DECOMPOSITIONS INTO PRODUCTS OF IDEMPOTENTS*

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Abstract. The purpose of this note is two-fold: (1) to study when quasi-Euclidean rings, regular rings, regular separative rings have the property (*) that each right (left) singular element is a product of idempotents (2) to consider the question: "when a singular nonnegative square matrix is a product of nonnegative idempotent matrices?" The importance of the class of quasi-Euclidean rings in connection with the property (*) is given in [2] where it is shown that every singular matrix over a right and left quasi-Euclidean domain is a product of idempotents, generalizing the results of Erdos [6] for matrices over fields and that of Laffey [14] for matrices over commutative Euclidean domains. We have shown in this paper that quasi-Euclidean rings appear among many interesting classes of rings and hence they are in abundance. We analyze the properties of triangular matrix rings and upper triangular matrices with respect to the decomposition into product of idempotents and show, in particular, that nonnegative nilpotent matrices are products of nonnegative idempotent matrices. We study as to when each singular matrix is a product of idempotents in special classes of rings. Regarding the second question for nonnegative matrices, bounds are obtained for a rank one nonnegative matrix to be a product of two idempotent matrices. It is shown that every nonnegative matrix of rank one is a product of three idempotent matrices. For matrices of higher orders, we show that some power of a group monotone matrix is a product of idempotent matrices.

Key words. Singular matrices, product of idempotents, Quasi-Euclidean rings, Nonnegative singular matrices.

AMS subject classifications. 15A23, 15A33, 15A48.

1. Definitions and Terminologies.

An element in a ring R is called von Neumann regular, known as regular (for convenience) if there exists $x \in R$ such that $axa = a$. If each element of a ring R is regular then R is called a regular ring. An element in a ring R is called unit regular if there exists a unit $u \in R$ such that $aua = a$. If each element of a ring R is unit regular then R is called a unit regular ring. Note that an idempotent is always a unit regular element. A module M is called an exchange module if for any module decomposition $K = M' \oplus N = \oplus_{i \in I} X_i$ with $M \cong M'$ there exist submodules Y_i of X_i such that $K = M' \oplus (\oplus_{i \in I} Y_i)$. Ring theoretic characterization of an exchange ring is that a ring R is an exchange ring if for each element

*Received by the editors on Month x, 200x. Accepted for publication on Month y, 200y Handling Editor: .

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$a \in R$, there exists an idempotent $e \in R$, such that $e \in Ra$ and $1 - e \in R(1 - a)$. Clearly, every regular ring is an exchange ring. A ring R is called a separative ring, if for all finitely generated projective $R-$modules A and B, $2A \cong A \oplus B \cong 2B$ implies $A \cong B$. A ring R is called strongly separative if if for all finitely generated projective $R-$modules A and B, $2A \cong A \oplus B$ implies $A \cong B$, A ring R is called of stable range 1 if $aR + bR = R$ implies that there exists an element x in R such that $a + bx$ is a unit. An ordered pair (a, b) is called a quasi-Euclidean pair, if there exits $q_i \in R$ and $r_i \in R$, such that $a = bq_1 + r_1, b = r_1q_2 + r_2, \ldots, r_{i-1} = r_iq_{i+1} + r_{i+1}, \ldots, r_{n+1} = 0$, $1 < i \leq n$. A ring R is called quasi-Euclidean if each ordered pair $(a, b) \in R^2$ is a quasi-Euclidean pair. A ring is directly finite if $ab = 1, a \in R, b \in R$ implies $ba = 1$. A ring is said to have IP property if each singular element in the sense of non-zero divisor is a product of idempotents. A ring is said to have $(IP)_n$ if each $n \times n$ singular matrix over R is a product of idempotents. A ring R is called right Hermite if for each $a, b \in R$, there exists invertible matrix P such that $(a, b)P =(r, 0)$, for some $r \in R$. A ring R is called right Bézout if the sum of any two principal right ideals is a principal right ideal. An $n \times n$ matrix is called elementary if it is of the form $I_n + ce_{ij}$ for $i \neq j$. A ring R is a GE_2 ring if each invertible 2×2 matrix is a product of elementary matrices and diagonal matrices with invertible entries. It was shown in Alahmadi et al. [2] that a ring is right quasi-Euclidean if and only if it is right Hermite and GE_2.

A ring is called projective-free if each finitely generated module is free, of unique rank. A local ring is projective-free. By a domain we mean a ring (not necessarily commutative) with no nonzero zero divisors. A ring R is Dedekind finite if for any $a, b \in R$, $ab = 1$ implies that $ba = 1$. A matrix with real entries is called nonnegative (positive) if each entry is nonnegative (positive). For an element a in a ring R, $l(a)$ and $r(a)$ will stand for the left and right annihilators of a respectively. For a matrix $A \in M_n(R)$ we will also use, when convenient, $l(A)$ and $r(A)$ for meaning vectors annihilating A on the left or on the right respectively.

2. Triangular matrix rings and IP.

Let us start this section with a brief analysis of the different IP concepts.

