DECOMPOSITION OF SINGULAR ELEMENTS OF AN ALGEBRA INTO IDEMPOTENTS, A SURVEY

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Abstract. In this paper, we provide a survey of some of the prominent works that were published from the 1960's related to the problem of writing a singular matrix as a product of idempotent matrices. The question of the decomposition of a nonunit *element* of a regular ring is also described in particular the results of Hannah and O'Meara occupies a full section. The problem of decomposition of nonnegative matrices as product of nonnegative idempotents ends the paper. Some new results and techniques are also offered.

Introduction

This survey article is meant to highlight the important discoveries on the question of presenting non-units of an algebra as a product of idempotents. No attempt is made to give full details, but we direct the interested reader to the references while giving some new proofs.

In the introduction we will go over the history of the problem and give the main discoveries. Open questions are also listed at the end.

This area of research was prompted by the work of Howie ([35]) who showed that any non-injective mapping from a finite set into itself is a product of idempotents. J. A. Erdos ([23]) considered $n \times n$ matrices over a field and showed that any singular matrix is a product of idempotents. This decomposition into idempotents of matrices or, more generally, of elements of any given Algebra, caught the attention of a number of mathematicians.

Laffey ([37], Theorem 1 and 2) proved that this decomposition into idempotents holds for singular matrices over division rings and commutative euclidean domains.

Dawlings (cf. [14]) refined the result obtained by Erdos and showed that any singular linear map of an n-dimensional vector space to itself is a product of at most n idempotent linear maps of rank $n-1$.

Ballantine ([11]) showed that if F is a field and $k, n \in \mathbb{N}$, then a matrix $A \in M_n(F)$ can be written as a product of k idempotent matrices if and only if $rank(I - A) \leq k.nullity(A)$.

Reynolds and Sullivan (cf. [46]) studied the question of the minimal number of idempotents needed in the decomposition process for linear transformations of an infinite dimensional vector space.

V. Gould [16], [17], [26] and her coauthors have also considered the problem of decomposition into idempotents using the techniques and machinery of Semigroups. In particular, jointly with Fountain (cf. [24]) she studied the case of matrices over

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principal ideal domains and, using semigroup theory, obtained a series of equivalent conditions for the existence of the decompositions into idempotents. This seminal work was pursued by Ruitenburg (cf [47]) who studied the case of matrices over Bézout domains.

Reynolds([47]), Sullivan([49]), Fountain([24]), Ballantine([11]) Ruitenburg([47]), Dandan-Gould-Quinn ([17]), Hannah and O'Meara ([30],[31]), Alahmadi-Lam-Jain-Leroy([5]), among others continued the analysis of the problem.

The results of Ballantine, Reynolds and Sullivan on the number of idempotents needed in a decomposition of an element were improved considerably by Hannah-O'Meara ([31], Proposition 1.1) in the following theorem: Let R be a regular ring and let S be the multiplicative subgroup generated by idempotents. Then $\Delta(S) \leq index(R)$, where $\Delta(S)$ denotes the depth of the semigroup S.

As a consequence they showed that in a regular ring, if a is a nilpotent element such that $a^n = 0$, then a is a product of n idempotents.

The contents of our paper are as follows:

In the first section, we consider matrices over arbitrary rings, not necessarily commutative, and prove some new results.

The second section deals with matrices over division rings and valuation rings or more generally, over local rings. We provide new proofs for decomposition of singular matrices over such rings into idempotents.

In the third section we consider quasi-Euclidean domains and give a proof that a singular matrix A (that is $l(A) \neq 0$) is a product of idempotents.

The main results proved by Hannah and O'Meara on von Neumann regular rings whose non units can be decomposed into idempotents are given in Section 4.

The last section 5 deals with the question whether a nonnegative real matrix (like stochastic matrix, (0-1) matrix or totally positive matrix) is a product of nonnegative idempotents.

1. Singular matrices over general rings

In addition to considering rings over which every singular matrix is a product of idempotent matrices, we look at the special types of singular matrices that always admit such a decomposition (whatsoever is the base ring). Such matrices include strictly upper triangular matrices, quasi permutation matrices, or quasi elementary matrices (the latter kind of matrices are considered here for the first time). These matrices can sometimes be used as a tool to prove that other matrices are product of idempotent matrices.

Let us first define two important families of matrices.

Definitions 1. (1) *A matrix $A \in M_n(R)$ with coefficients in a ring R will be called a quasi-permutation matrix if each row and each column of A has at most one nonzero element.*

(2) *An elementary matrix is either a permutation matrix or a matrix of the form $I + ae_{ij}$. A quasi elementary matrix is a matrix obtained by replacing at least one row of an elementary matrix by the zero row.*

Remarks 2. (a) *A quasi-permutation matrix can be singular and, in this case, it has at least one zero row and one zero column. We will mainly work with rows but the analogous properties for columns also hold (acting on the right with given permutation matrices).*

(b) *Of particular importance in our discussions are the quasi-permutation matrices, denoted by $P_{\sigma,l}$, $\sigma \in S_n$, $l \in \{1, \ldots, n\}$ obtained from P_σ by changing the nonzero element of the l^{th} row of P_σ to 0. We thus have*

$$P_{\sigma,l} = \sum_{\substack{i=1 \\ i \neq l}}^{n} e_{i,\sigma(i)}.$$

(c) *We observe that the $\sigma(l)^{th}$ column of the matrix $P_{\sigma,l}$ is the only column that is zero.*

(d) *It may be of interest to the reader to know the following main steps often used to prove that a matrix is a product of idempotents.*
- *The first step is to check whether left and right annihilators are nonzero.*
- *The second step is to try to show that the given matrix is conjugate to a matrix having a zero row (or column). This always happens if the base ring is a Hermite domain.*
- *Having a zero row, we can assume, using quasi permutation matrices, that the last row (or column) is zero.*
- *Assuming, for instance, that the last row is zero, we try to use an induction on the $(n-1) \times (n-1)$ top left matrix block, either directly or after some more permutations on the columns.*

These steps are used on different occasions in the paper.

