that conditions (1) $\div$ (6) are equivalent and that the extension of $\sigma$ to $Q(R)$ is an automorphism of $Q(R)$.

Now the equivalence of statements (4), (7) and (8) is given by Theorem 3.6 in [12].

Remark 2.8. (1). Notice that due to Proposition 2.1(2), the assumption in (7) and (8) of the above theorem that $\sigma^n$ is an inner automorphism of $Q(R)$, for some $n \geq 1$, can be replaced by a condition that $\sigma|_{Z(R)}$ is an automorphism of $Z(R)$ of finite order.

(2). One can also replace the ring $Q(R)$ in the above theorem by a localization $RS^{-1}$ where $S$ denotes a multiplicatively closed set consisting of all powers of a suitably chosen central $\sigma$-invariant element.

Theorem 2.7 says that if $R[x; \sigma, \delta]$ is a PI ring, then the extension of $\sigma$ to $Q(R)$ has to be an automorphism of $Q(R)$. It is not a surprise as, by Corollary D[2], every endomorphism of a semiprime Noetherian PI ring $R$ which is identity on $Z(R)$ is always an automorphism of $R$. Nevertheless, when $R$ is a prime PI ring (so it satisfies the ACC on annihilators), then the injective endomorphism $\sigma$ of $R$ does not have to be onto when $R[x; \sigma, \delta]$ is a PI ring.

Example 2.9. Let $R$ and $\sigma$ be as in Example 2.2. Then $R$ is a prime ring, the injective endomorphism $\sigma$ is not onto and $R[y; \sigma]$ satisfies a polynomial identity, since $R[y; \sigma] \subseteq M_2(Q[x])[y; \sigma] \simeq M_2(Q[x])[z]$.

Theorem 2.10. Let $R$ be a prime ring, $\sigma$ an injective endomorphism and $\delta$ a $\sigma$-derivation of $R$. Then $R[x; \sigma, \delta]$ is a PI ring if and only if $R$ is PI, $\sigma|_{Z(R)}$ is an automorphism of $Z(R)$ of finite order and one of the following conditions holds:

1. If $\text{char} R = 0$,

$$Q(R)[x; \sigma, \delta] \simeq \begin{cases} Q(R)[x; \sigma] & \text{if } \sigma|_{Z(R)} \neq \text{id}_{Z(R)} \\ Q(R)[x] & \text{else} \end{cases}$$

2. If $\text{char} R = p \neq 0$,

$$Q(R)[x; \sigma, \delta] \simeq \begin{cases} Q(R)[x; \sigma] & \text{if } \sigma|_{Z(R)} \neq \text{id}_{Z(R)} \\ Q(R)[x; d] & \text{else} \end{cases}$$
where $d$ is a suitable derivation of $Q(R) = RZ(R)^{-1}$ such that:

(a) $d(R) \subseteq R$

(b) there exist elements $q_i = 1, \ldots, q_l \in Q(R)$ such that $\sum_{i=0}^{l} q_id^{\mu_{r}}$ is an inner derivation of $Q(R)$

Proof. Suppose $R[x; \sigma, \delta]$ is a PI ring. Then $R$ is PI and Theorem 2.7 and Remark 2.8 show that $Q(R)[x; \sigma, \delta]$ is a PI ring, $\sigma$ is an automorphism of $Q(R)$ and $\sigma|_{Z(R)}$ is an automorphism of the center $Z(R)$ of finite order.

Suppose that $\sigma|_{Z(R)} \neq \text{id}|_{Z(R)}$ and let $c \in Z(R)$ be such that $\sigma(c) \neq c$. Then it is well-known that $\delta$ is an inner $\sigma$-derivation of $Q(R)$ adjoint to the element $a = (c - \sigma(c))^{-1}\delta(c)$. Then $Q(R)[x; \sigma, \delta] = Q(R)[x-a; \sigma] \simeq Q(R)[x; \sigma]$.

Suppose $\sigma|_{Z(R)} = \text{id}|_{Z(R)}$. Then, by Proposition 2.1(2), $\sigma$ is an inner automorphism of $Q(R)$, say induced by an invertible element $c \in Q(R)$, i.e. $\sigma(r) = crc^{-1}$, for any $r \in Q(R)$. Since $Q(R)$ is a central localization of $R$ we can write $c^{-1} = uz^{-1}$ for some $u \in R$ and $z \in Z(R)$ and we have $\sigma(r) = u^{-1}ru$ for all $r \in Q$. Then $u\delta = d$ is a derivation of $Q(R)$ such $d(R) \subseteq R$ and $Q(R)[x; \sigma, \delta] = Q(R)[ux; \text{id}, u\delta] \simeq Q(R)[x; d]$.

Applying Theorem 2.7 to the PI ring $Q(R)[x; d]$ we know that this ring contains a monic nonconstant semi-invariant polynomial. Now the thesis is a consequence of Proposition 2.8 and Lemma 2.3 from [12] and the fact that $Q(R)[x; d]$ is isomorphic to $Q(R)[x]$, provided $d$ is an inner derivation.

Conversely, if $R$ is a prime PI ring and $\sigma^n$ is the identity on the center, then Theorem 2.7(8), Remark 2.8 and the hypothesis made on $Q(R)[x; \sigma, \delta]$ shows that $R[x, \sigma, \delta]$ is PI.