DEFINITION 2.1. *Let R be a ring with unity. The ring R is said to be*

(a) *right (left) IP if for any $a \in R$, $r(a) = \{x \in R \mid ax = 0\} \neq (0)$ $(l(a) = \{x \in R \mid xa = 0\} \neq (0))$ implies that a is a product of idempotents. An IP ring is a left and right IP ring.*

(b) *a weak IP (a.k.a. IC rings) if any element $a \in R$ such that $r(a) \neq (0) \neq l(a)$ is a product of idempotents.*

Obviously an element a that is a product of idempotents different from the unity has the property that both its left and right annihilators are nonzero. We give some properties linking the concepts defined above.

LEMMA 2.2.

(a) *If R is right or left IP then it is weak IP.*
(b) *Any integral domain or boolean ring is right left and weak IP.*
(c) *Any unit regular ring which is weak IP is two-sided (i.e. left and right) IP.*
(d) *Any two-sided IP ring is Dedekind finite.*

Proof. Statements (a), (b) are obvious. Statement (c) is a direct consequence of the fact that in a unit regular ring an element is a right zero divisor if and only if it is a left zero divisor. Since a product of idempotents different from 1 is obviously a left and right zero divisor, an element of a two-sided IP ring is a right zero divisor if and only if it is a left zero divisor. Statement (d) is then clear. □

Idempotent matrices over projective-free rings have special properties. We exploit this fact in the following proposition. Let us recall that a ring R has the IBN property if every free R-module has unique rank (the notion is left right symmetric) and it is directly finite if $ab = 1, a \in R, b \in R$ implies $ba = 1$.

PROPOSITION 2.3. *Suppose that the $n \times n$ matrix ring $M_n(S)$ over a projective-free ring S has the IP property. Then S is a domain.*

Proof. The hypothesis means that every singular $n \times n$ matrix can be written as a product of idempotent matrices. Since the only idempotent elements of S are 0 and 1, we may assume that $n > 1$.

Let $0 \neq a \in S$ such that $r(a) \neq (0)$. Set $A = \begin{pmatrix} a & 0 \\ 0 & I_{n-1} \end{pmatrix}$. By hypothesis $A = E_1 E_2 \dots E_m$, where E_i are idempotents. By Cohn's result (cf. Proposition 0.4.7, [5]), each idempotent E_i is of the form $P_i Q_i$ where P_i and Q_i are matrices over S of respective sizes $n \times r_i$ and $r_i \times n$ (for some $r_i < n$) such that $Q_i P_i = I_{r_i}$. We thus get $A = P_1 \Gamma Q_m$ where $\Gamma = Q_1 \cdots P_m \in M_{r_1 \times r_m}(S)$. Write, $P_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ and $Q_m = (\delta \;\; \epsilon)$, where α is a row of length r_1, δ is a column of size r_m and β, ϵ are block matrices of appropriate sizes.

Then by comparing entries in the representation of A we obtain $a = \alpha \Gamma \delta$, $\alpha \Gamma \epsilon = (0, \dots, 0)$, $\beta \Gamma \delta = (0, \dots, 0)^T$, $I_{n-1} = \beta \Gamma \epsilon$.

Set $P_m = \begin{pmatrix} x \\ y \end{pmatrix}$ where x is a row vector and y is a block matrix. Since $Q_m P_m = I_{r_m}$, $\delta x + \epsilon y = I_{r_m}$

Consider $\begin{pmatrix} ax \\ y \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = AP_m = P_1\Gamma Q_m P_m = P_1\Gamma = \begin{pmatrix} \alpha\Gamma \\ \beta\Gamma \end{pmatrix}$. Thus $ax = \alpha\Gamma$, $y = \beta\Gamma$. Since $I_{n-1} = \beta\Gamma\epsilon$, we obtain $y\ \epsilon = I_{n-1}$. Since projective-free rings have IBN and every matrix ring over a projective-free ring is directly finite (cf. [5] where this property is named weakly finite") we obtain that $r_m = n-1$ and $\epsilon y = I_{n-1}$. Furthermore, $\delta x + \epsilon y = I_{r_m} = I_{n-1}$ implies $\delta x = 0$ and then $a = \alpha\Gamma\delta = ax\delta = \alpha\Gamma\delta x\delta = 0$, as desired. □

We now turn to triangular matrix rings. The upper triangular matrix ring constructed over R will be denoted by $T_2(R)$. Let us first start with the following easy lemma. We state it for $T_2(R)$ but analogous results hold for $T_n(R)$.

Proposition 2.4. *Let R be any ring.*

(a) *If $T_2(R)$ is a right quasi-Euclidean ring, then R is also right quasi-Euclidean.*
(b) *If $T_2(R)$ is a right IP ring, then R is also right IP.*

Proof. (a) Let a, b be elements in a ring R. Consider the matrices

$$A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}.$$

By hypothesis, there exist 2×2 matrices $Q_1, \ldots, Q_{n+1} \in T_2(R)$ and $R_1, \ldots, R_n \in T_2(R)$ such that for any $0 \le i \le n-1$, we have $R_{i-1} = R_i Q_{i+1} + R_{i+1}$, and $R_{n-1} = R_n Q_{n+1}$ where $R_{-1} = A, R_0 = B$. It is easy to check that for any $-1 \le i \le n$ we have $(R_{i-1})_{11} = (R_i)_{11}(Q_{i+1})_{11} + (R_{i+1})_{11}$. Since $(R_{-1})_{11} = a$ and $(R_0)_{11} = b$, we conclude that the pair (a, b) is Euclidean.