(e) *It is also important to notice that sometimes there is a relation between presentations of a noninvertible matrix as a product of idempotent matrices and decompositions of an invertible matrix as a product of elementary matrices. These relations are studied, in particular, by Salce-Zanardo (cf. [48]) and Facchini-Leroy (cf. [20]).*

In the next theorem we gather some specimens of singular matrices that are always product of idempotent matrices.

Theorem 3. *The following singular matrices in $M_n(R)$, $n > 1$, are always product of idempotent matrices, for any ring R.*

(1) *A matrix $A \in M_n(R)$ of the form*

$$\begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} B & 0 \\ L & 0 \end{pmatrix}$$

where B is a product of idempotent matrices.

(2) *Any strictly upper (lower) triangular matrix.*

(3) *Any singular 0-1 matrix (that is a matrix with 0, 1 entries only).*

(4) *For any $\sigma \in S_n$ and any $1 \leq l \leq n$, the matrix $P_{\sigma,l}$ is a product of idempotent matrices.*

(5) *Any diagonal matrix with one zero row.*

(6) *Any quasi permutation matrix is a product of idempotent matrices.*

(7) *Any quasi elementary matrix is a product of idempotent matrices.*

Proof. If B is a product of idempotent matrices, say $B = E_1 \dots E_l$, then

$$\begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_r & C \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 & 0 \\ 0 & I_{n-r} \end{pmatrix} \cdots \begin{pmatrix} E_l & 0 \\ 0 & I_{n-r} \end{pmatrix}$$

Since every matrix on the right hand side is an idempotent matrix we get the desired conclusion.
The second kind of matrix is treated similarly working with columns instead of rows. We may warn the reader that a little care is needed, since our base rings are non commutative (cf. Remark 4).

(2) This is obtained using (1) and an by induction.

(3) If a (0-1)-matrix A is singular this matrix can be written as a product of idempotent matrices in $M_n(\mathbb{Z})$, say $A = E_1 \cdots E_l$ where $E_i^2 = E_i \in M_n(\mathbb{Z})$ since $\mathbb{Z}$ is a commutative Euclidean domain (cf. [37]). If φ stands for the usual map $\mathbb{Z} \longrightarrow R$ where $1 \mapsto e$ (e is the identity of R) and $\widehat{\varphi} : M_n(\mathbb{Z}) \longrightarrow M_n(R)$ be its natural extension, we have that $A = \widehat{\varphi}(A) = \widehat{\varphi}(E_1) \cdots \widehat{\varphi}(E_l)$. This gives the result since for any $1 \leq i \leq l$, $\widehat{\varphi}(E_i) \in M_n(R)$ is idempotent.

(4) This is a direct consequence of the Statement 3, since a matrix of the form $P_{\sigma,l}$ is clearly a singular (0-1) matrix.

(5) We proceed by induction on n. If $n = 2$, the diagonal matrix with a zero row can have only two forms as given below. For any $a \in R$:

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We now assume that $n > 2$. Let us first remark that if A is a diagonal matrix with its 1^{th} row zero, we have $A = P_{(l,n),l} P_{(l,n)} A$. Since by Statement (4) $P_{(l,n),l}$ is a product of idempotent matrices, it is enough to show that $P_{(l,n)}A$ is a product of idempotent matrices. If $l \neq n$, $P_{(l,n)}A$ is of the form

$$P_{(l,n)}A = \begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix}$$

where B is an $(n-1)\times(n-1)$ diagonal matrix with one row zero (the l^{th} row of B comes from the bottom row of A). In this case the induction hypothesis implies that B is a product of idempotent matrices and Statement (1) above finishes the proof. If $l = n$, the last row and last column of A are zero and $A = AP_{(n-1,n)}P_{(n-1,n),n}$. Since Statement (4) above shows that $P_{(n-1,n),n}$ is a product of idempotent matrices, we only need to show that $AP_{(n-1,n)}$ is a product of idempotent matrices. This matrix is of the form

$$AP_{(n-1,n)} = \begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix}$$

where B is a diagonal matrix with its last row (and column) zero. The induction hypothesis and Statement (1) give the proof.

(6) For a general quasi permutation matrix, it is enough to remark that such a matrix is a product of the form $diag(a_1, \dots, a_n)P_{\sigma,l}$, where the first matrix stands for a diagonal matrix having some of its entries equal to zero . Note $a_l = 0$. Both matrices of the right hand side are then product of idempotent matrices and hence any quasi-permutation matrix is a product of idempotent matrices.

(7) An $n \times n$ quasi elementary matrix A that is not a quasi permutation matrix must be obtained from a matrix $I_n + ae_{ij}$, $a \in R$, by replacing at least one row by the zero row. As we have seen in the proof of (5) above, we may assume that the zero row is the last one. Then the resulting matrix is itself an idempotent.

□

Remarks 4. (1) *One of the very useful techniques for obtaining a decomposition into idempotents is as follows. Let A be a matrix with l^{th} row as zero and let U be an invertible matrix. If U_l stands for the matrix obtained from U by making the l^{th} row as zero, then $A = U_l(U^{-1}A)$. If we know that U_l is a product of idempotent matrices then it is enough that $U^{-1}A$ is a product of idempotents. In particular, since singular quasi-permutation matrices are product of idempotents, when a matrix has a zero row (or column) we can always assume that this is the last row (or column).*

(2) *It is not true, in general, that the transpose of an idempotent matrix is an idempotent. For instance if R is a ring which is not Dedekind finite, we can find $a, b \in R$ such that $ba = 1$ but $ab \neq 1$. We then also have that $(ab)^2 = ab$ but $a^2b \neq a$. It is then easy to check that*

$$A := \begin{pmatrix} ab & a \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} ab & 0 \\ a & 0 \end{pmatrix}$$

are such that $A^2 = A$ but $B^2 \neq B$. Let us remark that the matrix B is, nevertheless, the product of two idempotent matrices, as the reader may check.

