Ore Extensions and V-domains

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Abstract. We give necessary and sufficient conditions for a skew polynomial ring $K[t;\sigma,\delta]$ over a division ring K to be a left V-domain. In particular, when this ring admits a unique simple left module, the conditions obtained include: 1) all polynomials are Wedderburn, 2) all $n\times n$ matrices over K are (σ,δ)-similar. We also provide necessary and sufficient conditions for this ring to be both left and right V-domain.

These results, that are indeed motivated by a long-standing open question whether a left V-domain is a right V-domain, provide clues towards finding a possible counterexample to this question or answering it in the affirmative for the ring $K[t;\sigma,\delta]$.

1. Introduction

Skew polynomial rings such as Weyl algebras and quantum groups have been a source of many interesting examples in noncommutative ring theory. In particular, differential polynomial ring over a "universal differential" field was the first example of a simple right-left V-domain and PCI-domain (A domain such that each proper cyclic right module is injective is called right PCI-domain). Although there exists an example of a nondomain which is a left V-ring but not a right V-ring, the question whether the property of being V-domain or PCI-domain is left-right symmetric remains open. The examples in the literature that relate to these two properties have been constructed by using either differential polynomial rings or localizations of twisted polynomial rings (Cf. [**C**],[$\mathbf{Os}_1$],[$\mathbf{Os}_2$]). In this paper, we obtain necessary and sufficient conditions for $K[t;\sigma,\delta]$ to be a left V-domain (equivalently, left PCI-domain).We also provide conditions for $K[t;\sigma,\delta]$ to be both right and left V-domain (PCI-domain).

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2. Notations and Definitions.

Throughout all rings have unity. A ring R is called a left V-ring if every simple left R-module is injective. A ring R is called a left PCI-ring if each proper cyclic left R-module is injective (i.e. Left R-modules R/I where I is a nonzero left ideal are injective). It is known that R is a left V-ring if and only if each left ideal is an intersection of maximal left ideals (Cf. Cozzens and Faith [**CF**]). Furthermore, a left PCI-ring R is known to be either semisimple artinian or simple left Noetherian left hereditary domain such that each proper cyclic left R-module is semisimple (Faith [**F**], Damiano [**D**]).

Let K be a division ring, σ be an endomorphism of K and δ be a σ-derivation of K (i.e. δ is an additive endomorphism of K such that for $a, b \in K$, we have $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$). Throughout the paper R will denote the ring of left polynomials $K[t;\sigma,\delta]$ whose elements are polynomials of the form $\sum_i a_i t^i$. Addition of these polynomials is natural and their multiplication follows usual multiplication rule with $ta = \sigma(a)t + \delta(a)$. We note that if σ is an automorphism then we can also view the ring $K[t;\sigma,\delta]$ as a (right) polynomial ring with multiplication induced by $at = t\sigma^{-1}(a) - \delta(\sigma^{-1}(a))$. If $\delta = 0$ then $K[t;\sigma,\delta]$ is the skew polynomial ring $K[t;\sigma]$ considered by Jacobson and others and if σ is the identity then $K[t;\sigma,\delta]$ is the standard differential polynomial ring $K[t;\delta]$. When $\delta = 0$ and $\sigma =$ identity, $K[t;\sigma,\delta]$ is the classical polynomial ring $K[t]$.

$K[t;\sigma,\delta]$ is a principal left ideal domain and right division algorithm holds, i.e. for any two polynomials $f(t)$ and $g(t)$ there exist polynomials $q(t)$ and $r(t)$ such that $f(t) = q(t)g(t) + r(t)$, where the degree of $r(t)$ is less than the degree of $g(t)$. The evaluation of a polynomial $f \in R$ at $c \in K$, denoted by $f(c)$, is the only element in K satisfying $f(t) - f(c) \in R(t-c)$. The rule for evaluating the product $f(t)g(t)$ of left polynomials $f(t), g(t) \in R = K[t;\sigma,\delta]$ at $c \in K$ is given by:

$$(fg)(c) = 0 \text{ if } g(c) = 0 \text{ and } (fg)(c) = f(c^{g(c)})g(c) \text{ if } g(c) \neq 0,$$

where, for $k \in K \setminus \{0\}$, c^k denotes $\sigma(k)ck^{-1} + \delta(k)k^{-1}$. It can be verified that $(c^k)^{k'} = c^{k'k}$, for any nonzero elements k, k' in K. An element $b \in K$ is (σ,δ)-conjugate to c if there exists $k \in K \setminus \{0\}$ such that $b = c^k$. The (σ,δ) conjugacy is an equivalence relation and we denote by $\Delta(c) = \{\sigma(k)ck^{-1} + \delta(k)k^{-1} \mid 0 \neq k \in K\}$ the (σ,δ) conjugacy class of $c \in K$. The conjugacy relation defined above is consistent with the classical conjugacy relation (i.e. when $\sigma = Id., \delta = 0$). In other words, in the classical case $\Delta(c) = \{kck^{-1} \mid 0 \neq k \in K\}$. For any polynomial $f \in R$ we denote by $V(f)$ the set of right roots of f, i.e. $V(f) = \{a \in K \mid f(a) = 0\}$. For more information on evaluation of a polynomial in $R = K[t;\sigma,\delta]$ the reader may consult [**L**] or [$\mathbf{La}_4$].

A monic polynomial f in R is called a Wedderburn polynomial, in short W-polynomial, if there exists a finite set $\{a_1, ..., a_n\} \subset K$ such that $Rf = \cap_{i=1}^n R(t - a_i)$. In other words, f is the monic least left common multiple of the polynomials $t - a_1, \ldots, t - a_n$. In particular, any monic linear polynomial is a W-polynomial and, following the usage in our previous works (Cf. [$\mathbf{LL}_5$], [**LLO**]), the element $1 \in K$ will be considered as a W-polynomial. The product formula mentioned above shows that the monic least left common multiple of $t - a, t - b$ with $a \neq b$ is the Wedderburn polynomial $f(t) = (t - b^{b-a})(t - a) = (t - a^{a-b})(t - b)$. Moreover it is known that a monic polynomial $f \in R$ is a W-polynomial if and only if f can be written as a product of linear polynomials and every quadratic monic factor of f is a W-polynomial. For further information on W-polynomials the reader is referred to [$\mathbf{LL}_4$],[$\mathbf{LL}_5$],[**LLO**].

A ring S is called an n-fir if each n-generated left ideal is free of unique rank $\leq n$. This property is left-right symmetric. We note that a domain S is a 2-fir if and only if, for all $a, b \in S$, $Sa+Sb$ is principal whenever $Sa \cap Sb \neq 0$. The left-right symmetry implies that a similar characterization using right principal ideals also holds. Since a skew polynomial ring $R = K[t;\sigma,\delta]$ over a division ring K is always a left principal ideal domain, it is a 2-fir. In particular, if $f, g \in R$ are such that $fR \cap gR \neq 0$ then $fR + gR$ is a principal right ideal. This observation will be used freely in the text.

An element of a ring is called an atom if it is nonunit and cannot be written as a product of nonunits. An integral domain is called atomic if every element other than zero or a unit is a product of atoms. If an element a of a ring S is such that $Sa = \cap_{i=1}^{n} Sf_i$, where $f_1, \ldots, f_n$ are atoms in S then a is called fully reducible. We remark that $p \in R = K[t,\sigma,\delta]$ is an atom if and only if p is irreducible.

The idealizer of a left (right) ideal I in a ring R is the largest subring of R in which I is a two-sided ideal. It is denoted $Idl(I)$.