(b) The proof is similar to that of the statement (1) above, since $r(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) = 0$ implies that $r(a) = 0$ and the $(1,1)$ entry of an idempotent upper triangular matrix must be an idempotent element in R. □

Remarks 2.5. Unfortunately the same proof does not work in the case of matrix rings. For instance we don't know if $M_2(R)$ being right quasi-Euclidean implies that R is such. The fact that there exist right principal matrix rings over non right principal rings (cf. [18]), leads to suspicion that the above proposition is not true if we consider $M_2(R)$ instead of $T_2(R)$. On the other hand it is well-known that if R is is right quasi-Euclidean then $M_n(R)$ is also right quasi-Euclidean (cf. [2]). The analogue for $T_n(R)$ is untrue. Indeed, $\mathbb{Z}$ is right quasi-Euclidean and since right quasi-Euclidean rings are right Bézout, the non principal noetherian ring $T_2(\mathbb{Z})$ is not quasi-Euclidean. Similarly the (right) IP property doesn't lift from R to $T_2(R)$ as is easily seen by considering $T_2(k[x, y])$ where k is a field.

As mentioned above, we do not know whether the IP property goes down from the matrix ring to the ground ring. However, if the ground ring is commutative then this is true.

Proposition 2.6. *If R is a commutative ring and if the matrix ring $M_n(R)$ has the property that each singular matrix is a product of idempotents, then the ground ring R has the same property.*

Proof. Let $a \in R$ be a singular element. Consider the matrix $A = \begin{pmatrix} a & 0 \\ 0 & I_{n-1} \end{pmatrix}$. Since this matrix is singular we can write $A = E_1 \cdots E_l$ where the matrices $E_1, \ldots, E_l$ are idempotent. Taking determinants on both sides we obtain $a = \det(E_1) \cdots \det(E_l)$. This yields the desired result since, for $i = 1, \ldots, l$, $\det(E_i)$ are idempotent elements. $\square$

Let us now turn to concrete nilpotent and strictly upper triangular matrices. The next proposition is valid *over any ring.*

Proposition 2.7. *Let R be a ring and $n \in \mathbb{N}$. Any strictly upper triangular matrix (i.e. having zeros on the main diagonal) $A \in T_n(R)$ is a product of n idempotent matrices.*

Proof. We proved this statement by induction on the size n of the strictly upper triangular matrix A. If $n = 1$, $A = 0$ and there is nothing to prove. If $n > 1$, we can write $A = \begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix}$, where B is an $(n-1) \times (n-1)$ upper triangular matrix $C \in M_{(n-1)\times 1}(R)$ and the bottom row consists of zeros. We have

$$A = \begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{n-1} & C \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}.$$

The first matrix on the right side is an idempotent matrix. The induction hypothesis shows that the matrix B is a product of $n-1$ idempotent matrices, say $B = E_1 \ldots E_{n-1}$. We thus get

$$A = \begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{n-1} & C \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 & 0 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} E_{n-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

This yields the proof of the proposition. $\square$

Remarks 2.8.

(a) It might be of interest to remark that in a regular ring every nilpotent element is a product of idempotents. This is a result of Hannah and O'Meara (cf. Lemma 1.2 in [9]).

(b) A ring R is an n-fir if all right (equivalently left) ideals with at most n generators are free and have unique rank. If R is an n-fir, then any nilpotent matrix $C \in M_n(R)$ is similar to a upper triangular matrix (cf [5], Proposition 2.3.16) and hence by the above result 2.7 such a matrix is a product of idempotent matrices.

As a prelude for the last section of this paper let us mention the following result about nonnegative matrices.

Proposition 2.9. *Let $A \in M_n(\mathbb{R})$ be a nonnegative nilpotent matrix. Then A is a product of* n *nonnegative idempotent matrices.*

Proof. Lemma 6 in [12] shows that there exists a permutation matrix P such that PAP^{-1} is nonnegative and strictly upper triangular. The above proposition shows that such a matrix is a product of n idempotent matrices and the proof of this proposition shows that, since A is nonnegative, these idempotent matrices are nonnegative as well. □

3. IP in Quasi-Euclidean regular rings. Let us start this section by the following proposition that shows how euclidean pairs belonging to the annihilator of a matrix A can force a matrix to be a product of idempotent matrices. This proposition also shows that knowing about the matrices of the form $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ being product of idempotents is an essential step towards proving that $M_2(R)$ is IP.