2. Matrices over division rings or local rings

Let us start by mentioning the following results obtained by Erdos for matrices with coefficients in fields and generalized by Laffey for matrices over division rings. We will give a proof of this result which is different and more direct than the original one.

The first part of the following result is a folklore. The second part can be found in [37].

Lemma 5 ([37])**.** *A matrix $A \in M_n(D)$, where D is a division ring, is invertible if and only if $l(A) = 0$ if and only if $r(A) = 0$. If $l(A) \neq 0$, then A is similar to a matrix having its last row equal to zero.*

Theorem 6 (J.A. Erdos, [23], Laffey [37], Theorem 1)**.** *Let D be a division ring and $A \in M_n(D)$ be a singular matrix. Then A is a product of idempotent matrices in $M_n(D)$.*

Proof. By virtue of the above Lemma 5, we may assume that A is of the form:

$$A = \begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix},$$

where B is an $(n-1) \times (n-1)$ matrix. If $n = 2$ and $b = B \neq 0$ then

$$\begin{pmatrix} b & c \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & b^{-1}c \\ 0 & 0 \end{pmatrix}$$

If $b = 0$ then the matrix is strictly upper triangular and the result follows from Theorem 3(2).

Now let us assume that $n > 2$ and that the result is true for singular matrices of size strictly less than n.

- If B is singular then by the induction hypothesis B is a product of idempotent matrices, say $B = E_1 \dots E_r$, for some $r \in \mathbb{N}$ this gives
$$A = \begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{n-1\times n-1} & C \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 & 0 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} E_r & 0 \\ 0 & 1 \end{pmatrix}.$$
This gives the desired conclusion.
- If B is invertible we can write
$$\begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ I_{n-1,n-1} & 0 \end{pmatrix} \begin{pmatrix} I_{n-1,n-1} & B^{-1}C \\ 0 & 0 \end{pmatrix}$$
On the right hand side of this equation, the second matrix is lower triangular, the third matrix is an idempotent, so it remains to show that the first matrix is a product of idempotents.

We just write
$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B' & D \\ 0 & 0 \end{pmatrix}$$
where $B' \in M_{n-1,n-1}(D)$ has its first column zero and D is a column vector. This means that B' is singular and the induction hypothesis implies that B' is in fact a product of idempotents, say $B' = E_1 \dots E_r$ where $E_i^2 = E_i$ for any $1 \leq i \leq r$. We then have
$$\begin{pmatrix} B' & D \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{n-1,n-1} & D \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 & 0 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} E_r & 0 \\ 0 & 1 \end{pmatrix}.$$
This yields the proof of the theorem. □

In the case of singular matrices, $A \in M_n(k)$ over a field k, a bound on the number of idempotent matrices was given by Ballantine:

Theorem 7 (Ballantine [11]). *Let F be an arbitrary field, n and k be arbitrary positive integers, and A be an $n \times n$ matrix over F. Then A is a product of k idempotent matrices over F if and only if $\rho(I - A) < k\nu(A)$, where ρ and ν denote the rank and the nullity, respectively.*

This was later improved by Hannah and O'Meara (cf. section 4) that includes a larger family of algebras.

Let us also mention that Dawlings (cf. [14]) showed that any singular linear map of an n-dimensional vector space to itself is a product of at most n idempotent linear maps of rank $n - 1$.

Laffey, in the paper mentioned above ([37]), also proved that the decomposition result in Theorem 6 is also true for matrices with coefficients in a Euclidean domain. The case of noncommutative quasi-Euclidean domain was treated by Alahmadi, Jain, Lam and Leroy (cf. the section on quasi-Euclidean domains)

Laffey also showed a kind of converse of the above Theorem 6. Let us mention explicitly his result.

Theorem 8 ([38]). *Let k be a field and A be a finite dimensional k-algebra. Suppose that each non-invertible element of A can be expressed as a product of idempotent elements in A. Then A satisfies one of the following:*

(1) A is isomorphic to $M_n(D)$, the algebra of $n\times n$ matrices over a division algebra D,where D is finite dimensional over F, for some $n\in\mathbb{N}$;
(2) $k = GF(2)$ and A is the direct sum of $s > 1$ copies of $GF(2)$;
(3) $k = GF(2)$, $A/J(A)$ is the direct sum $k \oplus k$ where $J(A)$ denotes the Jacobson radical of A and A has a homomorphic image A/I, for some ideal I, isomorphic to the algebra T_2, where T_2 is the upper 2×2 triangular matrices over k.

We will now consider the case of projective-free rings and, in particular, local rings. We will need the following lemma which must be well-known.

Lemma 9. *A local right Bézout domain is a right valuation domain.*

Proof. Let $a, b \in R$. We need to prove that either $aR \subseteq bR$ or $bR \subseteq aR$. We know that there exists $g \in R$ such that $aR + bR = gR$. Hence there exist $u, v, a_1, b_1 \in R$ such that $au + bv = g, a = ga_1$ and $b = gb_1$. This leads to $g(1 - a_1u - b_1v) = 0$. Since R is a domain we get that either $g = 0$ and then $a = b = 0$. If $g \neq 0$ then $1 - a_1u - b_1v = 0$. Since R is a local ring the second possibility implies that either $a_1u \in U(R)$ or $b_1v \in U(R)$. In the first case we get $aR + bR = gR = ga_1uR = auR \subseteq aR$ and hence $bR \subseteq aR$. Similarly in the second case we obtain $aR \subseteq bR$. □

It was shown in [2] that if R is a projective free ring over which matrices with nonzero left and right annihilators are product of idempotents then the ring is actually a domain. In fact R is a Bézout domain.