Remark 2.11. In case $\sigma$ is an automorphism of $R$, the statement (1) from the above Theorem is exactly the result of Jondrup [8], see also the book by Goodearl and Brown [3].

We have seen that the center $Z(R)$ of $R$ plays a crucial role in determining if an Ore extension satisfies a polynomial identity. This theme will be pursued further in the next result and in Section 3.

For any subring $A$ of $R$, $A^{\sigma, \delta}$ will denote the the subalgebra of $(\sigma, \delta)$-constants, i.e. $A^{\sigma, \delta} = \{a \in A \mid \sigma(a) = a \text{ and } \delta(a) = 0\}$. Notice that we do not require that $A$ is $\sigma$ or $\delta$ stable.

With the above notations we have:

Theorem 2.12. Suppose that $\sigma$ is an injective endomorphism of a prime PI ring $R$ with the center $Z$. Let $K$ denote the field of fractions of $Z^{\sigma, \delta}$. The following conditions are equivalent:

1. $R[x; \sigma, \delta]$ is a PI ring.
2. $\udim_{Z,\sigma,\delta}(Z) < \infty$

3. $\dim_K Q(R) < \infty$ and $Q(R) = R(Z_{\sigma,\delta})^{-1}$.

4. $\dim_K Q(R) < \infty$.

Proof. (1) $\Rightarrow$ (2) Suppose that $R[x;\sigma,\delta]$ is a PI ring. Then, by Theorem 2.10, $Q(R)[x;\sigma,\delta]$ is $Q(R)$-isomorphic to $Q(R)[x;\phi]$, where either $\phi = \sigma$ and $\phi(Z) = Z$ or $\phi$ is a derivation of $Q(R)$ such that $\phi(R) \subseteq R$. In particular, in any case we have $\phi(Z) \subseteq Z$ and we can consider $Z[x;\phi]$ as a subring of the PI ring $Q(R)[x,\phi]$. Now, Lemma 1.8(2) shows that $\udim_{Z,\phi}(Z)$ is finite.

Notice that:

\[ Z(Q(R))_{\sigma,\delta} = Q(R) \cap Z(Q(R)[x;\sigma,\delta]) = Q(R) \cap Z(Q(R)[x;\phi]) = Z(Q(R))^{\phi}. \]

Therefore, $Z_{\sigma,\delta} = Z \cap Z(Q(R))_{\sigma,\delta} = Z \cap Q(R)^{\phi} = Z^{\phi}$ and $\udim_{Z,\sigma,\delta}(Z) = \udim_{Z,\phi}(Z) < \infty$ follows.

(2) $\Rightarrow$ (3). Suppose that $\udim_{Z,\sigma,\delta}(Z) < \infty$. Then, making use of Lemma 1.7, we obtain: $\dim_K Z(Z_{\sigma,\delta})^{-1} = \udim_{Z,\sigma,\delta}(Z) < \infty$. This implies that the commutative domain $Z(Z_{\sigma,\delta})^{-1}$ is a field. Therefore $Z(Z_{\sigma,\delta})^{-1} = ZZ^{-1}$ and $Q(R) = RZ^{-1} = R(Z_{\sigma,\delta})^{-1}$. Since $Q(R)$ is finite dimensional over its center $ZZ^{-1} = Z(Z_{\sigma,\delta})^{-1}$ which is a finite dimensional field extension of $K$, we have $\dim_K Q(R) < \infty$.

The implication (3) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (1). Under the hypothesis (4), $Q(R)[x;\sigma,\delta]$ is a finitely generated module over the commutative polynomial ring $K[x]$. This yields that $Q(R)[x;\sigma,\delta]$ is a PI ring and so is $R[x;\sigma,\delta]$.

\[ \square \]

3 Semiprime Coefficient Ring

In this section we will investigate Ore extensions $R[x;\sigma,\delta]$ over a semiprime coefficient ring $R$ satisfying the ACC on annihilators. We will frequently use the following lemma, which is an obvious application of results of Cauchon and Robson from [5] (Cf. Lemma 1.1 to Lemma 1.4).

Lemma 3.1. Let $R = \bigoplus_{i=1}^{s} B_i$ be a decomposition of a semisimple ring $R$ into its simple components and let $\sigma,\delta$ be an injective endomorphism and a $\sigma$-derivation of $R$, respectively. Then:

1. There exists a permutation $\rho$ of the index set $\{1,\ldots,s\}$ such that $\sigma(B_i) \subseteq B_{\rho(i)}$ and $\delta(B_i) \subseteq B_i + B_{\rho(i)}$. 

2. If \( \{1, \ldots, s\} = \bigcup_{j=1}^{k} O_j \) is the decomposition of the index set into orbits under the action of the permutation \( \rho \) and \( A_j := \bigoplus_{i \in O_j} B_i \), then 
\[ R[x; \sigma, \delta] = \bigoplus_{j=1}^{k} A_j[x_j; \sigma|_{A_j}, \delta|_{A_j}] \].

3. Let \( j \in \{1, \ldots, k\} \) be such that \( |O_j| > 1 \), then \( \delta|_{A_j} \) is an inner \( \sigma|_{A_j} \)-derivation of \( A_j \). In particular, \( A_j[x_j; \sigma|_{A_j}, \delta|_{A_j}] \) is \( A_j \)-isomorphic to \( A_j[x_j; \sigma|_{A_j}] \).