Proposition 3.1. *Let R be a domain such that for any $a, b \in R$ the matrix $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is a product of idempotents. Suppose that a matrix $A \in M_2(R)$ is such that there exists a right quasi-Euclidean pair (x, y) with $(x, y)A = (0, 0)$. Then A is a product of idempotent matrices.*

Proof. Suppose that the matrix $A \in M_2(R)$ is such that $(x, y)A = (0, 0)$, where (x, y) is a right quasi-Euclidean pair. Then there exists an invertible matrix P such that $(x, y) = r(0, 1)P$. We thus have $r(0, 1)PA = (0, 0)$. Since R is a domain we get $(0, 1)PA = (0, 0)$ and hence $PA = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ for some elements $a, b \in R$. Right multiplying by P^{-1}, we get that $PAP^{-1} = \begin{pmatrix} a' & b' \\ 0 & 0 \end{pmatrix}$. The hypothesis shows that the matrix on the right hand side is a product of idempotent matrices. This yields the proof. □

Recall that an element a in a ring R is (von Neumann) regular if there exists $x \in r$ such that $a = axa$. A ring is called regular closed if the product of two regular elements is regular. Our aim is to characterize IP and quasi-Euclidean rings amongst regular rings.

PROPOSITION 3.2. *A regular IP ring is unit regular and hence quasi-Euclidean.*

Proof. Let R be a regular IP ring. If $a \in R$ is not a unit then either $r(a) \neq (0)$, or $l(a) \neq (0)$. The IP property then implies that a is a product of idempotents. In particular, it is a product of unit regular elements. Since, by Lemma 3.2 in [8], the product of unit regular elements of a regular ring is still unit regular, we conclude that a is unit regular and hence the ring R is a unit regular ring. The last statement is due to the fact that every unit regular ring is quasi-Euclidean (cf. [2]). □

REMARKS 3.3.

(a) It can be shown that the conclusion of the proposition is still valid for an IP ring with the condition that sum of direct summands of a ring is also a direct summand (this property is known in the literature as SSP property). This last condition is in fact equivalent to the fact used in the above proof, namely the product of unit regular elements is a unit regular element.
(b) One may consider simple noetherian QE rings. We know that the Weyl algebra $R = F[x, x^{-1}, \sigma]$ is a simple noetherian domain. This is QE and trivially IP. We do not know if this is $(IP)_2$. Recall that the 2×2 matrix ring over R is a principal right ideal ring (cf. [18])
(c) Hannah and O'Meara proved (cf. Theorem 2.9 in [8]) that if a ring R is either unit regular or right self-injective, then an element $a \in R$ is a product of idempotents if and only if

$$Rr(a) = l(a)R = R(1-a)R \qquad (*)$$

Coincidently, the same condition characterizes separative rings amongst regular ones (cf. [3]), more precisely: A regular ring R is separative if and only if each a satisfying (*) is unit regular.

We now turn to the characterization of right (left) Quasi-Euclidean rings amongst regular rings. A quasi-Euclidean ring being defined as ring for which every pair (a, b) is right (left) quasi- Euclidean. We start with the following examples which show, in particular, that in a regular ring quasi-Euclidean pairs are abundant.

EXAMPLES 3.4.

(a) Let a, b be any elements in a ring R such that $a \in (aR \circ b)R$, where for

$x, y \in R$, $x \circ y := x + y - xy$. Then, the pair (b, a) (and also (a, b)) is a quasi-Euclidean pair. Indeed let $x, y \in R$ be such that $a = (ax + b - axb)y$. We then have the euclidean divisions:

$$b = a(xb - x) + ax + b - axb \quad \textit{and} \quad a = (ax + b - axb)y.$$

In particular,

- If $a = axa$ is a regular element in R and b is such that $ba = axba$, or equivalently $(1 - ax)ba = 0$, then taking $y = a$ we conclude easily that the pair (a, b) is quasi-Euclidean.
- if $e = e^2, r \in R$ are such that $be = ebe$ then, taking $a = x = y = e$, we check that the pair (e, b) is quasi-Euclidean. In this respect let us recall that an idempotent $e \in R$ is said to be left semicentral if for any $b \in R$, we thus get that left semicentral idempotents form Euclidean pairs with every other element of the ring.

(b) Let $a, b \in R$ be such that $b(1 - b)a = 0$ and $c := a - ba$ is a regular element. Then the pair (a, b) is right Euclidean. Indeed if $t \in R$ is such that $c = ctc$, we consider the following divisions:

$$a = ba + c \quad b = ct + (b - ct) \quad \textit{and} \quad c = (b - ct)(-c)$$

In particular, in a regular ring any pair of elements containing an idempotent is a quasi-Euclidean pair (take $b = b^2$, in the above equalities).

Theorem 3.5. *Let R be a regular separative ring. The following are equivalent:*

(i) *R is right Hermite,*
(i') *R is left Hermite,*
(ii) *R is right QE,*
(ii') *R is left QE,*
(iii) *Every square matrix over R is equivalent to a diagonal matrix.*
(iv) *$R \oplus R \oplus X \cong R \oplus Y$ implies that $R \oplus X \cong Y$ for every finitely generated projective right R-modules X, Y.*
(v) *R is of finite stable range.*

Proof. Let us first notice that the stable range notion can be defined on the right and on the left, but a well-known theorem of Vaserstein shows that the right and left stable range of a ring coincide. This is why it is enough to deal with the statements related to the right different concepts.