Theorem 10. *Let R be a projective-free ring such that any 2×2 matrix with $l(A) \neq 0$ or $r(A) \neq 0$ is a product of idempotent matrices. Then R is a Bézout domain.*

Proof. To prove that the ring is a domain, we refer the reader to [2] Theorem 9. We will now show that R is right K-Hermite i.e. that for any $a, b \in R$ there exists $Q \in GL_2(R)$ such that $(a, b)Q = (c, 0)$. Let $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_2(R)$. Since $l(A) \neq 0$, we can write $A = TE$ for some $T, E \in \mathbb{M}_2(R)$ where $E^2 = E \neq I_2$. Then there exists $Q \in \mathrm{GL}_2(R)$ such that $(a, b) = (r, 0)\, Q$ for some $r \in R$ (and hence $aR + bR = rR$). In particular, if all matrices $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ are products of idempotents in $\mathbb{M}_2(R)$, R must be a right K-Hermite ring. A classical result of Amitsur [6] shows that this implies that R is a Bézout domain. □

Corollary 11. *Let R be a local ring. Suppose that every 2×2 matrix over R having nonzero right or left annihilator is product of idempotents. Then R must be a valuation domain.*

Proof. Since R is local, R is projective free. The above theorem shows that R is in fact a Bézout domain. Lemma 9 yields the desired conclusion. □

Fountain's results ([24]) were based on semigroup techniques. Inspired by Fountain's work Ruitenburg proved that the singular matrices with coefficients in a (not necessarily commutative) Bézout domain are product of idempotent matrices if and only if invertible matrices are product of elementary matrices.

Salce and Zanardo ([48]) also considered this relation between the two decompositions. They studied the case of commutative domains but their results were generalized to a noncommutative domains by Facchini and Leroy. To present the latter result we need to introduce a few notions.

Definitions 12.

- *Let A, B, C be three right R-modules and $\alpha\colon A \to B$, $\beta\colon B \to C$ be homomorphisms. We say that the pair (α, β) is a* consecutive pair *if $im(\alpha) \oplus \ker(\beta) = B$*
- *We say that a ring R is* right n-regular *if for every $n \times n$ invertible matrix $M = (b_{ij}) \in M_n(R)$ there exists some $i, j = 1, 2, \dots, n$ such that $r(b_{ij}) = 0$.*
- *Let r, n be integers, $0 \le r \le n$. For a ring R we define $\mathcal{F}_{n,r} := \{ A \subseteq^{\oplus} R^n_R \mid A \cong R^r_R$ and $R^n_R/A \cong R^{n-r}_R \}$*

Theorem 13 (Facchini-Leroy, [20])**.** *Let R be a ring with the IBN property and let $n \ge 1$ be a fixed integer. Suppose that R is m-right regular for every $m = 2, 3, \dots, n$ and that for any two decompositions of $R^n = A \oplus X = Y \oplus B$ with A, B free right R-modules of ranks, respectively, $n-1, 1$, the submodules X, Y are free right R-modules. Then the following conditions are equivalent.*

(S_n) *For every $r = 1, 2, \dots, n$ and every free direct summands $A \subseteq^{\oplus} R^n_R$ and $B \subseteq^{\oplus} R^n_R$, with A, B free R-modules of rank $r, n-r$ respectively, there exist direct-sum decompositions*

$$R^n_R = A_1 \oplus B_1 = A_2 \oplus B_1 = A_2 \oplus B_2 = \dots = A_k \oplus B_{k-1} = A_k \oplus B_k$$

with $A = A_1$ and $B = B_k$.

(HI_n) *For every $r = 1, 2, \dots, n$ and every free direct summands $A \subseteq^{\oplus} R^n_R$ and $B \subseteq^{\oplus} R^n_R$, with A, B free R-modules of rank $r, n-r$ respectively, there exists an endomorphism β of R^n_R with $\mathrm{im}(\beta) = A$ and $\ker(\beta) = B$, such that β is a product $\varepsilon_1 \dots \varepsilon_k$ where $\epsilon_1, \dots, \epsilon_k$ are consecutive idempotents $(\mathcal{F}_{n,n-1}, \mathcal{F}_{n,1})$-endomorphisms.*

($\mathrm{S}_{n,1}$) *For every free direct summands $A \subseteq^{\oplus} R^n_R$ and $B \subseteq^{\oplus} R^n_R$, of rank $n-1, 1$ respectively, there exist direct-sum decompositions*

$$R^n_R = A_1 \oplus B_1 = A_2 \oplus B_1 = A_2 \oplus B_2 = \dots = A_k \oplus B_{k-1} = A_k \oplus B_k$$

with $A = A_1$ and $B = B_k$.

($\mathrm{HI}_{n,1}$) *For every free direct summands $A \subseteq^{\oplus} R^n_R$ and $B \subseteq^{\oplus} R^n_R$, with A, B free R-modules of rank $n-1, 1$ respectively, there exists an endomorphism β of R^n_R with $\mathrm{im}(\beta) = A$ and $\ker(\beta) = B$, which is a product $\beta = \varepsilon_1 \dots \varepsilon_k$ of consecutive idempotent $(\mathcal{F}_{n,n-1}, \mathcal{F}_{n,1})$-endomorphisms.*

(GE_n) *Every invertible $n \times n$ matrix is a product of elementary matrices.*

Let us remark that the hypotheses of the above theorem are satisfied by a right and left Bézout domain. So that the equivalences of Theorem 13 generalize Ruitenburg results.

