DECOMPOSITION OF SINGULAR MATRICES INTO IDEMPOTENTS

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Abstract. In this paper we provide concrete constructions of idempotents to represent typical singular matrices over a given ring as a product of idempotents and apply these factorizations for proving our main results. We generalize works due to Laffey ([12]) and Rao ([3]) to noncommutative setting and fill in the gaps in the original proof of Rao's main theorems (cf. [3], Theorems 5 and 7 and [4]). We also consider singular matrices over Bézout domains as to when such a matrix is a product of idempotent matrices.

1. Introduction and definitions

It was shown by Howie [10] that every mapping from a finite set X to itself with image of cardinality $\leq cardX - 1$ is a product of idempotent mappings. Erdös [7] showed that every singular square matrix over a field can be expressed as a product of idempotent matrices and this was generalized by several authors to certain classes of rings, in particular, to division rings and euclidean domains [12]. Turning to singular elements let us mention two results: Rao [3] characterized, via continued fractions, singular matrices over a commutative PID that can be decomposed as a product of idempotent matrices and Hannah-O'Meara [9] showed, among other results, that for a right self-injective regular ring R, an element a is a product of idempotents if and only if $Rr.ann(a) = l.ann(a)R= R(1-a)R$.

The purpose of this paper is to provide concrete constructions of idempotents to represent typical singular matrices over a given ring as a product of idempotents and to apply these factorizations for proving our main results. Proposition 14 and Theorem 22 fill in the gaps in Rao's proof of a decomposition of singular matrices over principal ideal domains (cf. [3], Theorems 5 and 7), and simultaneously generalize these results. We show that over a local ring R (not necessarily commutative), if every 2×2 matrix A with $r.ann(A) \neq 0$ is a product of idempotent matrices, then R must be a domain (Theorem 9). We prove the existence of a decomposition into product of idempotents for any matrix A with $l.ann(A) \neq 0$, over a local domain (not necessarily commutative) with Jacobson radical $J(R) = gR$ such that $\cap_n J(R)^n = 0$ (Theorem 10).

Let R be a Bézout domain such that every 2×2 singular matrix is a product of idempotent matrices. Theorem 22 shows that if every 2×2 invertible matrix over R is a product of elementary matrices and diagonal matrices with invertible diagonal entries then every $n \times n$ singular matrix is a product of idempotent matrices;

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The converse of Theorem 22 is true for commutative Bézout domain, that is, if every $n \times n$ singular matrix over such a domain is a product of idempotent matrices then every 2×2 invertible matrix is a product of elementary matrices and diagonal matrices with invertible diagonal entries (Corollary 21). Finally, Theorem 24 studies the condition when each right singular element of the endomorphism ring of an injective module is a product of projections. This shows, in particular, that each linear transformation of a vector space, which is right singular, is a product of projections if and only if the vector space is finite-dimensional.

Let us now give the main definitions and fix our terminology.

All rings considered are nonzero rings with an identity element denoted by 1, and need not be commutative. A ring R is called a local ring if it has a unique maximal right ideal (equivalently, unique maximal left ideal). For example, power series ring $F[[x]]$ over a field F and the localization $Z_{(p)}$ of the ring of integers Z are local rings. A ring R is called projective-free if each finitely generated projective right (equivalently, left) module is free of unique rank. Every local ring is projective-free. A ring R is a principal right (left) ideal ring if each right (left) ideal is principal. A right R-module M is called injective if every R-homomorphism from a right ideal of R to M can be extended to an R-homomorphism from R to M. Clearly, every vector space over a field is injective. A ring R is called right self-injective if it is injective as a right R-module. A ring R is called von Neumann regular if for each element $a \in R$, there exists an element $x \in R$ such that $axa = a$. A ring R is called unit regular if for each element $a \in R$, there exists an invertible element u such that $aua = a$. A ring R is called Dedekind finite if for all $a, b \in R$, $ab = 1$ implies $ba = 1$.

An $n \times n$ matrix is called elementary if it is of the form $I_n + ce_{ij}$, $c \in R$ with $i \neq j$. A ring R has a stable range 1 if for any $a, b \in R$ with $aR + bR = R$, there exists $x \in R$ such that $a + bx \in U(R)$, where $U(R)$ is the set of invertible elements of R. A ring R is right (left) Bézout if any finitely generated right (left) ideal of R is principal. Hermite rings have been defined differently by different authors in the literature. Following Kaplansky, we call R to be a right (left) Hermite ring, if for any two elements $a, b \in R$ there exists a 2×2 invertible matrix P and an element $d \in R$ such that $(a, b)P = (d, 0)$ $(P(a, b)^t = (d, 0)^t)$. Lam ([14], section I, 4) calls this ring as K-Hermite ring. By a Hermite (Bézout) ring we mean a ring which is both right and left Hermite (Bézout). Amitsur showed that a ring R is a right (left) Hermite domain if and only if R is a right (left) Bézout domain. Theorem 16 in this paper provides an alternative proof of Amitsur's theorem.

A ring R is GE_2 if any invertible 2×2 matrix is a product of elementary matrices and diagonal matrices with invertible diagonal entries.

A right unimodular row is a row $(a_1, \ldots, a_n) \in R^n$ with the condition $\sum_{i=1}^n a_i R = R$. A right unimodular row is completable if it is a row (equivalently, the bottom row) of an invertible matrix.