(i)$\Rightarrow$(ii) Regular rings are exchange rings and a well-known theorem of Para et al. in [3] shows that exchange separative rings are GE rings. Since a ring is right QE if and only if it is right Hermite and GE, we thus get that a regular separative right Hermite ring is right QE.

(i)$\Leftrightarrow$(iii) and (iii)$\Leftrightarrow$(iv) can be found in [19].

(i)$\Leftrightarrow$(v) It is part of folklore that Hermite rings are of stable range bounded by 2 (cf.[19]). On the other hand for a separative regular ring the converse is true as proved in [3]. □

Remark 3.6. The separative hypothesis might seem to be very restrictive but let us mention that in fact, as far as we know, there is no known example of a regular ring which is not separative.

The first part of the following is a folklore.

Lemma 3.7. . *A simple right (or left) self-injective is unit regular and hence right (or left) QE and directly finite.*

Proof. Such a ring by Osofsky's theorem is regular. By invoking a result due to O'Meara (cf. Theorem 2.8, [8]), every singular element, equivalently, non-unit is a product of idempotents. Thus each non-unit is unit regular. Thus R is QE and Dedekind finite. □

Remark 3.8. Since the IP property in a ring implies that the ring is directly finite, the following example shows that a regular QE ring need not be IP ring.

Example 3.9. If R is the ring of linear transformations of an infinite dimensional vector space and if S is the socle, then using Litoff's Theorem (cf. [10], p.90) R is QE (without unity), and is regular but not directly finite.

Question 1: If a ring is nonsingular right QE, what about its maximal right ring of quotients?

We remark that if R is right QE, then the same is true for any Ore localization of R with respect to regular elements (cf. [2]).

Question 2: Let R be unit regular, hence right and left QE ring. Is the right max ring of quotients QE?

4. Nonnegative Matrices. In this section we shall be working in the real n-dimensional vector spaces. We are interested in representing **rank one** nonnegative matrices as product of nonnegative idempotent matrices. Let us collect in the following lemma some preliminary facts. For more details the reader may consult [11]. For two column vectors $x, y \in \mathbb{R}^n$, we denote $x \cdot y = y \cdot x$ the usual dot product, sometimes also written as $y^T x$. Notice that the $n \times n$ matrix xy^T is idempotent if and only if $x \cdot y = 1$.

Lemma 4.1. *Let $M \in M_n(\mathbb{R})$ be a nonnegative matrix of rank one. Then there exist nonnegative column vectors $a, b \in \mathbb{R}^n$ such that $M = ab^T$. Let β be the nonnegative real number given by $\beta = b^T a = a \cdot b$. Then:*

(a) *if $\beta = 0$, M is a product of two nonnegative idempotent matrices,*
(b) *if $\beta > 0$, $M = \beta xy^T$ where x, y are nonnegative vectors such that $y^T x = 1$ (i.e. the matrix xy^T is an idempotent matrix). In this case, the matrix M is a product of two nonnegative idempotent matrices of rank 1 if and only if there exist nonnegative vectors a,b,c,d such that $\beta xy^T = (ab^T)(cd^T)$ where $a = x$, $d = y$, $b \cdot a = 1$, $d \cdot c = 1$ and $\beta = b \cdot c$.*

Proof. The existence of nonnegative vectors $a, b \in \mathbb{R}^n$ such that $M = ab^T$ is obvious. Let us prove statement (a). We write $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$ and choose $i \in \{1, \ldots, n\}$ such that $a_i \neq 0$. Since $b^T a = 0$, we must have $b_i = 0$. Define $x = \sqrt{\alpha} a$, $y = \frac{1}{\sqrt{\alpha}} b$ and $e = \frac{1}{a_i \sqrt{\alpha}} e_i$, where $\alpha = b^T b$ and $e_i = (\delta_{i1}, \ldots, \delta_{in})$ is the standard unit basis vector of $\mathbb{R}^n$. We then have $y^T y = 1$, $e^T y = \frac{1}{\alpha a_i} e_i^T b = \frac{1}{\alpha a_i} b_i = 0$, $(y+e)^T y = y^T y + e^T y = 1$ and also $(y+e)^T x = y^T x + e^T x = b^T a + \frac{1}{\sqrt{\alpha} a_i} e_i^T x = \frac{a_i}{a_i} = 1$. This shows that $x(y+e)^T$ and yy^T are nonnegative idempotent matrices. One also has $(x(y+e)^T)(yy^T) = x((y+e)^T y)y^T = xy^T = ab^T = M$. This yields the proof.