Ruitenburg's paper inspired many other works. In particular, some similar results have been proved for quasi-Euclidean rings (cf. [5]). Another direction was pursued by Reynolds and Sullivan ([46]) who studied the decomposition problem for the linear transformations of an infinite dimensional vector space. Using the ring structure of the algebra of linear transformations Hannah and O'Meara generalized the results to von Neumann regular rings (cf. section 4).

3. Quasi-Euclidean domains

As we mentioned above the case of commutative Euclidean domains was studied by Laffey. This was generalized in [5] where noncommutative quasi-Euclidean rings are treated. We recall the results from that paper which are relevant to our discussion and give a complete proof of the decomposition theorem. This proof didn't appear in [5]. Firstly, let us recall the definition of a Euclidean pair.

Definition 14. *An ordered pair (a, b) over any ring R is said to be a right Euclidean pair if there exist elements (q_1, r_1), ..., $(q_{n+1}, r_{n+1}) \in R^2$ (for some $n \geq 0$) such that $a = bq_1 + r_1$, $b = r_1q_2 + r_2$, and*

$$(*) \qquad r_{i-1} = r_iq_{i+1} + r_{i+1} \text{ for } 1 < i \leq n, \text{ with } r_{n+1} = 0.$$

The notion of a left Euclidean pair is defined, similarly. A ring is right quasi-Euclidean if all its pairs are right Euclidean pairs.

We begin this section with the following useful though quite straightforward proposition. This was not explicitly mentioned in [5] but will be used in the proof of the theorem 22.

Proposition 15. *Let R be a right (resp. left) quasi-Euclidean ring. Then*

1. *for any $(a_1, \ldots, a_n) \in R^n$, there exist an invertible matrix $Q \in GL_n(R)$ and an element $r \in R$ such that $(a_1, \ldots, a_n)Q = (r, 0, \ldots, 0)$ (resp.$Q(a_1, \ldots, a_n) = (0, \ldots, 0, r)$), and*
2. *for any matrix $A \in M_n(R)$ there exists an invertible matrix $Q \in GL_n(R)$ such that AQ (resp. QA) is a lower (resp. upper) triangular matrix.*

Proof. We will prove the first statement by induction on n. For the case $n = 2$ we remark that, with the notations as in the above definition, we can translate the fact that (a, b) is a Euclidean pair by the following equation

$$(a, b) = (b, r_1)\begin{pmatrix} q_1 & 1 \\ 1 & 0 \end{pmatrix} = (r_1, r_2)\begin{pmatrix} q_2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} q_1 & 1 \\ 1 & 0 \end{pmatrix} = \cdots = (r_n, 0)P(q_{n+1}) \ldots P(q_1)$$

where, for $q \in R$, we have $P(q) = \begin{pmatrix} q & 1 \\ 1 & 0 \end{pmatrix}$. This yields the result.

Let us now consider an element $(a_1, \ldots, a_n) \in R^n$ with $n > 2$. The case $n = 2$ shows that there exist an invertible matrix $Q_1 \in GL_2(R)$ and an element $s \in R$ such that $(a_{n-1}, a_n)Q_1 = (s, 0)$. This gives that

$$(a_1, a_2, \ldots, a_n)\begin{pmatrix} I_{n-2} & 0 \\ 0 & Q_1 \end{pmatrix} = (a_1, \ldots, a_{n-2}, s, 0)$$

By induction there exists $Q_2 \in GL_{n-1}(R)$ such that $(a_1, \ldots, a_{n-2}, s)Q_2 =(r, 0 \ldots, 0)$. This gives

$$(a_1, a_2, \ldots, a_n)\begin{pmatrix} I_{n-2} & 0 \\ 0 & Q_1 \end{pmatrix}\begin{pmatrix} Q_2 & 0 \\ 0 & 1 \end{pmatrix} = (r, 0 \ldots, 0),$$

as desired.

Let $A = (a_{ij}) \in M_n(R)$ and, for $1 \leq i \leq n-1$, define $A_i = (a_{i,i}, a_{i,i+1}, \ldots, a_{i,n})$. By Statement (1) we can find $P_i \in GL_{n-i+1}(R)$ and $r_i \in R$ such that $A_iP_i = (r_i, 0, \ldots, 0)$. For $i = 1$, we put $Q_1 = P_1$ and for $i > 1$, we define $Q_i = \begin{pmatrix} I_{i-1} & 0 \\ 0 & P_i \end{pmatrix}$. Then we easily obtain that $AQ_1Q_2 \cdots Q_{n-1}$ is a lower triangular matrix. □

The next Theorem was given in [5] and we omit the proof.

Theorem 16. *Let $(a, b) \in R^2$ be a right Euclidean pair. Then the matrix*

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

is a product of idempotent matrices.

Remark 17. (1) *If the pair (a, b) is left Euclidean instead, a similar decomposition into products of idempotents holds for the matrix $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$. We leave the details to the reader.*

(2) *In Theorem 16, the condition that the matrix is a product of idempotents is only necessary but not sufficient for (a, b) to be a Euclidean pair. To see this, let $\theta = \sqrt{-5}$ and $R = \mathbb{Z}[\theta]$ be the full ring of algebraic integers in the number field $\mathbb{Q}[\theta]$. The Dedekind domain R has class number 2, and its class group is generated by the ideal $-2R + (\theta + 1) R$ (see* [40, Example 2.19D]*). The matrix $E = \begin{pmatrix} -2 & \theta + 1 \\ \theta - 1 & 3 \end{pmatrix}$ over R has trace 1 and determinant 0, and so $E^2 = E$. Thus, $A := \begin{pmatrix} -2 & \theta + 1 \\ 0 & 0 \end{pmatrix}$ has a idempotent factorization, namely, $diag\,(1, 0)\, E$. However, the ideal $-2R + (\theta + 1) R$ is (by choice) not a principal ideal. In particular, $(-2, \theta + 1)$ is not a Euclidean pair over R, according to Theorem 6 (a).*

The following theorem appears as Theorem 20 in [5].