An element a in a ring R will be called right (left) singular if $r.ann\ (a) \neq 0$ $(l.ann\ (a) \neq 0)$. An element is singular if it is both left and right singular. $U(R)$ will denote the set of invertible elements of a ring R. $M_{n\times m}(R)$ stands for the set of $n \times m$ matrices over the ring R. The ring of $n \times n$ matrices over R will be denoted by $M_n(R)$. The group of $n \times n$ invertible matrices over R is denoted by $GL_n(R)$.

2. Preliminaries.

We begin with an elementary lemma which works like our reference table for the proofs of our results. Note that one can obtain additional factorizations from a given factorization into idempotent matrices by taking conjugations.

Lemma 1. *(Table of factorizations) Let R be any ring and let $a, b, c \in R$. Then*

(a) $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$,

(a') $\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$,

(b) $\begin{pmatrix} a & ac \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & c \\ 0 & 0 \end{pmatrix}$,

(b') $\begin{pmatrix} a & 0 \\ ca & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix}$,

(c) $\begin{pmatrix} ac & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix}$,

(c') $\begin{pmatrix} ca & 0 \\ a & 0 \end{pmatrix} = \begin{pmatrix} 0 & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix}$,

(d) *with* $b \in U(R)$, $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b(b^{-1}a) & b \\ 0 & 0 \end{pmatrix}$ *can be factorized as in (c) and* $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} (ab^{-1})b & 0 \\ b & 0 \end{pmatrix}$ *can be factorized as in (c'),*

(e) *with* $a \in U(R)$, $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & a(a^{-1}b) \\ 0 & 0 \end{pmatrix}$ *can be factorized as in (b) and* $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ (ba^{-1})a & 0 \end{pmatrix}$ *can be factorized as in (b').*

In the next lemma, we consider factorizations of $n \times n$ matrices:

Lemma 2. *Let R be any ring and $A \in M_2(R)$ be either*

(a) *an elementary matrix,*

(b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

(c) *a diagonal matrix,*

(d) $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ *or* $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$, $a, b \in R$.

Then, for $n \geq 3$, the $n \times n$ matrix $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ *is a product of idempotent matrices, where zero blocs are of appropriate sizes.*

Proof. We will treat the case when $n = 3$. The general case is similar.

(a) Let us, for instance, choose an elementary matrix $A = I_2 + ae_{12}$.

$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. Then $\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & a & 1 \\ 0 & 1 & 0 \\ 0 & -a & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(b) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(c) $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1-a & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1-b & 0 \end{pmatrix}.$

(d) Let us consider the case $\begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$ □

Lemma 3. *Let R be a ring. If each right (left) singular element is a product of idempotents, then R is Dedekind finite.*

Proof. Let $a, b \in R$ be such that $ab = 1$. Then $l.ann(a) = 0$. If $r.ann(a) = 0$ then $a(ba-1) = 0$ implies $ba = 1$, and we are done. In case $r.ann(a) \neq 0$ then by hypothesis a is a product of idempotents. This implies that $l.ann(a) \neq 0$, a contradiction. Therefore, $r.ann(a) = 0$ and so as above $ba = 1$. □

The following lemma is well-known (cf. [11], Theorem 7.1).

Lemma 4. *If R is right (left) Hermite domain then each right (left) unimodular row is completable.*

Lemma 5. *Let $A \in M_n(R)$ be a square matrix with coefficients in a right Bézout domain R. Let $0 \neq u \in R^n$ be such that $uA = 0$. Then there exists an invertible matrix $P \in GL_n(R)$ such that PAP^{-1}has its last row equal to zero.*

Proof. By hypothesis, for some $u \in R^n$ $uA = 0$. Since R is a right Bézout domain we may assume that the vector u is right unimodular. Since right Bézout domain are right Hermite we know that there exists an invertible matrix P such that the last row of P is the vector u. Of course, this implies that the last row of PA is the zero row and this is true as well for the last row of PAP^{-1}. □

Next, we list some properties and results for rings with stable range 1 which will be referred to in the proofs. Let us first mention a well-known theorem by Vaserstein which shows that the notion of stable range is left-right symmetric.

Lemma 6. *Let $a, a', b, b', x, d \in R$ and $u \in U(R)$ be such that $a + bx = du$, $a = da'$ and $b = db'$. Then*

(a) $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = E \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}$, *where E is a product of idempotent matrices,*

(b) *There exists an invertible matrix $P \in M_2(R)$ such that*

$$\begin{pmatrix} a & b \\ -x & 1 \end{pmatrix} P = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. (a) Indeed we have: $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u & b' \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}$ and the first two matrices on the right side are products of idempotent matrices as shown in the table of factorizations given in Lemma 1.

(b) The matrix P is given by $P = \begin{pmatrix} u^{-1} & -u^{-1}b' \\ xu^{-1} & 1 - xu^{-1}b' \end{pmatrix}$. □

Rings with stable range 1 possess many properties. The next lemma mentions two of them that are particularly relevant to our study.