The first part of statement (b) is easily checked if we define $x = \frac{1}{\sqrt{\beta}} a$ and $y = \frac{1}{\sqrt{\beta}} b$. Suppose now that we can express βxy^T with $y^T x = 1$, $x \geq 0$, $y \geq 0$, as a product of two nonnegative rank one matrices and write:

$$\beta xy^T = (ab_1^T)(c_1 d^T), \tag{4.1}$$

where $b_1.a = 1 = d.c_1$. This implies by postmultiplying with y that $\beta(y^T y)x = (b_1^T c_1).(d^T y)a$ and so x and a are parallel vectors. Set $a = k_1 x$. Similarly, $d = k_2 y$, where $k_1 > 0$ and $k_2 > 0$

Equation 4.1 gives $\beta xy^T = (k_1 x b_1^T c_1 k_2 y^T) = ((b_1^T c_1) k_1 k_2) xy^T$.

By equating scalars, $\beta = (b_1^T c_1) k_1 k_2$. Choose $b = b_1 k_1$ and $c = c_1 k_2$. We then obtain

$(xb^T)(cy^T) = x(b^T c)y^T = (b^T c)xy^T = k_1 k_2 b_1^T c_1 xy^T = \beta xy^T$. Moreover $b^T x = k_1 b_1^T x = b_1^T (k_1 x) = b_1^T a = 1$, similarly $y^T c = 1$ and $\beta = b \cdot c$. This proves that by suitably choosing the vectors b and c , we can express βxy^T as a product of idempotents (xb^T) and (cy^T) where $\beta = b \cdot c$.

The fact that the conditions stated in (b) are sufficient is clear. □

We shall characterize those nonnegative β for which the rank one matrix $M = \beta xy^T$, where x and y are nonnegative vectors, is a product of two nonnegative idempotent matrices. The above lemma 4.1 shows that we can restrict our study to the case when $\beta > 0$.

We now introduce some notation which will be helpful in the proofs of the lemmas and theorems that follow.

- Let V be an n-dimensional real vector space with $n > 1$. V_+ shall denote the set of all nonnegative vectors in V and V_{++} shall denote the set of all positive vectors of V.
- For $0 \neq v \in V_+$, define $S_v = \{w \in V_+ \mid v \cdot w = 1\}$ Notice that $w \in S_v$ if and only if vw^T is an idempotent.
- For any nonzero $v \in V$ we denote its orthogonal space by $v^\perp$. Remark that $\tilde{v} = \frac{1}{v \cdot v} v \in S_v$ and it is easy to see that $S_v \subseteq \tilde{v} + v^\perp$.
  Since for any fixed vector v and arbitrary vector $w \in S_v$, vw^T is an idempotent, w may be called an *idempotent companion* of v. Note that $v \in S_w$ if and only if $w \in S_v$.
- For $x, y \in V$, define
  $$\Delta_{x,y} = \{\beta \in \mathbb{R} \mid b \cdot c = \beta \text{ for some } (b, c) \in S_x \times S_y \ \}$$
- For a pair of idempotent companions x, y of nonnegative vectors, define $\beta_i(x, y) = 1/(x_i y_i)$ if both x_i, y_i are nonzero and ∞ otherwise. We may shorten the notation to just β_i when (x, y) are fixed.
- We also define $\mathrm{Supp}(x, y) = \{i \mid x_i y_i \neq 0\}$.

Lemma 4.1 shows that a nonnegative matrix of rank one $M = \beta xy^T$, where $x \cdot y = 1$, decomposes as a product of two nonnegative idempotent matrices of rank one if and only if $\beta \in \Delta_{x,y}$. The next theorem gives a description of $\Delta_{x,y}$ and hence gives exact characterization when such a decomposition is possible.

Theorem 4.2. *Given a pair of nonnegative idempotent companions $x, y \in V_+$, we set $\beta^*(x, y) = \max\{\beta_i(x, y) \mid \text{ for } i = 1, \cdots, n\}$.*

Then we have:

(a) *If $\beta^*(x, y) = \infty$ and Supp(x, y) has at least two elements, then $\Delta_{x,y} = [0, \infty)$.*
(b) *If $\beta^*(x, y) < \infty$, then $\Delta_{x,y} = [0, \beta^*(x, y)]$.*
(c) *If Supp(x, y) is a singleton, then $\Delta_{x,y} = [1, \infty)$. Note that if we want to express $M = \beta xy^T$ as a product of two idempotents then $\beta \in \Delta_{x,y}$ takes no values in the interval $[0, 1)$.*

Theorem 4.3. *Suppose that (x, y) is a pair of idempotent companions. Then βxy^T can always be expressed as a product of three idempotent matrices for every nonnegative $\beta \in \mathbb{R}$.*

Proof of Theorem 4.2

(a) Suppose that for three different i, j, k in $\{1, 2, \cdots, n\}$ we have : $x_i y_i \neq 0 \neq x_j y_j$ and $x_k y_k = 0$.

We have exactly three cases:

- $x_k = 0, y_k \neq 0$. Then we see that
$$x\left(\frac{1}{x_i}e_i + 2y_k\beta e_k\right)^T\left(\frac{1}{2y_k}e_k + \frac{1}{2y_j}e_j\right)y^T = \beta xy^T.$$
Thus $\beta \in \Delta_{x,y}$ for all $0 \leq \beta \in \mathbb{R}$. This clearly shows that $\Delta_{x,y} = [0, \infty)$.
- $x_k = y_k = 0$. Here we see that
$$x\left(\frac{1}{x_i}e_i + \beta e_k\right)^T\left(e_k + \frac{1}{y_j}e_j\right)y^T = \beta xy^T.$$
Thus $\beta \in \Delta_{x,y}$ for all $0 \leq \beta \in \mathbb{R}$. This clearly shows that $\Delta_{x,y} = [0, \infty)$.
- $y_k = 0, x_k \neq 0$. Follows similar to the first item above.