For any ring S, $\mathrm{Q}^r_{\max}(S)$ ($\mathrm{Q}^l_{\max}(S)$) denotes the right (left) maximal quotient ring of S. For a right nonsingular ring S, $\mathrm{Q}^r_{\max}(S)$ is always von Neumann regular right selfinjective. If, in addition, S is an integral domain then $\mathrm{Q}^r_{\max}(S)$ is also simple.

3. Preliminaries

Let us start with the following general lemma which may also be of independent interest.

LEMMA 3.1. *Let f, g be nonzerodivisors in a ring R. Then the following are equivalent:*

(i) $0 \longrightarrow R/Rf \xrightarrow{g_r} R/Rfg \longrightarrow R/Rg \longrightarrow 0$ *splits, where g_r stands for the right multiplication by g.*

(ii) $1 \in Rf + gR$.

(iii) $0 \longrightarrow R/gR \xrightarrow{f_l} R/fgR \longrightarrow R/fR \longrightarrow 0$ *splits, where f_l stands for the left multiplication by f.*

(iv) $Idl(Rf) \subseteq Rf + gR$.

(v) $Idl(gR) \subseteq Rf + gR$.

PROOF. (i)$\Rightarrow$(ii). By hypothesis there exists a map $\phi : R/Rfg \longrightarrow R/Rf$ such that $\phi \circ g_r = Id._{R/Rf}$. Let $y \in R$ be such that $\phi(1 + Rfg) = y + Rf$. We then have $(\phi \circ g_r)(1 + Rf) = gy + Rf = 1 + Rf$, i.e. $gy - 1 \in Rf$. This gives that there exists $x \in R$ such that $gy + xf = 1$; proving (ii).
(ii)$\Rightarrow$(iii). If $x, y \in R$ are such that $xf + gy = 1$, we define $\phi : R/fgR \longrightarrow R/gR : 1 + fgR \mapsto x + gR$. This map is well define since $xfg = g(1 - yg) \in gR$. Moreover $(\phi \circ f_l)(u + gR) = xfu + gR = (1 - gy)u + gR = u + gR$. This means that ϕ is a splitting of f_l, as desired.
The proof of (iii)$\Rightarrow$(ii) is similar to the proof of (ii)$\Rightarrow$ (i).
(ii)$\Leftrightarrow$(iv). Let us write $1 = xf + gy$, for some $x, y \in R$. If $r \in Idl(Rf)$ then there exists $r' \in R$ such that $fr = r'f$ and we get $r = xfr + gyr = xr'f + gyr \in Rf + gR$.

This shows that $Idl(Rf) \subseteq Rf + gR$. The reverse implication is clear since $1 \in Idl(Rf)$.
(ii)$\Leftrightarrow$(v) is proved similarly. $\square$

THEOREM 3.2. *Let R be an atomic left principal ideal domain. Then the following are equivalent:*

(i) *R is a left PCI domain.*
(ii) *R is a left V-domain.*
(iii) *$R = Rp + gR$ for every atom $p \in R$ and every element $g \in R \setminus \{0\}$.*
(iv) *$R = Rf + gR$ for every $f, g \in R \setminus \{0\}$.*
(v) *$R = Rf + I$ for every $f \neq 0$ and every nonzero right ideal I.*
(vi) *$1 \in Rf + I$ for every $f \neq 0$ and every nonzero right ideal I.*
(vii) *$1 \in Rf + gR$ for every $f, g \in R \setminus \{0\}$.*
(viii) *$R/Rfg \cong R/Rf \times R/Rg$ for every $f, g \in R \setminus \{0\}$.*
(ix) *$R/fgR \cong R/fR \times R/gR$ for every $f, g \in R \setminus \{0\}$.*
(x) *$1 \in Rp + qR$ for every atoms $p, q \in R$.*
(xi) *$1 \in Rp + I$ for every nonzero right ideal I and every atom $p \in R$.*
(xii) *$Idl(Rp) \subseteq Rp + qR$ for every atoms $p, q \in R$.*
(xiii) *$Idl(qR) \subseteq Rp + qR$ for every atoms $p, q \in R$.*
(xiv) *All products pq, where p, q are atoms, are fully reducible.*
(xv) *All nonunit elements of R that are nonzero are fully reducible.*