Lemma 7. *Let R be a ring with stable range 1. Then*

(a) *R is GE_2, and*
(b) *any unimodular row (a, b) is completable.*

Proof. (a) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible matrix with coefficients in R. We thus have, in particular, that $aR + bR = R$ and the stable range 1 hypothesis shows that there exists $x \in R$ such that $a + bx = u \in U(R)$. Let us put $v := d - (c + dx)u^{-1}b$. We then have

$$A = \begin{pmatrix} b & u \\ d & c+dx \end{pmatrix} \begin{pmatrix} -x & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} u & 0 \\ c+dx & v \end{pmatrix} \begin{pmatrix} u^{-1}b & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -x & 1 \\ 1 & 0 \end{pmatrix}.$$

Since A is invertible, v is a unit. This finally gives us

$$A = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v^{-1}(c+dx) & 1 \end{pmatrix} \begin{pmatrix} u^{-1}b & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -x & 1 \\ 1 & 0 \end{pmatrix},$$

as required.
(b) If $aR + bR = R$ then there exists $x \in R$ such that $a + bx = u \in U(R)$. In this case Lemma 6 shows that the unimodular row (a, b) is completable. □

3. Local Rings

Firstly, as a consequence of our table of factorizations in Lemma 1, we give a very simple proof of the celebrated theorem that every singular matrix over a division ring is a product of idempotent matrices. The proof given below is for a singular 2×2 matrix over a division ring. However, as a consequence of Theorem 22, the proposition holds for any $n \times n$ singular matrix.

Proposition 8. *Every 2×2 singular matrix over a division ring can be factorized as a product of idempotent matrices.*

Proof. Let $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ be a singular matrix. Then the columns $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ are linearly dependent. Suppose $\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \alpha$.

Then $\begin{pmatrix} a & a\alpha \\ b & b\alpha \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix}$. If $b = 0$, then Lemma 1 gives factorization of the first factor whereas the second factor is already an idempotent. If $b \neq 0$, then one can use Lemma 1 (d) to conclude the result. □

Next, we show that if each right (resp. each left) singular matrix over a local ring R is a product of idempotent matrices then the ring R must be a domain. Let us recall that a local ring is projective-free. For an idempotent matrix $E \in M_n(R)$, $n > 1$, where R is projective-free, there exist matrices $A \in M_{n\times r}(R)$ and $B \in M_{r\times n}(R)$ with $r < n$ such that $E = AB$ and $BA = I_r$ (See Cohn [5], Proposition 0.4.7, p. 24).

Theorem 9. *Let R be a local ring such that each right (resp. each left) singular 2×2 matrix over R can be expressed as a product of idempotents. Then R is a domain.*

Proof. We assume that every right singular matrix is a product of idempotents. Let $a \in R$. Suppose $r.ann(a) \neq 0$. Since the matrix

$$A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

is right singular it can be expressed as a product of idempotent matrices, say $A = E_1 \dots E_n$. Since a belongs to the Jacobson radical of R, it cannot be itself an idempotent and hence we must have $n > 1$. The property of idempotent matrices recalled in the paragraph preceding this theorem shows that A can be written as $A = P_1Q_1 \dots P_nQ_n$ where $Q_i \in M_{1\times 2}(R), P_i \in M_{2\times 1}(R)$ are such that $Q_iP_i = 1$. Set $P_1 = (\alpha, \beta)^t$, $Q_1P_2Q_2 \dots P_n = \gamma \in R$ and $Q_n = (\delta, \epsilon)$. Then $a = \alpha\gamma\delta$, $0 = \alpha\gamma\epsilon$, $0 = \beta\gamma\delta$, and $1 = \beta\gamma\epsilon$. Let us set $P_n = (x, y)^t$. Since $Q_nP_n = 1$, we obtain $\delta x + \epsilon y = 1$. Furthermore,

$$\begin{pmatrix} ax \\ y \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \gamma \begin{pmatrix} \delta & \epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha\gamma \\ \beta\gamma \end{pmatrix}.$$

This leads to $ax = \alpha\gamma$ and $y = \beta\gamma$. We then easily get $1 = \beta\gamma\epsilon = y\epsilon$ and since R is Dedekind finite we also have $\epsilon y = 1$. This leads consecutively to $\delta x = 0$, $ax = \alpha\gamma\delta x = 0$, $\alpha\gamma = ax = 0$ and finally $a = \alpha\gamma\delta = 0$, as desired. □

The following theorem gives sufficient conditions for singular 2×2 matrices over local rings to be a product of idempotents.

Theorem 10. *Let R be a local domain such that its radical $J(R) = gR$ with $\cap_n (J(R))^n = 0$. Let S be the 2×2 matrix ring over R. Then each matrix $A \in S$ with $l.ann(A) \neq 0$ is a product of idempotent matrices.*

Proof. Since $J = gR$ with $\cap_n(J(R))^n = 0$, we note that for any nonzero elements $x, y \in R$ there exist positive integers n, l such that $x = g^n u$ and $y = g^l v$, for some invertible elements $u, v \in U(R)$, where $U(R)$ denotes the set of invertible elements in R. If $n \geq l$ we can write $x = yc$ with $c := v^{-1}g^{n-l}u$. Clearly, $c \neq 0$. Since $l.ann(A) \neq 0$, we can assume that there exists $(x, y) \neq (0, 0)$ such that $(x, y)A = (0, 0)$. Furthermore, since $x = yc$, $y \neq 0$ and R is a domain, we have $(c, 1)A = (0, 0)$. This shows that UA has bottom row zero where $U = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ and so does the matrix UAU^{-1}. Since for every pair $(x, y) \neq (0, 0)$, one of them is a multiple of the other by invoking Lemma 1 (b), we obtain that $A = U^{-1}E_1 ... E_kU$, where E_i are idempotents and hence $A = (U^{-1}E_1U)(U^{-1}E_2U)...(U^{-1}E_kU)$ is a product of idempotents. □

Remark 11. *If the matrix A is such that $r.ann(A) \neq 0$ then the same proof will hold if we assume $J = Rg$ and $\cap_{i\geq 0} J(R)^i = 0$.*

4. Construction of Idempotents and Representation of singular matrices

The following lemma completes our "table" of Lemma 1 in an interesting way. The lemma proved below provides a further useful tool while working with idempotent matrices over a projective-free ring.