Theorem 18. *Let R be a right quasi-Euclidean domain and let $A \in \mathbb{M}_2(R)$ be such that $l.ann\,(A) \neq 0$. Then A is a product of idempotent matrices.*

The following lemmas will be needed in the proof of the next theorem.

Lemma 19. *If $A \in M_n(R)$ where R is a right Euclidean domain and $l.ann(A) \neq 0$, then A is similar to a matrix with its last row zero.*

Lemma 20. *Let R be a right quasi-Euclidean ring and suppose that $E \in M_n(R)$ is an idempotent matrix. Then there exists an invertible matrix $P \in M_n(R)$ such that $PEP^{-1} = \begin{pmatrix} I_h & 0 \\ 0 & 0 \end{pmatrix}$, for some $1 \leq h \leq n$.*

Proof. Let us first remark that for any $1 \leq l < n$ and for any $C \in M_{l \times n}(R)$, we have

$$\begin{pmatrix} I_l & C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_l & -C \\ 0 & I_{n-l} \end{pmatrix} \begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_l & C \\ 0 & I_{n-l} \end{pmatrix}$$

This shows that an idempotent matrix $\begin{pmatrix} I_l & C \\ 0 & 0 \end{pmatrix}$ is always similar to an idempotent of the desired form.

Let us now consider an idempotent matrix Y, $I \neq Y^2 = Y \in M_n(R)$. The above lemma shows that we may assume that Y has its last row zero. Let us write:

$$Y = \begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} \quad \text{where } B \in M_{n-1,n-1}(R) \text{ and } C \in M_{n-1,1}(R)$$

From $Y^2 = Y$ we get that $B^2 = B$ and $BC = B$. If $B = I_{n-1,n-1}$ it is easy to conclude the result by the above paragraph. if $B \neq I_{n-1,n-1}$, the induction hypothesis shows that there exists an invertible matrix Q such that QBQ^{-1} has the desired form. Thus there exists $l \in \mathbb{N}$ such that:

$$\begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Q^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix} & QC \\ 0 & 0 \end{pmatrix}$$

From the first paragraph of this proof easily leads to the desired conclusion. □

The following proposition is Proposition 22 in [5].

Proposition 21. *Let R be a right quasi-Euclidean domain and $A \in \mathbb{M}_n(R)$. Then $l.ann\,(A) \neq 0$ implies that $r.ann\,(A) \neq 0$.*

We now arrive at the main result of this section which was mentioned in [5] without proof.

Theorem 22. *Let R be a right and left quasi-Euclidean domain. Then every matrix $A \in \mathbb{M}_n(R)$ with $l.ann\,(A) \neq 0$ (equivalently, $r.ann\,(A) \neq 0$) is a product of idempotent matrices.*

Proof. We proceed by induction on n. If $n = 2$ the result is clear by the Theorem 18. So suppose that $n > 2$ and let $(a_1, \ldots, a_n)A = (0, \ldots, 0)$. Since R is a right Euclidean domain, it is a right Bézout domain and we may assume that $\sum_{i=1}^n a_i R = R$. It follows from Proposition 15(1) that there exists a matrix $Q \in GL_n(R)$ such that its last row is $(a_1, \ldots, a_n)$. By straightforward computations we get that

$$QAQ^{-1} = \begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} := A'$$

where $B \in M_{n-1}(R)$ and C is a column in R^{n-1} while the last row is the zero row. According to Proposition 15 there exists an invertible matrix Q such that QB is upper triangular. Moreover we have

$$A' = \begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} Q^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} QB & QC \\ 0 & 0 \end{pmatrix}$$

Consider the first matrix on the right hand side. It has a zero column and Remark 4 (1) shows that we can permute the last two columns. An easy induction and Theorem 3 shows that this matrix is a product of idempotent matrices. We just have to deal with the second matrix on the right hand side. This shows that we can assume that the matrix B in A' is in fact upper triangular.
Consider the $(n-1) \times (n-1)$ matrix in the lower right corner of A'. Its bottom row is zero and hence the induction hypothesis implies that this matrix is a product $E_1 \ldots E_r$ of idempotent matrices $E_i^2 = E_i$. Since $E_r^2 = E_r$ we have

$$A' = \begin{pmatrix} d_1 & (d_2, \ldots, d_n) \\ 0 & E_1 \cdots E_r \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & E_1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ 0 & E_r \end{pmatrix} \begin{pmatrix} d_1 & (d_2, \ldots, d_n) \\ 0 & E_r \end{pmatrix}$$

This implies that we only have to show that the last matrix, say M, on the right hand side is a product of idempotent matrices. Lemma 20 shows that there exists an invertible matrix $P \in Gl_{n-1}(R)$ and a natural number $0 \leq h \leq n-1$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} M \begin{pmatrix} 1 & 0 \\ 0 & P^{-1} \end{pmatrix} = \begin{pmatrix} d_1 & (d_2, \ldots, d_n) \\ 0 & PE_rP^{-1} \end{pmatrix} = \begin{pmatrix} d_1 & (d_2, \ldots, d_n) \\ 0 & \begin{pmatrix} I_h & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