PROOF. (i)$\Rightarrow$(ii) is obvious.
(ii)$\Rightarrow$(iii). Since R is a left V-domain, left simple modules are injective, and so divisible. On the other hand, since R is an atomic left principal ideal domain, the simple left modules are of the form R/Rp, where p is an atom. But then, for every nonzero $g \in R$, $g(R/Rp) = R/Rp$. This yields $gR + Rp = R$, proving statement (iii).
(iii)$\Rightarrow$(iv). Let p, r be atoms and g be a nonzero element of R. By hypothesis we have $R = Rp + gR$ and $R = Rr + gR$. Using these two equalities we have $R = (Rp + gR)r + gR = Rpr + gRr + gR = Rpr + gR$. Since R is atomic, an induction on the length of f implies that $Rf + gR = R$ for all $f, g \in R$.
(iv)$\Rightarrow$(i). $Rf + Rg = R$ implies that $g(R/Rf) = R/Rf$. Thus every cyclic module R/Rf is divisible and hence injective because R is a left principal ideal domain.
Since the equivalence (iv)$\Leftrightarrow$(v) is obvious. We have thus shown that statements (i) to (v) are equivalent.
The implication (v)$\Rightarrow$(vi) is clear.
The equivalences (vi)$\Leftrightarrow$(vii) $\Leftrightarrow \ldots \Leftrightarrow$(ix) are either obvious or direct consequences of Lemma 3.1.
$(vii) \Rightarrow (x)$ is obvious.
(x)$\Rightarrow$(vii) can be obtained in the same way as the implication (iii)$\Rightarrow$(iv).
The equivalences (x)$\Leftrightarrow$(xi) $\Leftrightarrow \ldots \Leftrightarrow$(xiii) are either obvious or direct consequences of Lemma 3.1.
(xiii)$\Rightarrow$(xiv). Since $(xiii)$ is equivalent to (x) it is enough to show that if $1 \in Rp + qR$ then pq is fully reducible. Therefore, suppose there exist $u, v \in R$ such that $up + qv = 1$. Notice that this implies that $qv \notin Rp$ and $v \notin Rp$. Left multiplying the equality $up + qv = 1$ by p we get $pup + pqv = p$. Hence $pqv \in Rp \cap Rqv = Rp'qv$ for some $p' \in R$. We thus have $p \in Rp'$. Observe that, since $qv \notin Rp$, p' cannot be a unit. Since p is an atom we conclude that $Rp = Rp'$. Let

us put $Rp \cap Rv = Rp''v$ for some $p'' \in R$. Note that p'' is an atom. We then have $Rp'qv = Rp \cap Rqv = Rp \cap Rv \cap Rqv = Rp''v \cap Rqv = (Rp'' \cap Rq)v$. This leads to $Rpq = Rp'q = Rp'' \cap Rq$. This shows that pq is fully reducible, as required.
(xiv)$\Rightarrow$(xv). Let f be a nonunit element of $R\backslash 0$. Since R is assumed to be an atomic principal ideal domain, f can be written as $f = up_1p_2 \cdots p_n$, where $p_1, \dots, p_n$ are atoms. The number n of atoms appearing in such a factorization of an element f is independent of the factorization chosen and is denoted $l(f)$. We argue by induction on the length $l(f)$ of an element $f \in R$. If $l(f) = 1$, f is an atom and hence fully reducible. If $l(f) = n > 1$, let us write $f = f'p_n$ where p_n is an atom and $f' \in R$ is such that $l(f') = n-1$. Hence f' is fully reducible and there exist atoms $p_1, \dots, p_{n-1}$ such that $Rf' = \cap_{i=1}^{n-1} Rp_i$. We have $Rf = Rf'p_n = \cap_{i=1}^{n-1} R(p_ip_n) = \cap_i Rp'_i$ where the last equality comes from the fact that $p_1p_n, \dots, p_{n-1}p_n$ are fully reducible by the induction hypothesis. Thus f is fully reducible.
(xv)$\Rightarrow$(ii). Since R is left principal, any left ideal I of R is of the form Rf for some element f. (xv) shows that every right ideal is an intersection of maximal right ideals, i.e., R is a left V-domain. $\square$