Lemma 12. *Let a, b, c, d be elements in a ring R.*

(a) *If $ca + db = 1$, then the matrix*

$$E = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}$$

is an idempotent matrix. If R is a domain and the matrix E is nonzero, then the converse is also true.

(b) *The matrix $\begin{pmatrix} ab+u & a \\ 0 & 0 \end{pmatrix}$, u a unit, is a product of idempotent matrices.*

(c) *If there exists $x \in R$ such that $a + bx \in U(R)$ then the matrix*

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

is a product of idempotent matrices.

Proof. (a) This is easily checked.
(b) $\begin{pmatrix} ab+u & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^{-1}ab+1 & u^{-1}a \\ -b(u^{-1}ab+1) & -bu^{-1}a \end{pmatrix}$.
(c) By hypothesis, there exist $x \in R$ and $u \in U(R)$ such that $a + bx = u$. Hence $va = vb(-x) + 1$ where $v = u^{-1}$. Using our previous table the conclusion follows since one can write $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} va & vb \\ 0 & 0 \end{pmatrix}$. Statement (b) above now yields the result. □

Remark 13. *The form of the 2×2 idempotent matrix that appears in Lemma 12 (a) is the only kind to consider in the case when the ring R is projective-free. Indeed in this case any 2×2 idempotent matrix A can be written as $A = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix}$ with the condition that $ca + db = 1$ (cf. the comments before Theorem 9).*

In view of this remark we look at the representation of a singular 2×2 matrix as product of idempotent matrices of the form PQ^t where P and Q are columns vectors such $Q^t P = 1$.

The next proposition translates the decomposition of a singular 2×2 matrix into a product of idempotents in terms of a family of equations. This generalizes Rao's theorem ([3], Theorem 5) to noncommutative domains and at the same time fills in the gaps in his original arguments (cf. [4]).

Proposition 14. *Let a, b be nonzero elements in a domain R such that $aR + bR = R$. Then the following are equivalent:*

(i) *There exist an integer $n > 0$ and elements $a_i, b_i, c_i, d_i \in R$, $i = 1, \ldots, n$ such that $a_1 = c_1 = 1$, $b_1 = 0$, $c_n = a$, $d_n = b$, $c_i a_i + d_i b_i = 1$, $1 \le i \le n$ and $c_i a_{i+1} + d_i b_{i+1} = 1$, $1 \le i \le n-1$.*

(ii) *There exist an integer $n > 0$ and elements $a_i, b_i, c_i, d_i \in R$, $1 \le i \le n$, such that the matrix $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ can be written as a product $E_1 \ldots E_n$ of idempotent matrices $E_i^2 = E_i$, where $E_i = \begin{pmatrix} a_i c_i & a_i d_i \\ b_i c_i & b_i d_i \end{pmatrix} = \begin{pmatrix} a_i \\ b_i \end{pmatrix} (c_i, d_i)$.*

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Proof. $(i) \Rightarrow (ii)$: Lemma 12 (a) shows that for $1 \leq i \leq n$, the matrix $E_i = \begin{pmatrix} a_ic_i & a_id_i \\ b_ic_i & b_id_i \end{pmatrix}$ is an idempotent. Moreover, we have

$$E_1 \cdots E_n = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} (c_1, d_1) \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} (c_2, d_2) \cdots \begin{pmatrix} a_n \\ b_n \end{pmatrix} (c_n, d_n),$$

and since $c_ia_{i+1} + d_ib_{i+1} = 1$, $1 \leq i \leq n-1$ and $a_1 = 1$, $b_1 = 0$, $c_n = a$, $d_n = b$, we obtain $E_1E_2 \cdots E_n = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$.

$(ii) \Rightarrow (i)$: We will construct elements a'_i, b'_i, c'_i, d'_i satisfying the conditions stated in (i). Since R is a domain and $E_i \neq 0$, Lemma 12 (a) shows that for any $1 \leq i \leq n$ we have $c_ia_i + d_ib_i = 1$. We can thus write

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} (c_1, d_1) \cdots \begin{pmatrix} a_n \\ b_n \end{pmatrix} (c_n, d_n), \text{with } c_ia_i + d_ib_i = 1, \ \ 1 \leq i \leq n.$$

If s stands for the product $s := \prod_{i=1}^{n-1}(c_ia_{i+1} + d_ib_{i+1})$, then we have $a = a_1sc_n$ $b = a_1sd_n$, $b_1sc_n = 0$ and $b_1sd_n = 0$. Since R is a domain, we easily get $b_1 = 0$ and $a_1c_1 = 1 = c_1a_1$. Thus $E_1 = \begin{pmatrix} 1 & a_1d_1 \\ 0 & 0 \end{pmatrix}$. We set $a'_1 = c'_1 = 1$, $b'_1 = 0$, $d'_1 = a_1d_1$. Then we have

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1, d'_1) \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \cdots \begin{pmatrix} a_n \\ b_n \end{pmatrix} (c_n, d_n), \text{ with } c_ia_i + d_ib_i = 1 \text{ for } 1 \leq i \leq n.$$