4. There exists an \( m \geq 1 \) such that \( \sigma^m(B_i) \subseteq B_i \), for all \( i \in \{1, \ldots, s\} \).

Let us first consider the case of an Ore extension of endomorphism type.

**Proposition 3.2.** Let \( \sigma \) be an injective endomorphism of a semiprime PI ring \( R \) satisfying the ACC on annihilators. The following conditions are equivalent:

1. \( R[x; \sigma] \) is a PI ring.

2. The restriction \( \sigma|_{Z(R)} \) is an automorphism of \( Z(R) \) of finite order.

3. There exists \( n \geq 1 \) such that \( \sigma^n \) is identity on the center \( Z(R) \) of \( R \).

4. There exists a regular element \( u \in Z(R) \) such that \( \sigma(u) = u \) and \( \sigma \) is an automorphism of the localization \( RS^{-1} \) of finite inner order, where \( S \) denotes the set of all powers of \( u \).

5. \( \sigma \) is an automorphism of \( Q(R) \) of finite inner order.

**Proof.** By Lemma 1.5, \( \sigma \) can be extended to an injective endomorphism of \( Q(R) = RZ(R)^{-1} \). Let \( Q(R) = \bigoplus_{i=1}^{s} B_i \) be a decomposition of \( Q(R) \) into its simple components.

(1) \( \Rightarrow \) (5). Suppose \( R[x; \sigma] \) is a PI ring. Thus, by Proposition 1.6, \( Q(R)[x; \sigma] \) also satisfies a polynomial identity. By Lemma 3.1, there exists \( m \geq 1 \), such that all components \( B_i \) are \( \sigma^m \)-stable. Therefore, \( Q(R)[x; \sigma^m] \supseteq Q(R)[x; \sigma] \) is also a PI ring and
\[
Q(R)[x; \sigma^m] = \bigoplus_{i=1}^{s} B_i[x_i; \sigma^m] \simeq \bigoplus_{i=1}^{s} B_i[x_i; \sigma^m].
\]

This shows that, for any \( 1 \leq i \leq s \), \( B_i[x_i; \sigma^m] \) satisfies a polynomial identity and Theorem 2.7(8) applied to each simple component \( B_i \) yields that there exists \( k \geq 1 \) such that \( \sigma^n \), where \( n = mk \), is an inner automorphism of \( Q(R) \), i.e. (5) holds.

Since \( Q(R) = RZ(R)^{-1} \), the implication (5) \( \Rightarrow \) (4) can be proved using the same argument as in the proof of Proposition 2.1(2).
The implication (4) ⇒ (3) is clear and (3) ⇒ (2) is a direct consequence of Lemma 1.4(1).

(2) ⇒ (1). Suppose that (2) holds and let \( n \) denote the order of \( \sigma|_{Z(R)} \). Then \( \sigma^n(B_i) \subseteq B_i \), for \( 1 \leq i \leq s \). Then, by Lemma 1.4 and the theorem of Skolem-Noether, \( \sigma^n|_{B_i} \) is an inner automorphism of \( B_i \), for any \( i \). Hence \( \sigma^n \) is an inner automorphism of \( Q(R) \) and the subring \( Q(R)[x^n] \subseteq Q(R)[x;\sigma] \) is isomorphic to a polynomial ring \( Q(R)[y] \), so it satisfies a polynomial identity. This implies that \( Q(R)[x;\sigma] \) is a PI ring, as \( Q(R)[x;\sigma] \) is a finitely generated free module over its subring \( Q(R)[x^n] \) and (1) follows. \( \square \)

As an immediate application of the above proposition, Corollary 1.3 and Proposition 1.6 we easily get the following:

**Corollary 3.3.** Let \( R \) be a semiprime ring with ACC on annihilators and \( \sigma \) an injective endomorphism of \( R \). If \( R[x;\sigma,\delta] \) is a PI ring then \( Q(R)[x;\sigma,\delta] \) is a PI ring and \( \sigma \) is an automorphism of the semisimple ring \( Q(R) \) of finite inner order.

Thus while investigating the PI property of \( R[x;\sigma,\delta] \), the crucial case is when the ring \( R \) is semisimple. Moreover, by Lemma 3.1(2), one may restrict attention to the case when \( \sigma \) acts transitively on the set of all simple components \( B_i \) of \( R = \bigoplus_{i=1}^{s} B_i \).

**Proposition 3.4.** Let \( R = \bigoplus_{i=1}^{s} B_i \) be a decomposition of a semisimple PI ring into simple components and \( \sigma \) an injective endomorphism of \( R \). Suppose that \( \sigma \) acts transitively on the set of simple components. Then the following conditions are equivalent:

1. \( R[x;\sigma,\delta] \) is a PI ring.
2. \( \sigma \) is an automorphism of \( R \) of finite inner order and one of the following conditions holds:
   
   (a) \( R[x;\sigma,\delta] \) is isomorphic either to \( R[x;\sigma] \) or to \( R[x] \).
   