(b) We now have $x_i \neq 0 \neq y_i$ for all $i = 1, \cdots, n$. Set $v_i = \frac{1}{x_i}e_i$ and $w_i = \frac{1}{y_i}e_i$ for all $i = 1, \cdots, n$. We note that
$$\frac{1}{x_iy_i}xy^T = \left(xv_i^T\right)\left(w_iy^T\right) \text{ for all } i = 1, \cdots n$$
Thus $\beta_i(x,y) = \frac{1}{x_iy_i}$ are clearly in $\Delta_{x,y}$.

Note that every $v \in S_x$ is uniquely expressible as $v = \sum_{i=1}^n \lambda_i v_i$ where $\lambda_i \geq 0$. Moreover, the condition $x \cdot v = 1$ implies that $\sum_{i=1}^n \lambda_i = 1$. After a similar observation with S_y, we claim that for any $(b,c) \in S_x \times S_y$ we have $b = \sum_i p_i v_i$ and $c = \sum_i q_i w_i$ such that $\sum_i p_i = \sum_i q_i = 1$ and thus
$$b \cdot c = \sum_{i=1}^n \beta_i(x,y)p_iq_i.$$

Let j be chosen such that $\beta_j(x,y) \geq \beta_i(x,y)$ for all $i = 1, 2, \cdots, n$. We **claim** that
$$\max\{b \cdot c \mid (b,c) \in S_x \times S_y\} = \beta_j(x,y) = \beta^*(x,y).$$

Let $b = \sum_i p_i v_i$ and $c = \sum_i q_i v_i$ be a pair of idempotent companions in $S_x \times S_y$. We have:
$$b \cdot c = \sum_i (p_iq_i)(v_i \cdot w_i)$$
and since $v_i \cdot w_i = \beta_i(x,y) \leq \beta_j(x,y)$ for all i, we get:
$$\sum_i (p_iq_i)\beta_i(x,y) \leq \left(\sum_i (p_iq_i)\right)\beta_j(x,y) \leq \beta_j(x,y).$$

The last inequality follows from Cauchy-Schwartz: $(\sum_i (p_i q_i)) \leq 1$. By rearranging the indices, we may assume that $j = 1$, so that $\beta_1(x, y) = \beta^*(x, y)$. It remains to show that all values in $[0, \beta_1]$ are in $\Delta_{x,y}$.

Let $t \in [0, 1] \subset \mathbb{R}$.

Consider $(b(t), c(t)) = (\frac{t}{x_1} e_1 + \frac{1-t}{x_2} e_2, \frac{1}{y_1} e_1) \in S_x \times S_y$. It is clear that $b(t) \cdot c(t) = t\beta_1$ and thus as t varies in $[0, 1]$ we get the interval $[0, \beta_1] \subset \Delta_{x,y}$.

(c) In this case, by reordering the numbering, we may assume that $x_1 y_1 = 1$ and $x_i y_i = 0$ for all $i \geq 2$. Then any $(b, c) \in S_x \times S_y$ satisfies $b = \frac{1}{x_1} e_1 + v$ and $c = \frac{1}{y_1} e_1 + w$ where v, w are vectors with nonnegative coefficients in the span of $e_2, \cdots c_n$. It follows that $b \cdot c \geq 1$. This completes the proof of Theorem 4.2.

Proof of Theorem 4.3 Let (x, y) be a pair of nonnegative idempotent companions and $0 \leq \beta \in \mathbb{R}$. We show that $\beta \in S_x \times S_y$.

By renumbering if necessary, we may assume that $x_1 y_1 \neq 0$. There are exactly three possibilities.

1. $y_2 = 0$.
2. $x_2 = 0$.
3. $x_2 \neq 0 \neq y_2$.

We give explicit description of each of the three idempotents whose product equals $\beta x y^T$.

Case 1. Observe that

$$\beta x y^T = \left[x \left(\frac{1}{x_1} e_1 \right)^T \right] \left[(\beta x_1 e_1 + e_2) e_2^T \right] \left[\left(e_2 + \frac{1}{y_1} e_1 \right) y^T \right]$$

Case 2. Observe that

$$\beta x y^T = \left[x \left(\frac{1}{x_1} e_1 + e_2 \right)^T \right] \left[e_2 (e_2 + \beta y_1 e_1)^T \right] \left[\left(\frac{1}{y_1} e_1 \right) y^T \right]$$

Case 3. Observe that

$$\beta x y^T = \left[x \left(\frac{1}{x_1} e_1 \right)^T \right] \left[(\beta x_1 y_2 e_1 + e_2) e_2^T \right] \left[\left(\frac{1}{y_2} e_2 \right) y^T \right]$$

This completes the proof of Theorem 4.3.