By comparing the entries on both sides we get $a = rc_n, b = rd_n$, where $r := (a_2 + d'_1b_2)\prod_{i=2}^{n-1}(c_ia_{i+1} + d_ib_{i+1})$. By hypothesis, there exist $x, y \in R$ such that $ax + by = 1$. This implies $rc_nx + rd_ny = 1$. This shows that $r \in U(R)$. Set $u_1 = (a_2 + d'_1b_2) \in U(R)$, $a'_2 = a_2u_1^{-1}$, $b'_2 = b_2u_1^{-1}$, $c'_2 = u_1c_2$ and $d'_2 = u_1d_2$. The matrix E_2 can be written $E_2 = \begin{pmatrix} a'_2 \\ b'_2 \end{pmatrix} (c'_2, d'_2)$. As per our definition $c'_1 = 1$ and so we have $c'_1a'_2 + d'_1b'_2 = a_2u_1^{-1} + d_1b_2u_1^{-1} = 1$. Once again Lemma 12 (a) shows that $c'_2a'_2 + d'_2b'_2 = 1$ (this can of course, be checked directly, as well). We then define $u_2 := (1, d_1)E_2 \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} = (1, d_1) \begin{pmatrix} a'_2 \\ b'_2 \end{pmatrix} (c'_2, d'_2) \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} = c'_2a_3 + d'_2b_3 = u_1(c_2a_3 + d_2b_3) \in U(R)$ (since u_2 is a factor of r). Set $a'_3 = a_3u_2^{-1}$, $b'_3 = b_3u_2^{-1}$, $c'_3 = u_2c_3$ and $d'_3 = u_2d_3$. The matrix E_3 can be written as $E_3 = \begin{pmatrix} a'_3 \\ b'_3 \end{pmatrix} (c'_3, d'_3)$. This gives $c'_3a'_3 + d'_3b'_3 = 1$ and $c'_2a'_3 + d'_2b'_3 = u_1(c_2a_3 + d_2b_3)u_2^{-1} = 1$. We continue this process by defining $u_3 := (1, d_1)E_2E_3 \begin{pmatrix} a_4 \\ b_4 \end{pmatrix} = c'_3a_4 + d'_3b_4 = u_2(c_3a_4 + d_3b_4)$, $a'_4 = a_4u_3^{-1}$, $b'_4 = b_4u_3^{-1}$, $c'_4 = u_3c_4$, $d'_4 = u_3d_4$ and so on. In general, we define for any $1 \leq i \leq n-1$, $u_i := u_{i-1}(c_ia_{i+1} + d_ib_{i+1})$, a factor of r and hence $u_i \in U(R)$. Set $a'_{i+1} := a_{i+1}u_i^{-1}, b'_{i+1} = b_{i+1}u_i^{-1}, c'_{i+1} := u_ic_{i+1}$ and $d'_{i+1} := u_id_{i+1}$. The elements $a'_i, b'_ic'_i, d'_i$, where $i \geq 2$, together with $a'_1 = 1 = c'_1$, $b'_1 = 0$, $d'_1 = a_1d_1$ will satisfy the required equalities. □

Corollary 15. *Let $a, b \in R$ be elements in a projective-free domain R such that $aR + bR = R$. Then the matrix $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is a product of idempotent matrices if and only if there exist an integer $n > 0$ and elements $a_i, b_i, c_i, d_i \in R$, $i = 1, \ldots, n$*

such that $a_1 = c_1 = 1$, $b_1 = 0$, $c_n = a$, $d_n = b$, $c_ia_i + d_ib_i = 1$, $1 \le i \le n$ *and* $c_ia_{i+1} + d_ib_{i+1} = 1$, $1 \le i \le n-1$.

Proof. By the above proposition we know that the conditions mentioned in the corollary are sufficient. Since any idempotent 2×2 matrix with coefficients in a projective-free domain is of the form $\begin{pmatrix} a \\ b \end{pmatrix}(c, d)$ with $ca + db = 1$, the implication $(ii) \Rightarrow (i)$ in above Proposition shows that the conditions are also necessary. □

5. Singular matrices over Bézout domains

We first mention the classical facts that for any two elements a, b in a right Bézout domain R both $aR + bR$ and $aR \cap bR$ are principal right ideals and such a domain is a right Ore domain. The next theorem is due to Amitsur [1]. We provide a different proof of this theorem. This proof is inspired by results of Cohn (cf. [6]).

Theorem 16. *A domain* R *is right Hermite if and only if it is right Bézout.*

Proof. Suppose R is right Hermite. Then for $a, b \in R$ there exist $d \in R$ and $P \in GL_2(R)$ such that $(a, b)P = (d, 0)$. Hence we have $dR \subseteq aR + bR$. Since we also have $(d, 0)P^{-1} = (a, b)$, we conclude that $aR + bR = dR \simeq R$. This yields R is right Bézout.
Conversely, suppose R is right Bézout and so it is a right Ore domain. Let $a, b \in R$. We first consider the case when $aR + bR = R$. We know $aR \cap bR$ is a principal right ideal, say, mR. Let $x, y, u, v \in R$ be such that $ax + by = 1$ and $au = m = bv$. We then obtain $a(xa - 1) = -bya \in mR$ and so there exists $c \in R$ such that $xa - 1 = uc$, $vc = -ya$. Similarly, from $axb = b(1 - yb) \in mR$, we get $d \in R$ such that $xb = ud$ and $1 - yb = vd$. Let us then consider the matrices

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad X := \begin{pmatrix} x & -u \\ y & v \end{pmatrix}.$$

We can check that the above relations give $XA = I$ and so $AX = I$, since R is embeddable in a division ring. This, in turn, leads to $(a, b)X = (1, 0)$.
In the general case, we have $aR + bR = dR$. We can write $a = da'$, $b = db'$ and then since R is a domain, $a'R + b'R = R$. So R is right Bézout as shown above. □

Definition 17. *We say that a ring* R *has the* IP_2 *property if every* 2×2 *singular matrix is a product of idempotent matrices.*

Of course, every ring for which singular matrices are products of idempotent matrices has IP_2. In particular, every commutative euclidean domain has the IP_2 property as shown by Laffey (cf. [12]).