   (b) \( R \) is a simple ring of a nonzero characteristic \( p \) and \( R[x;\sigma,\delta] \simeq R[x;d] \) where \( d \) is a derivation of \( R \) such that there exist elements \( q_i = 1, \ldots, q_l \in R \) such that \( \sum_{i=0}^{l} q_id^i \) is an inner derivation of \( R \).

**Proof.** If \( R \) is simple, i.e. \( R = B_1 \), then the proposition is a direct consequence of Theorem 2.10.

If \( R \) is not simple then, by Lemma 3.1(3), \( R[x;\sigma,\delta] \simeq R[x;\sigma] \) and the proposition is a consequence of Proposition 3.2. \( \square \)
Corollary 3.5. Let $R$ be a semisimple PI ring with an injective endomorphism $\sigma$. The following conditions are equivalent:

1. $R[x; \sigma, \delta]$ is a PI ring.

2. The center of $R[x; \sigma, \delta]$ contains a nonconstant polynomial with invertible leading coefficient.

3. udim$_{Z(R)_{\sigma, \delta}}(Z(R))$ is finite.

Proof. (2) $\Rightarrow$ (1). Let $f \in R[x; \sigma, \delta]$ be a nonconstant polynomial from the center of $R[x; \sigma, \delta]$ with invertible leading coefficient. Then the subring $R[f] \subseteq R[x; \sigma, \delta]$ satisfies a polynomial identity and $R[x; \sigma, \delta]$ is a finitely generated left module over $R[f]$. This implies that $R[x; \sigma, \delta]$ is a PI ring.

(1) $\Rightarrow$ (2). Let $R = \bigoplus_{i=1}^s B_i = \bigoplus_{j=1}^k A_j$, where $A_j = \bigoplus_{i \in \mathcal{O}} B_i$, be a decomposition of $R$ described in Lemma 3.1. Then, by the same lemma, we have $R[x; \sigma, \delta] = \bigoplus_{j=1}^k A_j[x_j; \sigma_j, \delta_j]$, where $\sigma_j = \sigma|_{A_j}$ and $\delta_j = \delta|_{A_j}$. Hence, by Proposition 3.4, the PI ring $T_j = A_j[x_j; \sigma_j, \delta_j]$ is $A_j$-isomorphic to one of the following rings: $A_j[x]$, $A_j[x; \sigma_j]$, where $\sigma_j$ is an automorphism of $A_j$ of finite inner order or $A_j[x; d_j]$, where $d_j$ is a derivation of a simple ring $A_j$.

Suppose $T_j \simeq A_j[x; \sigma_j]$ and let $n \geq 1$ and $u \in A_j$ be an invertible element such that, for any $a \in T_j$, $\sigma^n(a) = u^{-1}au$. It is known that, eventually replacing $n$ by $n^2$ and $u$ by $u\sigma(u) \ldots \sigma^{n-1}(u)$, we may additionally assume that $\sigma(u) = u$. Then $f_j = ux^n$ is a polynomial from the center of $T_j$.

Suppose that $T_j \simeq A_j[x; d_j]$, where $A_j$ is a simple ring. Then, by Theorem 2.7, $T_j$ contains a nonconstant central polynomial $f_j$ with an invertible leading coefficient.

The above shows that, for any $1 \leq j \leq k$, $T_j = A_j[x_j; \sigma_j, \delta_j]$ contains a nonconstant central polynomial $f_j$ with invertible leading coefficient. Since a power of a central element is again central, we may choose the polynomials $f_j$’s in such a way that $\deg f_i = \deg f_j$, for all $1 \leq i, j \leq k$. Then the polynomial $f = \sum_{j=1}^k f_j$ belongs to the center of $R[x; \sigma, \delta]$, is nonconstant and the leading coefficient of $f$ is invertible.

(1) $\Leftrightarrow$ (3). We will continue to use the notation as in the proof of (1) $\Rightarrow$ (2). Notice that $Z(R) = \bigoplus_{j=1}^k Z(A_j)$ and $Z(R)^{\sigma, \delta} = \bigoplus_{j=1}^k Z(A_j)^{\sigma_j, \delta_j}$. Hence udim$_{Z(R)_{\sigma, \delta}}(Z(R)) = \sum_{j=1}^k$ udim$_{Z(A_j)_{\sigma_j, \delta_j}}(Z(A_j))$. This means that, without losing generality, we may assume that $R = A_1$, i.e. $\sigma$ acts transitively on the simple components of $R$.

If $R$ is simple, then the equivalence (1) $\Leftrightarrow$ (3) is given by Theorem 2.12.

Suppose $R$ is not simple. Then, by Lemma 3.1(3), $R[x; \sigma, \delta]$ is $R$-isomorphic to $R[x; \sigma]$. Since $Z(R)^{\sigma, \delta} = Z(R[x; \sigma, \delta]) \cap R$ and $Z(R)^\sigma = Z(R[x; \sigma]) \cap R$, we may replace $R[x; \sigma, \delta]$ by $R[x; \sigma]$. 


If \( R[x; \sigma] \) satisfies a polynomial identity then Lemma 1.8(2a), applied to \( Z(R)[x; \sigma]\), shows that \( \text{udim}_{Z(R)^\sigma}(Z) < \infty \).