Corollary 3.3. *Let R be an atomic left and right principal ideal domain. Then R is a left V-domain if and only if it is a right V-domain.*

Proof. This is clear from the fact that statement (x) in Theorem 3.2 is left-right symmetric. $\square$

Corollary 3.4. *Let $R = K[t;\sigma,\delta]$ be an Ore extension over a division ring K. Then the following are equivalent:*

(i) *R is a left V-domain and all irreducible polynomials are linear.*
(ii) *All monic polynomials are Wedderburn polynomials.*

4. $K[t;\sigma,\delta]$ as Left V-Domain.

In this section we will obtain necessary and sufficient conditions for an Ore extension $R = K[t;\sigma,\delta]$ to be a left V-domain.

Lemma 4.1. *For $R = K[t;\sigma,\delta]$ the following statements are equivalent:*

(i) *For all nonzero elements $f \in R$, $(fR + Rf) \cap K \neq 0$.*
(ii) *R is simple.*
(iii) *For all nonzero elements $f, g \in R$, $(fR + Rg) \cap K \neq 0$.*

Proof. (i)$\Rightarrow$(ii). Let f be a nonzero element in R. By hypothesis there exists a nonzero element $\alpha \in (Rf + fR) \cap K$. Thus $\alpha \in RfR$ and we get $RfR = R$.
(ii)$\Rightarrow$(iii). If R is simple then, for nonzero elements $f, g \in R$, we have $R = RgR \subseteq (fR + Rg)R$. This gives that $(fR + Rg)R = R$ and hence $fR + Rg$ must contain a polynomial of degree 0.
(iii)$\Rightarrow$(i) is clear. $\square$

Let $f = \sum_{i=0}^n a_it^i \in R = K[t;\sigma,\delta]$ be a monic polynomial ($a_n = 1$). In order to characterize the injectivity of the cyclic left $R = K[t;\sigma,\delta]$-module R/Rf, let us denote by C_f the usual companion matrix of the monic polynomial $f = \sum_{i=0}^n a_it^i$.

We thus have

$$C_f = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}$$

Considering R/Rf as a left K-vector space, the element $\sum_{i=0}^{n-1} v_i t^i + Rf \in R/Rf$ is represented by the vector $v = (v_0, v_1, \ldots, v_{n-1}) \in K^n$ and the action of t on v is given by $t.v = (\sigma(v_0), \ldots, \sigma(v_{n-1}))C_f + (\delta(v_0), \ldots, \delta(v_{n-1}))$. This action, denoted by T_f, is a pseudo linear transformation i.e. T_f is additive and we have $T_f(\alpha v) = \sigma(\alpha)T_f(v) + \delta(\alpha)v$ where $\alpha \in K$. For example, $T_{t-a} : K \longrightarrow K$ is given by $T_{t-a}(x) = \sigma(x)a + \delta(x)$. Notice that $T_t = \delta$ and, in case $\delta = 0$, $T_{t-1} = \sigma$. For $g = \sum_{i=0}^{l} a_j t^j \in R$ and $f \in R$, we also consider $g(T_f) = \sum_{i=0}^{l} a_j (T_f)^j$. For more details on pseudo linear transformations the reader is referred to [**L**]. Let us now mention the following technical result to be used later.