Lemma 18. *A left (right) Bézout domain with stable range* 1 *has the* IP_2 *property.*

Proof. Let $A \in M_2(R)$ be a singular matrix. By Lemma 5 we may assume that the matrix has a bottom row equal to zero. Since matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ are products of idempotent matrices (cf. Lemma 1), we may further assume that the first row of A is unimodular. The hypothesis of stable range 1 and Lemma 12 (c) show that R has the IP_2 property. □

In the case of a commutative Bézout domain we can replace the stable range 1 hypothesis by the $IP2$ property and still get strong conclusions as shown in Proposition 19 and Corollary 21. Indeed proposition 19 provides a relationship between a decomposition of a singular matrix into a product of idempotent matrices and Bézout (equivalently, Hermite) domains.

Proposition 19. *Let a, b be elements in a commutative Bézout domain R with $aR + bR = R$. Then the following are equivalent.*

(i) $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ *is a product of n idempotent matrices.*

(ii) *There exist elements $r_0, r_1 \dots r_{2n-2} \in R$ such that*

$$(a, b) = (1, 0) \begin{pmatrix} r_{2n-2} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_{2n-3} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} r_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Proof. $(i) \Rightarrow (ii)$: Since a right Bézout domain is projective-free, Corollary 15 shows that there exist elements $a_i, b_i, c_i, d_i \in R$, $i = 1, \dots, n$ such that $a_1 = c_1 = 1$, $b_1 = 0$, $c_n = a$, $d_n = b$, $c_i a_i + d_i b_i = 1$ for $1 \le i \le n$ and $c_i a_{i+1} + d_i b_{i+1} = 1$, $1 \le i \le n-1$. Write

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} (c_1, d_1) \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} (c_2, d_2) \cdots \begin{pmatrix} a_n \\ b_n \end{pmatrix} (c_n, d_n).$$

Let us put $r_{2n-2} = c_n d_{n-1} - c_{n-1} d_n$ and $r_{2n-3} = a_{n-1} b_n - a_n b_{n-1}$. We then write, successively,

$$\begin{pmatrix} c_n & d_n \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_n & d_n \\ b_n & -a_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r_{2n-2} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_n & -a_n \\ c_{n-1} & d_{n-1} \end{pmatrix},$$

and

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} c_n & d_n \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r_{2n-2} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_{2n-3} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_{n-1} & d_{n-1} \\ b_{n-1} & -a_{n-1} \end{pmatrix}.$$

Continuing this process we put, for $1 \le i \le n-1$, $r_{2(n-i)} = c_{n-i+1} d_{n-i} - c_{n-i} d_{n-i+1}$ and $r_{2(n-i)-1} = a_{n-i} b_{n-i+1} - a_{n-i+1} b_{n-i}$. With these notations one gets:

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r_{2n-2} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} r_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & d_1 \\ b_1 & -a_1 \end{pmatrix},$$

where $c_1 = 1$, $b_1 = 0$, $a_1 = 1$. Hence

$$\begin{pmatrix} c_1 & d_1 \\ b_1 & -a_1 \end{pmatrix} = \begin{pmatrix} 1 & d_1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -d_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This, finally, yields that

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r_{2n-2} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} r_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

completing the proof with $r_0 = -d_1$.

$(ii) \Rightarrow (i)$: We are given $2n - 1$ elements r_i, $0 \le i \le 2n - 2$, and we want to produce $4n$ elements a_j, b_j, c_j, d_j, $1 \le j \le n$, satisfying the relations given in the Proposition 14. Let us show how to retrace our steps. First, we have $a_1 = 1 = c_1$, $b_1 = 0$, $d_1 = -r_0$, and $a_1 c_1 + d_1 b_1 = 1$. Suppose that we have constructed a_j, b_j, c_j, d_j satisfying the necessary relations for all $1 \le j \le i$ and let us show how to construct $a_{i+1}, b_{i+1}, c_{i+1}, d_{i+1}$. We determine a_{i+1} and b_{i+1} via the system of equations:

$$a_i b_{i+1} - b_i a_{i+1} = r_{2i-1}, \tag{1}$$

$$d_i b_{i+1} + c_i a_{i+1} = 1. \tag{2}$$

Since $a_i c_i + d_i b_i = 1$, the above system has a unique solution. To determine c_{i+1} and d_{i+1} we use the following equations,

$$d_i c_{i+1} - c_i d_{i+1} = r_{2i}, \tag{3}$$

$$a_{i+1} c_{i+1} + b_{i+1} d_{i+1} = 1. \tag{4}$$

Then $d_i b_{i+1} + c_i a_{i+1} = 1$ gives that the above system has a unique solution. □

The next corollary gives another proof of Lemma 2 in Laffey's paper [12].