Suppose now, that \( \text{udim}_{Z(R)^\sigma}(Z) \) is finite. Recall that \( R = \bigoplus_{i=1}^s B_i \) and, by Lemma 3.1(1), there is \( n \geq 1 \), such that \( \sigma^n(B_i) \subseteq B_i \), for all \( i \) and \( \text{udim} R[x; \sigma^n] = \bigoplus_{i=1}^s B_i[x_i; \tau_i] \), where \( \tau_i = \sigma^n|_{B_i} \). Since \( Z(R) \subseteq Z(R)^\sigma \), \( \text{udim}_{Z(R)^\sigma}(Z) < \infty \) and, consequently, \( \text{udim}_{Z(B_i)\tau_i}(Z(B_i)) < \infty \), for any \( 1 \leq i \leq s \). Therefore, Theorem 2.12 applied to Ore extensions \( B_i[x_i; \tau_i] \) shows that \( \text{udim} R[x; \sigma^n] = \bigoplus_{i=1}^s B_i[x_i; \tau_i] \) is a PI ring. Since \( R[x; \sigma] \) is a finitely generated module over its subring \( R[x^n] \) which is itself isomorphic to the PI ring \( R[x; \sigma^n] \), we conclude that \( R[x; \sigma] \) is a PI ring.

The following lemma is of crucial importance for the forthcoming theorem.

**Lemma 3.6.** Suppose that the ring \( R[x; \sigma, \delta] \) satisfies a polynomial identity, where \( R \) is a semiprime ring with the ACC on annihilators and \( \sigma \) is an injective endomorphism of \( R \). Let \( Z \) denote the center of \( R \) and \( Q = Q(R) \). Then:

1. \( Q = R(Z^{\sigma, \delta})^{-1} \)
2. If an element \( a \in Z^{\sigma, \delta} \) is regular in \( Z^{\sigma, \delta} \), then \( a \) is regular in \( R \).
3. \( Z(Q) = Z(Z^{\sigma, \delta})^{-1} \) and \( Z(Q)^{\sigma, \delta} = Z^{\sigma, \delta}(Z^{\sigma, \delta})^{-1} \)

**Proof.** By Proposition 1.6, we can extend \( \sigma \) and \( \delta \) to the classical semisimple quotient ring \( Q = Q(R) \) of \( R \). Let \( Q = \bigoplus_{i=1}^s B_i = \bigoplus_{j=1}^k A_j \), where \( A_j = \bigoplus_{i \in \mathcal{O}_j} B_i \), be decompositions of \( Q \) described in Lemma 3.1. Then, by the same lemma, we have \( Q[x; \sigma, \delta] = \bigoplus_{j=1}^k A_j[x_j; \sigma_j, \delta_j] \), where \( \sigma_j = \sigma|_{A_j} \) and \( \delta_j = \delta|_{A_j} \).

(1). Since \( Q = RZ^{-1} \), in order to show that \( Q = R(Z^{\sigma, \delta})^{-1} \), it is enough to prove that any regular element \( z \) from the center of \( R \) is invertible in \( R(Z^{\sigma, \delta})^{-1} \).

Let \( z \in Z \) be regular in \( Z \). By Corollary 3.3, \( \sigma|_Z \) is an automorphism of \( Z \) of finite order \( n \geq 1 \). Then, the element \( w = z\sigma(z) \cdots \sigma^{n-1}(z) \in Z \) is regular and \( \sigma(w) = w \). From this we easily deduce that \( z \) is invertible in \( R(Z^{\sigma})^{-1} \). This means that \( Q = R(Z^{\sigma})^{-1} \). Therefore, we may assume that our regular element \( z \) belongs to \( Z^{\sigma} \).

Recall that \( Q = \bigoplus_{j=1}^k A_j \), where \( \sigma(A_j) \subseteq A_j \) and \( \delta(A_j) \subseteq A_j \), for all \( j \). Thus, in particular, \( Z(Q)^{\sigma} = \bigoplus_{j=1}^k Z(A_j)^{\sigma_j} \) and we can present our element \( z \) in the form \( z = z_1 + \cdots + z_k \), where \( z_j \in Z(A_j)^{\sigma_j} \), for \( 1 \leq j \leq k \).

Notice that Lemma 3.1(3) (when \( A_j \) is not simple) and Theorem 2.10 (when \( A_j \) is a simple ring) imply that the restriction \( \delta|_{A_j} = \delta_j \) is always an inner \( \sigma_j \)-derivation of \( A_j \) but the case \( A_j \) is a simple ring of characteristic \( p_j \neq 0 \) and \( \sigma_j|_{A_j} = \text{id}|_{Z(A_j)} \). In the later case \( \delta_j(z_j^{p_j}) = p_j z_j^{p_j-1} \delta_j(z_j) = 0 \), as
Suppose that \( u \in \text{Z}(A_j) = \text{Z}(A_j)^{\sigma_j} \). When \( \delta_j \) is an inner \( \sigma_j \)-derivation, then \( \delta_j(z_j) = 0 \), as \( z_j \in \text{Z}(A_j)^{\sigma_j} \). In this case we set \( p_j = 1 \). Then \( \delta_j(z_j^{p_j}) = 0 \), for any \( 1 \leq j \leq k \). Let \( m = \prod_{j=1}^{k} p_j \). Then, putting together the above information, we get \( \delta(z^m) = \delta_1(z_1^m) + \ldots + \delta_k(z_k^m) = 0 \). This shows that the regular element \( z^m \) belongs to \( \text{Z}^{\sigma,\delta} \) and proves that \( z \) is invertible in \( R(\text{Z}^{\sigma,\delta})^{-1} \), i.e. \( Q \) is a localization of \( R \) with respect to regular elements from \( \text{Z}^{\sigma,\delta} \).