Lemma 4.2. *Let $R = K[t; \sigma, \delta]$ be an Ore extension over a division ring K. Then, for $f \in R$, and $0 \neq y \in K$, $f(\delta)(y) = f(0^y)y$.*

Proof. Evaluating $f(t)y$ at 0 and using the product formula stated in the Notations and Definitions (Section 2), we get $(f(t)y)(0) = f(0^y)y$. On the other hand the evaluation at 0 is also the independent term $f(\delta)(y)$ of the polynomial $f(t)y$. This completes the proof. □

Lemma 4.3. *Let f be a nonzero element in $R = K[t; \sigma, \delta]$. Then the following are equivalent:*

(i) *R/Rf is injective.*

(ii) *$g(T_f)$ is onto for all nonzero polynomials $g \in R$.*

Proof. (i)⇔(ii). Since R is a left principal domain, the injectivity of R/Rf is equivalent to its divisibility. Now, the divisibility of R/Rf is equivalent to the fact that for every nonzero $g \in R$, $g.R/Rf = R/Rf$. This means that the action of $g(t). = g(t.)$ is onto. As explained in the paragraph before lemma 4.2 the action of $t.$ on R/Rf corresponds to the action of the operator T_f on K^n, where $n = deg f$. This gives the result. □

Our methods give criteria for certain cyclic left modules to be injective modules. B. Osofsky considered injective modules over skew polynomial rings $k[t; \sigma]$, where k is a commutative field [$\mathbf{Os}_1$]. The previous lemmas lead to a generalization of one of her results to the case when k is a division ring.

Corollary 4.4. *Let σ be an endomorphism of a division ring K and let R denote the skew polynomial ring $R = K[t; \sigma]$. Then $R/R(t-1)$ is injective if and only if for any $p \in K[t; \sigma]$, $p(\sigma)$ is onto.*

Proof. Since $\delta = 0$ we have, as mentioned in the paragraph preceding Lemma 4.2, $T_{t-1} = \sigma$. Lemma 4.3 then yields the desired result. □

Let us recall that the inner σ-derivation on K induced by a, denoted $\delta_{a,\sigma}$, is defined by $\delta_{a,\sigma}(x) = ax - \sigma(x)a$, for $x \in K$.

Lemma 4.5. *Let f, a be elements in R and K respectively. Then*

(a) *$a \in fR + Rt$ iff $a \in imf(\delta)$.*
(b) *$fR + Rt = R$ iff $f(\delta)$ is onto.*
(c) *R/Rt is divisible iff for any $f \in R$, $f(\delta)$ is onto. In particular, if $\delta = 0$, R/Rt is never divisible.*
(d) *$R/R(t-a)$ is divisible iff for any $f \in R$, $f(\delta - \delta_{a,\sigma})$ is onto.*

Proof. (a) Suppose $a \in fR + Rt$ and let $g, h \in R$ be such that $fg + ht = a$. Comparing independent terms on both side we get $f(\delta)(g_0) = a$ where g_0 stands for the independent term of g. Conversely, suppose there exists $c \in K$ such that $f(\delta(c)) = a$. This implies that the independent term of $f(t)c$ is a, i.e. $f(t)c - a \in Rt$, as required.
(b) Since $fR + Rt = R$ if and only if $K \subseteq Rt + fR$, the statement (a) above gives us immediately the conclusion.
(c) We know R/Rt is divisible if and only if for any $f \in R$, $fR + Rt = R$ and (b) above gives us the first statement. The particular case is clear.
(d) This is a consequence of (c) using the fact that, for any $a \in K$, one has $K[t;\sigma,\delta] = K[t-a;\sigma,\delta-\delta_{a,\sigma}]$ □

Let us give some examples:

Example 4.6. A well known example of Cozzens (cf.[**C**] or [**CF**], chapter 5) is the following: let k be a commutative field and δ be derivation of k. Assume that k is differentially closed. Then $R = k[t; Id., \delta]$ is a left and right V-domain.

Of course, not every simple skew polynomial ring over a division ring is a V-domain.