Corollary 20. *Let R be a euclidean domain. Then R has the IP_2 property.*

Proof. We have to show that any singular matrix $A \in M_2(R)$ is a product of idempotent matrices. Lemma 5 shows that we may assume A is of the form $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. Let $d \in R$ be such that $aR + bR = dR$ and write $a = da'$, $b = db'$ for some $a', b' \in R$. We have

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & 0 \end{pmatrix}.$$

Since matrices $\begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}$ are always product of idempotent matrices, we may assume without loss of generality that $aR + bR = R$. The euclidean algorithm provides sequences of elements $q_0, q_1, \ldots, q_n, q_{n+1}$ and $r_0, r_1, \ldots, r_n$ in R such that $-b = aq_0 + r_0$, $a = r_0 q_1 + r_1$, $\ldots, r_{n-2} = r_{n-1} q_n + 1$, $r_{n-1} = q_{n+1}$. We then have:

$$(-b, a) = (a, r_0) \begin{pmatrix} q_0 & 1 \\ 1 & 0 \end{pmatrix} = (r_0, r_1) \begin{pmatrix} q_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_0 & 1 \\ 1 & 0 \end{pmatrix},$$

and finally

$$(-b, a) = (1, 0) \begin{pmatrix} q_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_n & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} q_0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Right multiplying this equality by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and using Proposition 19, we conclude that the matrix $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is a product of idempotent matrices, as required. □

Corollary 21. *If R is a commutative Bézout domain with the IP_2 property then every 2×2 invertible matrix is a product of elementary matrices and diagonal matrices with invertible diagonal entries.*

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R)$. We thus have $aR + bR = R$ and the IP_2 property shows that the matrix $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is a product of n idempotent matrices, for some n. Proposition 19 then shows that $(a, b) = (1, 0)U$, where

$$U = \begin{pmatrix} r_{2n-2} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_{2n-3} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} r_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in GL_2(R).$$

Since for $r \in R$ we have:

$$\begin{pmatrix} r & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

we conclude that matrices of the form $\begin{pmatrix} r & 1 \\ 1 & 0 \end{pmatrix}$ and hence also the matrix U are products of elementary and diagonal matrices with invertible diagonal entries. Let us write $AU^{-1} = \begin{pmatrix} 1 & 0 \\ c' & d' \end{pmatrix}$ for some $c', d' \in R$. We then have $A = \begin{pmatrix} 1 & 0 \\ c' & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & d' \end{pmatrix} U$. This shows that A is a product of elementary matrices and diagonal matrices with invertible diagonal entries, as desired. □

We say that a ring R has the IP property if every singular square matrix over R can be written as a product of idempotent matrices. Of course, the IP property implies the IP_2 property. Theorem 22 shows that in certain situations the converse is true, that is: IP_2 property implies IP property. The proof of this result follows the pattern of Laffey's proof [12].

Theorem 22. *Let R be a Bézout domain satisfying the IP_2 property. Then every singular matrix is a product of idempotent matrices if R has the GE_2 property. In particular, a Bézout domain with stable range 1 has the IP property.*

Proof. Let $A \in M_n(R)$ be a singular matrix. Lemma 5 shows that we may assume that the bottom row of A is zero. Let us write

$$A = \begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix},$$

where $B \in M_{n-1}(R)$ and other matrices are of appropriate sizes.

We now proceed by induction on n. The case $n = 1$ is trivial since R is a domain. If $n = 2$ this is the IP_2 property. Let $n \in \mathbb{N}$ be such that $n > 2$. Write

$$A = \begin{pmatrix} I_{n-1} & C \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}.$$

If B is singular, we apply induction hypothesis on B and we obtain A as a product of idempotents. So let us assume, B is nonsingular. Since R is left Bézout and hence left Hermite, by invoking GE_2 we can find a sequence of elementary matrices $E_1, \ldots, E_l \in M_{n-1}(R)$ such that $D := E_1 \cdots E_l B$ is an upper triangular matrix. For $M \in M_{n-1}(R)$, we define $\widehat{M} \in M_n(R)$ by

$$\widehat{M} := \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \in M_n(R).$$

We then have

$$A = \begin{pmatrix} E_1^{-1} \cdots E_l^{-1} D & C \\ 0 & 0 \end{pmatrix} = \widehat{E_1^{-1}} \widehat{E_2^{-1}} \cdots \widehat{E_l^{-1}} \begin{pmatrix} D & E_l \cdots E_1 C \\ 0 & 0 \end{pmatrix}.$$

Lemma 2 shows that $\widehat{E_1^{-1}}, \widehat{E_2^{-1}}, \cdots, \widehat{E_l^{-1}}$ are products of idempotent matrices. We thus have to show that the matrix

$$\begin{pmatrix} D & E_l \cdots E_1 C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} d_1 & \begin{pmatrix} d_2 & .. & .. & d_n \end{pmatrix} \\ 0 & D_1 \end{pmatrix}$$

is a product of idempotents. The last row of the triangular matrix D_1 is zero and hence our induction hypothesis shows that there exist idempotent matrices

$Y_1, \dots, Y_m$ such that $D_1 = Y_1 \cdots Y_m$. We may assume $Y_m \neq I_{n-1}$ and, since $D_1 Y_m = D_1$, we can write

$$\begin{pmatrix} d_1 & \begin{pmatrix} d_2 & .. & .. & d_n \end{pmatrix} \\ 0 & D_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & D_1 \end{pmatrix} \begin{pmatrix} d_1 & (d_2 \cdots d_n) \\ 0 & Y_m \end{pmatrix}.$$