(2) Let \( 1 = e_1 + \ldots + e_k \) be a decomposition of \( 1 \in Q \) into a sum of central primitive idempotents (i.e. \( B_i = e_i R \), for all \( i \)).

Let us fix an element \( a \in \text{Z}^{\sigma,\delta} \) which is regular in \( \text{Z}^{\sigma,\delta} \) and \( b \in \text{Z} \) such that \( ab = 0 \).

Assume that \( b \neq 0 \). Then, there exists an index \( s \) such that \( ae_s = 0 \). Eventually changing the numeration, we may assume that \( s = 1 \) and \( A = \{ e_1, \ldots, e_1 \} \) is the orbit of \( e_1 \) under the action of \( \sigma \) on the set \( \{ e_i \mid 1 \leq i \leq k \} \). Since \( \sigma(a) = a \), we have \( ac = 0 \), where \( c = \sum_{e_i \in A} e_i \). Observe that the element \( c \) is \( (\sigma, \delta) \)-invariant.

By the statement (1), \( Q = R(\text{Z}^{\sigma,\delta})^{-1} \). Therefore, there exist an element \( z \in \text{Z}^{\sigma,\delta} \) regular in \( R \) and \( 0 \neq u \in R \) such that \( c = uz^{-1} \). Using the fact that the elements \( c \) and \( z \) are central \( (\sigma, \delta) \)-invariant and \( z \) is regular in \( R \), one can check that \( u \in \text{Z}^{\sigma,\delta} \). Since \( a \) is regular in \( \text{Z}^{\sigma,\delta} \) we obtain \( 0 \neq au = acz = 0 \). This contradiction shows that \( b = 0 \) and this completes the proof of (2).

The statement (3) is a direct consequence of the fact that \( \text{Z}(Q) = ZZ^{-1} \) and statements (1) and (2).

Now we are in position to prove the main theorem of this section:

**Theorem 3.7.** Suppose that \( R \) is a semiprime PI ring with the ACC on annihilators and \( \sigma \) is an injective endomorphism of \( R \). Let \( Z \) denote the center of \( R \) and \( Q = Q(R) \). The following conditions are equivalent:

1. \( R[x; \sigma, \delta] \) is a PI ring.
2. \( \text{udim}_{\text{Z}^{\sigma,\delta}}(Z) = \text{udim}_{\text{Z}(Q)^{\sigma,\delta}}(Z(Q)) \) is finite.
3. The center of \( R[x; \sigma, \delta] \) contains a nonconstant polynomial with a regular leading coefficient.

If one of the above equivalent conditions holds, then every regular element from \( \text{Z}^{\sigma,\delta} \) is regular in \( R \), \( Q = R(\text{Z}^{\sigma,\delta})^{-1} \) and \( \text{Z}(Q)^{\sigma,\delta} = \text{Z}^{\sigma,\delta}(\text{Z}^{\sigma,\delta})^{-1} \).

**Proof.** (1) \( \Rightarrow \) (2). Suppose (1) holds. Then, by Proposition 1.6, \( Q[x; \sigma, \delta] \) is a PI ring and Corollary 3.5 shows that \( \text{udim}_{\text{Z}(Q)^{\sigma,\delta}}(Z(Q)) < \infty \). The equality \( \text{udim}_{\text{Z}^{\sigma,\delta}}(Z) = \text{udim}_{\text{Z}(Q)^{\sigma,\delta}}(Z(Q)) \) is a direct consequence of Lemmas 3.6 and 1.7.
(2) ⇒ (3). Suppose (2) holds. Then, by Corollary 3.5 applied to the ring $Q$, the ring $Q[x; \sigma, \delta]$ satisfies a polynomial identity and there exists a nonconstant polynomial $f$ in the center of $Q[x; \sigma, \delta]$ with invertible leading coefficient. In particular, $R[x; \sigma, \delta]$ is also a PI ring and Lemma 3.6 shows that $Q = R(Z^{\sigma, \delta})^{-1}$. Hence, there exists an element $z \in Z^{\sigma, \delta}$ regular in $R$, such that $zf \in R[x; \sigma, \delta]$. Clearly $zf$ is central in $R[x; \sigma, \delta]$ and has a regular leading coefficient.

(3) ⇒ (1). Notice that $Z(R[x; \sigma]) \subseteq Z(Q[x; \sigma, \delta])$ and every regular element in $R$ is invertible in $Q$. Thus, the statement (3) together with Corollary 3.5 show that the ring $Q[x; \sigma, \delta]$ satisfies a polynomial identity. This gives the thesis.

The above shows that conditions (1) ÷ (3) are equivalent. The remaining statements from the theorem are direct consequences of Lemma 3.6.

Let us remark that the assumption in the above theorem, that the ring $R$ satisfies the ACC condition on annihilators, is essential.