It is thus sufficient to consider the case when M is of the form of the last matrix on the right.
We have $1 \leq h \leq n-1$. If $h = n-1$ we would get that E_r is the identity matrix. So we may assume that $0 < h < n-1$. Once again if $h < n-2$ then our matrix has the last two rows equal to zero. In this case the induction and Theorem 3 (1) yields the result. So we may assume that $h = n-2$. For $a \in R$ and $2 \leq i \leq n-1$, let us denote $F_i(a)$ the matrix $I_n + ae_{1,i}$ and $\widehat{F_i(a)}$ (resp. $\widehat{I_n}$) the matrix obtained from $F_i(a)$ (resp. I_n) by replacing its last row by a zero row. Now we have, for $2 \leq i \leq n-1$, $M = \widehat{I_n}M = \widehat{F_i(d_i)}F_i(-d_i)M$. Theorem 3 shows that the matrix $\widehat{F_i(d_i)}$ is a product of idempotent matrices and hence we only need to consider matrices of the form $F_i(-d_i)M$, i.e. we may assume that, in the matrix M described above, we have $d_2 = d_3 = \cdots = d_{n-1} = 0$. Now conjugating first the matrix M with $P_{2,n}$ leads to a matrix having its second row zero. This means that our matrix is now of the form:

$$\begin{pmatrix} d_1 & d_n & (0,\ldots,0) & 0 \\ 0 & 0 & (0,\ldots,0) & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(Where X stands for a $((n-3)\times(n-3)$ matrix). Permuting again the second row and the last row we easily conclude, thanks to the induction hypothesis, that the top left $(n-1)\times(n-1)$ corner matrix is a product of idempotents. Theorem 3 (1) then yields the result. $\square$

Let us recall that an IP ring is one over which singular matrices are product of idempotents. Here is a kind of converse of the above Theorem for von Neumann regular rings.

Proposition 23 ([1], Theorem 3.2)**.** *An IP ring which is von Neumann regular is unit regular and hence quasi-Euclidean.*

Lezowski ([37]) defined Euclidean rings (not necessarily domains) using functions φ from R to $\mathbb{Z}$. Our definition of quasi-Euclidean rings, although similar is independent of the existence of the function φ. He proved that a matrix ring $M_n(R)$ with $n > 1$ over a commutative ring is right Euclidean if and only if the ring R is a principal ideal ring. We may remark that this result is not true for right quasi-Euclidean rings. For example, consider a commutative unit regular ring which is not principal ideal ring. Matrix rings over this ring are quasi-Euclidean. We refer the reader to [41] for more details.

4. Hannah and O'Meara

Hannah and O'Meara published several interesting results on the decomposition of nonunit elements of a regular ring into idempotents ([30], [31], [42], [43],[44]). We will present here some of their results.

Theorem 24 (Hannah and O'Meara [30])**.** *If an element a of a regular ring R is a product of k idempotents then $(1-a)R \leq k.rann(a)$.*

In the same paper they proved a converse for the unit regular rings:

Proposition 25 ([30]). *Let $a \in R$ be an element of a unit regular ring and let k be a positive integer. Then a is a product of k idempotents if and only if $(1-a)R \lesssim k.rann(a)$.*

This leads to the following

Corollary 26 ([30]). *An element a in a unit regular ring is a product of idempotents if and only if $R.rann(a) = R(1-a)R$.*

In fact it is well-known that in a regular ring a product of unit regular element is unit regular. This was first proved in [30] and immediately implies that in a regular rings a product of idempotents is always unit regular. This result is very useful and was used in particular in [7] .

Hannah and O'Meara also proved the following remarkable result:

Theorem 27 (Theorem 2.9 in [30]). *Let R be one of the following rings: (i) unit regular, (ii) right continuous, or (iii) a factor ring of a right self-injective ring. Then a is a product of idempotents if and only if*

$$R.rann(a) = R(1-a)R = lann(a).R$$

In a more recent paper O'Meara (cf. [44]) proved other interesting results related to the decomposition into product of idempotents in a regular ring which is separative. A ring R is separative if for all finitely generated projective modules, A, B

$$A \oplus A \simeq A \oplus B \simeq B \oplus B \quad \text{implies} \quad A \simeq B$$

Equivalently, $nA \cong nB$ for all $n > 1$ implies $A \cong B$

Theorem 28 (Proposition 2.4, [44]). *Let R be a regular ring. Then the separativity of R is equivalent to the fact that an element is a product of idempotents if and only if $R.rann(a) = R(1-a)R = lann(a).R$*

It is worthy to mention that no example of a regular ring that is not separative is known. This is certainly one of the most important open problems in regular rings.

Let us come to another work by Hannah and O'Meara ([31]). This work generalizes Reynolds and Sullivan's work and also the Ballantine theorem on the number of idempotents necessary to present a singular element.

For a semigroup S generated by idempotents the depth $\Delta(S)$ of S is the minimum number of idempotents needed to express a general element as a product of idempotents.

Theorem 29 (Proposition 1.1 and Theorem 1.3 in [31]). *Let R be a regular ring and let S be the multiplicative semigroup generated by all its idempotents. Then $\Delta(S) \leq Index(S)$. If R is Dedekind finite, then the equality holds.*

The equality in the above theorem holds when R is right self injective and is not a product $R = R_1 \times R_2$ of a ring R_1 with finite index and a regular ring R_2 of type III. (cf. Goodearl's book [27] for the classification of regular rings)

5. Nonnegative Matrices

It is natural to wonder, in the context of real matrices, if a singular nonnegative real matrix is always a product of nonnegative idempotent matrices. We will see that this is not true in general, but there are some positive results.

Firstly let us mention the form of nonnegative idempotent matrices (cf. [21]).