The first left factor matrix on the right hand side is a product of idempotent matrices. Thus we only need show that the second factor matrix on the right hand side is a product of idempotent matrices. Now, since R is projective-free we know that Y_m is similar to a diagonal matrix with only ones and zeros on the diagonal (cf. [5], Proposition 0.4.7.). We claim that the number, say h, of ones on this diagonal is strictly positive or, in other words, we claim that $Y_m \neq 0$. Indeed, if $Y_m = 0$ then $D_1 = D_1 Y_m = 0$ and a row of the matrix D is zero (since $n \geq 3$). Hence $D = E_1 \cdots E_l B$ is singular, this implies that B is singular, a contradiction since B is non singular. We are thus reduced to show that a matrix of the form

$$\begin{pmatrix} d_1 & \begin{pmatrix} d_2 & .. & .. & d_{h+1} \end{pmatrix} & ... & d_n \\ 0 & I_h & 0 & 0 \\ .. & .. & .. & .. \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

for some $h > 0$ is a product of idempotent matrices. This matrix is similar to the following:

$$\begin{pmatrix} I_h & 0 & 0 & \cdots & 0 \\ (d_2, \dots, d_{h+1}) & d_1 & d_{h+2} & \cdots & d_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Performing row elementary operations on the first $h+1$ rows we reduce the above matrix to the following:

$$\begin{pmatrix} I_h & 0 & 0 & \cdots & 0 \\ 0 & d_1 & d_{h+2} & \cdots & d_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I_h & 0 \\ 0 & * \end{pmatrix},$$

where the bloc matrix * is an $(n-h) \times (n-h)$ matrix with the last row zero. Since these operations can be accomplished by multiplying on the left by products of idempotent matrices the induction hypothesis applied to the matrix * completes the proof.

The proof of the particular case is clear since Lemma 7 and 18 show that stable range 1 implies GE_2 and IP_2, respectively. □

As a special case of the above theorem, the following corollary, parts (a) and (c), gives Laffey's theorem (cf. [12]) and Rao's theorem (cf. [3], Theorem 2), respectively.

Corollary 23. *Let R be a domain which is any one of the following types:*

(a) *a euclidean domain,*
(b) *a local domain such that its radical $J = Rg = gR$ with $\cap Rg^n = 0$,*
(c) *a commutative principal ideal domain with IP_2, or*
(d) *a local Bézout domain.*

Then every singular matrix over R is a product of idempotent matrices (in other words, R has the IP property).

Proof. (a) It is clear that a euclidean domain is Bézout. On the other hand, Corollaries 20 and 21 show that R has the IP_2 as well as the GE_2 property. Therefore, by the above theorem R has the IP property.

(b) It is clear that a local ring has stable range 1 and by our hypothesis R is a valuation domain (as in the proof of Theorem 10), and hence a Bézout domain. The particular case mentioned in the above theorem yields the result.

(c) This follows from the above theorem and from Corollary 21.

(d) Since a local domain has stable range 1, the result follows from the particular case of the theorem. □

6. Endomorphisms of Injective Modules

Finally, we consider the endomorphism ring S of an injective module M. Recall, from the introduction, that an element $s \in S$ is called right singular if $rann(s) \neq 0$. We know that if any ring has the IP property, then it need not be of stable range 1. However, for the endomorphism ring of an injective (or even quasi-injective) module, we have the following theorem. Its proof is straightforward.

Theorem 24. *Let M_R be an injective module and $S = End(M_R)$. If each right singular element $s \in S$ can be expressed as a product of idempotents, then S has stable range 1.*

Proof. Let $J = J(S)$ denote the Jacobson radical of $S = End(M_R)$. Lemma 3 shows that S is Dedekind finite. It is a folklore that S/J is also Dedekind finite. We provide its proof for reference only. For $x \in S$, let us write $\overline{x} := x+J$. If $a, b \in S$ are such that $\overline{a}\overline{b} = \overline{1}$ then $1 - ab \in J$ and hence $1-(1-ab) = ab \in U(S)$, the set of units of S. Since S is Dedekind finite, we also have $ba \in U(S)$. Thus there exists $c \in S$ such that $bac = 1$ and we get $\overline{bac} = \overline{1}$. Since $(\overline{b}\overline{a} - \overline{1})\overline{b} = \overline{0}$, we obtain by post multiplying this by $\overline{ac}$, $\overline{b}\overline{a} = \overline{1}$, as desired. It is well-known that S/J is a regular right self-injective ring (cf. [13], Theorem 13.1). Because a von Neumann regular right self-injective Dedekind finite ring is a unit regular ring (cf. [8], Theorem 9.17), it follows that $S/J(S)$ is a unit regular ring. This implies $S/J(S)$ has stable range 1.

We now prove S has stable range 1. Let $aS + bS = S$. Then $(a + J(S))(S + J(S)) + (b + J(S))(S + J(S)) = S + J(S)$. This gives $(a + J(S)) + (b + J(S))(u + J(S))$ is invertible for some $u + J(S)$ in $S/J(S)$. This implies that there exists $v \in S$ such that $(a + bu)v - 1 \in J(S)$ and hence $(a + bu)v$ is invertible. Since S is Dedekind finite $a + bu$ is invertible. This concludes the proof that S is of stable range 1. □

Remark 25. *Since the endomorphism ring of an infinite dimensional vector space is not of stable range* 1*, it follows that every right singular (equivalently non monomorphism) endomorphism can be expressed as a product of projections if and only if the vector space is of finite dimension.*

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