**Example 3.8.** Let $R = \prod_{i=1}^{\infty} \mathbb{C}$, where $\mathbb{C}$ denotes the field of complex numbers, and let $\sigma$ be the automorphism of $R$ which is the complex conjugation on every component $\mathbb{C}$ of $R$. Then $R[x; \sigma]$ is a PI ring, $R^\sigma = \prod_{i=1}^{\infty} \mathbb{R}$ and $\text{.udim}_{R^\sigma}(R)$ is infinite.

4 \hspace{1cm} \textbf{Noetherian Coefficient Ring}

Throughout this section $\sigma$ stands for an arbitrary, not necessarily injective, endomorphism of a ring $R$. $\mathcal{B}(R)$ denotes the prime radical of $R$.

The following result is due to Mushrub (Cf.[13]):

**Lemma 4.1.** Suppose that $R$ is a noetherian ring and $\ker \sigma \subseteq \mathcal{B}(R)$, then $\sigma(\mathcal{B}(R)) \subseteq \mathcal{B}(R)$.

**Proposition 4.2.** Suppose $R$ is noetherian and $\ker \sigma \subseteq \mathcal{B}(R) = B$. Then $\sigma^{-1}(B) = B$ and $\sigma$ induces an injective endomorphism $\bar{\sigma}$ of $R/B$.

**Proof.** By Lemma 4.1, $\sigma(B) \subseteq B$. Thus $\sigma$ induces an endomorphism $\bar{\sigma}$ of $R/B$.

By assumption, $\ker \sigma$ is a nilpotent ideal of $R$. Then, it is easy to check that also $\sigma^{-1}(I)$ is a nilpotent ideal, for any nilpotent ideal $I$ of $R$. Thus, in particular, $\sigma^{-1}(B) \subseteq B$. Then $B \subseteq \sigma^{-1}(\sigma(B)) \subseteq \sigma^{-1}(B) \subseteq B$ and $\sigma^{-1}(B) = B$ follows. This, in turn, implies that $\bar{\sigma}$ is injective. \hfill \Box

**Proposition 4.3.** Suppose that $\sigma$ is an endomorphism of a noetherian PI ring $R$ such that $\ker \sigma \subseteq B = \mathcal{B}(R)$. Then $R[x; \sigma]$ is a PI ring if and only if the
restriction of \( \bar{\sigma} \) to the center \( Z \) of \( R/B \) is an automorphism of \( Z \) of finite order.

Proof. By Proposition 4.2, \( \sigma \) induces an injective endomorphism \( \bar{\sigma} \) of \( R/B \). Since \( \sigma(B) \subseteq B \) and \( B \) is nilpotent, \( B[x;\sigma] \) is a nilpotent ideal of \( R[x;\sigma] \). Therefore, as \( (R/B)[x;\bar{\sigma}] \) is isomorphic to \( R[x;\sigma]/B[x;\sigma] \), \( R[x;\sigma] \) is a PI ring iff \( (R/B)[x;\bar{\sigma}] \) is PI. Now the thesis is a direct consequence of Proposition 3.2.

If \( \sigma \) is an automorphism of \( R \), then \( \sigma(B(R)) = B(R) \) and the above proposition is exactly Corollary 10[6], in this case.

Notice that, by Lemma 1.5(2), the ring \( (R/B)[x;\bar{\sigma}] \) from the proof of the above theorem is semiprime. Therefore we have:

Remark 4.4. Suppose that \( \sigma \) is an endomorphism of a noetherian PI ring \( R \) such that \( \ker \sigma \subseteq B(R) \). Then \( B(R[x;\sigma]) = B(R)[x;\sigma] \).

It is known that if the ring \( R \) is not noetherian, then \( B(R[x;\sigma]) \) is not necessarily an extension of an ideal of the coefficient ring \( R \) even when \( \sigma \) is an automorphism of \( R \). However, it was observed by Pascaud and Valette in [14] that \( B(R[x;\sigma]) = B(R)[x;\sigma] \), provided \( \sigma \) is an automorphism of \( R \) and the Ore extension \( R[x;\sigma] \) satisfies a polynomial identity.

The following lemma is crucial in considering the case when \( \ker \sigma \) is not included in the radical \( B(R) \) of \( R \).

Lemma 4.5. Let \( I \) be an ideal of \( R \) such that \( \sigma(I) \subseteq I \). Suppose that \( R \) and \( (R/I)[x;\sigma] \) are PI rings. Then \( T = R[x;\sigma]/(I[x;\sigma]x) \) is also a PI ring.

Proof. Since \( \sigma(I) \subseteq I \), \( \sigma \) induces an endomorphism, also denoted by \( \sigma \), on the factor ring \( R/I \).

Notice that \( R \cap I[x;\sigma]x = 0 \), so we can consider \( R \) as a subring of \( T \). Then \( I \) is also an ideal of \( T \) and \( T/I \) is isomorphic to \( (R/I)[x;\sigma] \).

Let \( (x) \) denote the ideal of \( T \) generated by the natural image of \( x \) in \( T \). Then \( T/(x) \simeq R \). Therefore, as \( I \cap (x) = 0 \), there exists an embedding of \( T \) into \( R \oplus (R/I)[x;\sigma] \) which, by assumption, is a PI ring.