Theorem 30. *Let I be a nonnegative idempotent matrix of rank k. There exists a permutation matrix P such that*

$$PIP^{-1} = \begin{pmatrix} J & JB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ AJ & AJB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{with} \quad J = diag(J_1, \dots, J_k)$$

where J_i are positive idempotent matrices of rank one (that is, $J_i = x_i y_i^t$ where x_i,y_i are positive vectors and $y_i^t x_i = 1$) and A, B are arbitrary nonnegative matrices of appropriate sizes. Conversely every matrix of the above form is an idempotent of rank k.

This section is heavily based on the recent paper [4]. We will only mention the main results.

Proposition 31. *Let $A \in M_n(\mathbb{R}^+)$, $n > 1$, be a nonnegative matrix of rank 1 or 2. Then A is a product of nonnegative idempotent matrices.*

Let us mention that in the case of a matrix of rank 1, it was shown in [1] that three idempotent matrices were sufficient.

We now give an example of a nonnegative singular matrix that cannot be written as a product of nonnegative idempotent matrices.

Example 32. *Consider the 4×4 matrix*

$$A_\alpha := \begin{pmatrix} \alpha & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \alpha \\ \alpha & 0 & \alpha & 0 \\ 0 & \alpha & 0 & \alpha \end{pmatrix}, \quad \textit{where } \alpha \in \mathbb{R}^+,\ \alpha \neq 0.$$

If $A_\alpha = E_1 \cdots E_n$ is a product of nonnegative idempotent matrices $E_1, E_2, \dots, E_n \in M_4(\mathbb{R}^+)$, we get that $A_\alpha = A_\alpha E_n$. For $i = 1, 2, 3, 4$, let us write (x_i, y_i, z_i, t_i) for the i^{th} row of the matrix E_n. Equating the entries of the rows on both sides we get, after simplification,

For the first row $1 = x_1 + x_2$, $1 = y_1 + y_2$, $0 = z_1 + z_2$ and $0 = t_1 + t_2$.

For the second row $0 = x_3 + x_4$, $0 = y_3 + y_4$, $1 = z_3 + z_4$ and $1 = t_3 + t_4$

For the third row $1 = x_1 + x_3$, $0 = y_1 + y_3$, $1 = z_1 + z_3$ and $0 = t_1 = t_3$

For the fourth row $0 = x_2 + x_4$, $1 = y_2 + y_4$, $0 = z_2 + z_4$ and $1 = t_2 + t_4$

Since all the real numbers x_i, y_i, z_i, t_i must be nonnegative it is easy to conclude from the above equations that the only solution will be $E_n = Id$. This shows that the matrix A_α does not have a presentation as product of nonnegative idempotent matrices. In fact considering $\alpha = 1/2$ we remark that $A_{1/2}$ is a doubly stochastic matrix. Considering the matrix $A_{1/2}A_{1/2}^T$ we may also conclude that even a non-negative symmetric stochastic matrix cannot always be presented as the product of nonnegative idempotent matrices.

Despite this example there exist special nonnegative matrices which always admit a decomposition into product of nonnegative idempotent matrices. We will conclude this paper giving two such examples. Let us first recall the form of nonnegative matrices having nonnegative von Neumann inverse.

Proposition 33. *(cf. [36], Theorem 1 and Lemma 2) If a nonnegative square matrix A admits a nonnegative von Neumann inverse X (i.e. $A = AXA$), then there exists a permutation matrix P such that PAP^T is of the form*

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where C and D are nonnegative matrices of suitable sizes and J is a direct sum of matrices of the following three types:

(1) βxy^T where x, y are positive vectors and β is a positive real number.

(2)

$$\begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & 0 & \cdots & 0 \\ 0 & & \beta_{23}x_2y_3^T & 0 & \cdots & 0 \\ \vdots & & & \ddots & \cdots & \vdots \\ 0 & & & 0 & \cdots & \beta_{d-1,d}x_{d-1}y_d^T \\ \beta_{d1}x_dy_1^T & & & 0 & \cdots & 0 \end{bmatrix}$$

where for $1 \leq i \leq d$ the vectors x_i, y_j are positive and β_{ij} is a positive real number.

(3) the zero matrix.

Theorem 34. *Let $A \in M_n(\mathbb{R})$, $n \geq 2$, be a singular nonnegative matrix having a nonnegative von Neumann inverse X (i.e. $A = AXA$). Then A is a product of nonnegative idempotent matrices.*

We have seen in Theorem 3 that singular 0-1 matrices are always products of idempotent matrices. In fact, looking at such a matrix as a real matrix one can show much more:

Proposition 35. *Let $A \in M_n(\mathbb{R})$ be a singular nonnegative definite* 0-1 *matrix. Then A is a product of nonnegative idempotent matrices.*

Let us end this paper with a few questions.

Questions

(1) Is Theorem 22 true for $n = 1$, that is, is it true that singular elements of an indecomposable Quasi-Eucldean ring are product of idempotents?
(2) Is it true that a singular totally nonnegative matrix is a product of nonnegative idempotents? By a result of Cryer [13] and also by Goodearl-Lenagan [28], it is sufficient to consider a singular totally nonnegative upper or lower triangular matrix.
(3) When can we say that the matrix

$$A = \begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix}$$

where B is invertible is a product of idempotent matrices.

(4) Find classes of nonnegative singular matrices that are always product of nonnegative idempotents.
(5) Does there exists an example of a regular ring R and an element $a \in R$ which is not the product of idempotents and satisfies the condition: $Rr(a) = l(a)R = R(1-a)R$. An answer to this question would in fact solve one of the most fundamental problem in regular rings: does there exist regular non separative ring? (cf. p.129 of [7]).
(6) Let R be a ring, $U \in M_n(R)$ an invertible matrix and E, F idempotent matrices not equal to the identity. Is it true that the matrix EUF is a product of idempotent matrices.
(7) What is the structure of an artinian ring R such that every nonunit is a product of idempotents?

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