Our result has some impact on group-monotone matrices. Let us first recall that a group-monotone matrix A is a direct sum of matrices of following three types (some types may be absent) (see Theorem 1 [11]).

(I) βxy^T where $\beta > 0$, x, y are positive vectors such that $y^T x = 1$

(II) $d \times d$ block matrix of the form

$$\begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & ... & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & ... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ \beta_{d1}x_dy_1^T & 0 & 0 & ... & 0 \end{bmatrix}$$ where $\beta_{ij} > 0$, x_i, y_j are positive vectors such that $y_j^T x_i = 1$

(III) Zero matrix.

We can now state the following corollary.

Corollary 4.4. *Let A be a group-monotone matrix. Then a power of A is a product of* 3 *nonnegative idempotents matrices.*

Proof. Since d th power of the type (ii) matrix is a product of diagonal block matrix with blocks of rank one on the diagonal, it follows from the above Theorems 4.2 and 4.3 that A^d is always product of 3 idempotent matrices and is a product of 2 idempotent matrices under some conditions. □

We conclude this section by asking an open question.

Question 3: Let $A = \begin{pmatrix} 0 & \beta xy^T \\ \gamma uv^T & 0 \end{pmatrix}$ be a 2 x 2 block matrix having (1,2) and (2,1) entries as nonnegative rank one matrices and let the other two block entries be zero. Is this a product of nonnegative idempotent matrices?

We may mention that it suffices to answer the above question when (1,2) and (2,1) entries in the Question 3 are nonnegative idempotent matrices.

Acknowledgment. We wish to thank the referee for his/her careful reading and Prof. K. M. Prasad for his interest and useful comments. The third author would like to thank the King Abdulaziz University for the kind hospitality he received while working on this paper.

REFERENCES

[1] A. Alahmadi, S. K. Jain, and A. Leroy. Decomposition of singular matrices into idempotents. *Linear Multilinear Algebra*, 62:13–27, 2014.

[2] A. Alahmadi, S. K. Jain, T. Y. Lam and A. Leroy. Euclidean pairs and quasi-Euclidean rings. *Journal of Algebra*, 406:154–170, 2014.

[3] P. Ara, K. R. Goddearl, K. C. O'Meara, and E. Pardo. Separative cancellation for projective modules over exchange rings. *Israel Journal of Mathematics*, 105:105–137, 1998.

[4] K. P. S. Bhaskara Rao. Products of idempotent matrices. *Linear Algebra and its Applications*, 430:2690-2695, 2009.

[5] P. M. Cohn. *Free Ideal Rings and Localization in General Rings*. New Mathematical Monographs, No. 3, Cambridge University Press, Cambridge, 2006.

[6] J. A. Erdos. On products of idempotent matrices. *Glasgow Mathematical Journal*, 8: 118–122, 1967.

[7] K. R. Goodearl. *Von Neumann Regular Rings*. Krieger Publishing Company, Malabar, Florida, 1991.

[8] J. Hannah and K. C. O'Meara. Products of idempotents in regular rings, II. *Journal of Algebra*, 123:223–239, 1989.

[9] J. Hannah and K. C. O'Meara. Depth of Idempotent-Generated Subsemigroups of a Regular Ring. *Proceedings of London mathematical society*, 59, 464–462, 1989.

[10] N. Jacobson. *Structure of rings* American Mathematical Society, Providence RI, 1968.

[11] S. K. Jain, V. K. Goel and E. K. Kwak. Decomposition of nonnegative good monotone matrices. Transactions of the American Mathematical Society, 25 (2):371–385, 1980.

[12] S. K. Jain, V. K. Goel and E. K. Kwak. Nonnegative mth roots of nonnegative 0-symmetric idempotent matrices. *Linear Algebra and its applications*, 23:37–51 1979.

[13] I. Kaplansky. Elementary divisors and modules. *Transactions of the American Mathematical Society*, 66:464–491, 1949.

[14] T. J. Laffey. Products of idempotent matrices. *Linear and Multilinear Algebra*, 14:309–314, 1983), 309-314.

[15] T. Y. Lam. *A First Course in Noncommutative Rings, Second Edition* Graduate Texts in Math., Vol. **131**, Springer-Verlag, Berlin-Heidelberg-New York, 2001.

[16] T. Y. Lam. *Lectures on Modules and Rings.* Graduate Texts in Math., Vol. **189**, Springer-Verlag, Berlin-Heidelberg-New York, 1999.

[17] T. Y. Lam. A Crash Course on stable range, cancellation and exchange. *Journal of Algebra and its Applications*, 13, issue 03:301–343, 2004.

[18] J. C. McConnell and J.C. Robson. *Noncommutative Noetherian Rings*. Graduate Studies in Mathematics 30, Amer. Math. Soc., 2001.

[19] P. Menal and J. Moncasi. On regular rings with stable range 2. *Journal of pure and Applied Algebra*, 24:25–40 1982.

[20] A. Tuganbaev. *Rings close to regular.* Kluwer Academic Publishers, Mathematical and its Applications, Vol. 545 (2002).