Notice that an endomorphism \( \sigma \) of \( R \) induces an endomorphism of the factor ring \( R/\ker \sigma \). This endomorphism will also be denoted by \( \sigma \). The above lemma gives us immediately:

Corollary 4.6. Suppose that \( R \) is a PI ring. The following conditions are equivalent:

1. \( R[x;\sigma] \) is a PI ring;
2. for any $n \geq 0$, $(R/\ker \sigma^n)[x; \sigma]$ is a PI ring;

3. there exists $n \geq 0$ such that $(R/\ker \sigma^n)[x; \sigma]$ is a PI ring.

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are clear.

$(3) \Rightarrow (1)$. Suppose $(R/\ker \sigma^n)[x; \sigma]$ is PI for some $n \geq 0$. We may assume $n \geq 1$. Let $I = \ker \sigma^n$. Then, by Lemma 4.5, the ring $T = R[x; \sigma]/(I[x; \sigma]x)$ is PI. Since $I = \ker \sigma^n$, $x^nI = 0$. Therefore $(I[x; \sigma]x)^{n+1} = 0$. This implies that $R[x; \sigma]$ satisfies a polynomial identity, \hfill \Box

When $\sigma$ is an endomorphism, then $\{\ker \sigma^k\}_{k \geq 1}$ is an increasing sequence of ideals of $R$. Thus, when $R$ is noetherian, there is $n \geq 1$ such that $\ker \sigma^n = \ker \sigma^m$ for any $m \geq n$ and $\sigma$ induces an injective endomorphism, also denoted by $\sigma$, of the factor ring $R/\ker \sigma^n$.

**Theorem 4.7.** Suppose $R$ is a noetherian PI ring. Let $n \geq 1$ be such that $\ker \sigma^n = \ker \sigma^{n+1}$ and $R' = R/I$, where $I = \ker \sigma^n$. The following conditions are equivalent:

1. $R[x; \sigma]$ is a PI ring;

2. $R'[x; \sigma]$ is a PI ring;

3. $\bar{\sigma}$ is an automorphism of finite order on the center of $R'/B(R')$.

Proof. As we have seen in comments before the theorem, $\sigma$ induces an injective endomorphism of $R'$. Thus, Proposition 4.3, shows that the statements (2) and (3) are equivalent.

The equivalence $(1) \Leftrightarrow (2)$ is given by Corollary 4.6. \hfill \Box

We have seen in Proposition 4.2 that if $\ker \sigma \subseteq B(R)$ in a noetherian ring $R$, then $\sigma^{-1}(B(R)) = B(R)$. Hence $\ker \sigma^n \subseteq B(R)$, for any $n \geq 1$. This means that $R'/B(R') = R/B(R)$ in the theorem above, i.e. Proposition 4.3 and Theorem 4.7 coincide when $\ker \sigma \subseteq B(R)$.

Notice that, as the following standard example shows, the ring $R[x; \sigma]$ is not necessary noetherian even when it is PI and $R$ is noetherian.

**Example 4.8.** Let $K$ be a field and $\sigma$ a $K$-linear endomorphism of the polynomial ring $R = K[y_1, \ldots, y_n]$ given by $\sigma(y_1) = 0$ and $\sigma(y_k) = y_{k-1}$ for $k > 1$. It is easy to see that $R[x; \sigma]$ is neither left nor right noetherian. By Theorem 4.7, $R[x; \sigma]$ satisfies a polynomial identity. In fact, since $\ker \sigma^n = (y_1, \ldots, y_n) = I$, $(R/I)[x; \sigma] = K[x]$ and the proof of Theorem 4.7 shows that $R[x; \sigma]$ satisfies the identity $[x_1, x_2]^{n+1} = 0$. 
For any ring $R$, $\sigma$ induces an injective endomorphism of $R/I$ where $I = \sum_{k=1}^{\infty} \text{ker} \sigma^k$. Thus one could hope that an analog of Theorem 4.7 could hold at least in the case $I$ is a prime ideal of $R$ (then, by Proposition 2.5, statements (2) and (3) are equivalent and (1) always implies (2)). However this is not the case as the following example shows.

**Example 4.9.** Let $K$ be a field and $\sigma$ a $K$-linear endomorphism of the polynomial ring $R = K[y_i \mid i \geq 1]$ given by $\sigma(y_1) = 0$ and $\sigma(y_k) = y_{k-1}$ for $k > 1$. Then $I = \sum_{k=1}^{\infty} \text{ker} \sigma^k = (y_1, y_2, \ldots)$ and $R[x; \sigma]/I[x; \sigma] \simeq R/I[x; \sigma] \simeq K[x]$ is a PI ring.

We claim that $R[x; \sigma]$ does not satisfy a polynomial identity by showing that, for any $m \geq 2$ and $k \geq 1$, $R[x; \sigma]$ does not satisfy the identity $S_m(x_1, \ldots, x_m)^k$, where $S_m(x_1, \ldots, x_m)$ denotes the standard identity in indeterminates $x_1, \ldots, x_m$. To this end, let us fix $n = n(m,k)$ such that $n > mk$. Then $S = S_m(y_{nt}, \ldots, y_{n+m-1}t) = (y_n^m + f)t^m$ for some suitable $f \in R[x; \sigma]$ such that $\deg_{y_n}(f) < m$. The choice of $n$ implies that $y_n^m + f \notin \text{ker} \sigma^{(k-1)m}$. Hence $S^k \neq 0$ follows, as $R$ is a domain.

**References**