Example 4.7. Let $K = k(x)$ be the field of rational functions over a field k of characteristic zero. $R = k(x)[t; Id., d/dx]$ is a simple noetherian domain. Since d/dx is not onto on $k(x)$, R/Rt is not injective (Cf. Lemma 4.5) and hence R is not a left V-domain.

We end this section with some necessary condition for $R = K[t; S, D]$ to be a left V-domain.

Theorem 4.8. *Assume $R = K[t;\sigma,\delta]$ is a left V-domain. Then*
(1) *For any $a, b, c \in K$, the (σ,δ) metro equation*
$$bx - \sigma(x)a - \delta(x) = c$$
has a solution in K.
(2) *For any $n \in \mathbb{N}$, the map $\delta + n\sigma$ is onto.*
(3) *$End_R(R/Ra) \cong End(R/aR)$. These rings are division rings if and only if $a \in R$ is an atom.*

Proof. (1) Since R is a left V-domain, we have for all $a, b \in R$, $t - b$ divides $R/R(t-a)$ i.e. $R(t-a) + (t-b)R = R$. So, in particular, for any $c \in K$ there exists $u, v \in R$ such that $u(t-a) + (t-b)v = c$. Dividing v on the right by $t - a$ and writing $v = q(t-a) + d$ for some $q \in R$ and $d \in K$. We may thus assume that $v \in K$. Comparing the degrees we then conclude that $u \in K$ as well. The above equality gives us that $(u + \sigma(v))t + \delta(v) - ua - bv - c = 0$. This leads to $\sigma(v)a + \delta(v) - bv - c = 0$. This shows that v is a solution of the metro equation.
(2) This follows from (1) by putting $a = n$ and $b = 0$.

(3) Let us recall that $Idl(aR)$ and $Idl(Ra)$ stand for the idealizers of aR and Ra respectively. It is well known and easy to prove that if R is a domain then $Idl(Ra)/Ra \cong Idl(aR)/aR$. This shows that $End_R(R/Ra) \cong End_R(R/aR)$. Schur's lemma and the fact that R is a left principal ideal domain imply that $End_R(R/Ra)$ is a division ring if a is an atom.
Conversely if $a = p_1...p_n$ is a product of $n > 1$ atoms then Theorem 3.2 $(viii)$ implies that R/Ra is a product of n simple modules $R/Rp_i,\ 1 < i \leq n$ hence $End(R/Ra)$ cannot be a division ring. □

5. $K[t;\sigma,\delta]$ as a V-Domain With Unique Simple Left Module

We will now give characterizations of Ore extensions $R = K[t;\sigma,\delta]$ which are left V-domains and have a unique simple left module (up to isomorphisms). This will explain some features of examples given by Cozzens in ([**C**]). We assume throughout the section that $\delta \neq 0$.

First, we give general characterizations of Ore extensions R having a unique simple left R-module.

Proposition 5.1. *Let $R = K[t;\sigma,\delta]$ be an Ore extension over a division ring K where $\delta \neq 0$. Then the following are equivalent:*

(i) *R admits, up to isomorphisms, a unique simple left module.*
(ii) *Every monic irreducible polynomial is of the form $t - 0^y$ for some $y \in K \setminus \{0\}$.*
(iii) *For any nonconstant polynomial $f \in R$, $Ker(f(\delta)) \neq 0$, that is, each linear differential equation over K has a nonzero solution in K.*
(iv) *For any irreducible polynomial p, $Ker(p(\delta)) \neq 0$.*

Proof. (i) ⇒ (ii). Every simple left R-module is of the form R/Rp for some irreducible polynomial p. The uniqueness of the isomorphic class of simple left R-modules gives $R/Rp \overset{\varphi}{\cong} R/Rt$. Comparing the dimensions as left K vector spaces we see that $p = t-a$ for some $a \in K$. Write $\varphi(1+Rp) = \varphi(1+R(t-a)) = y+Rt, y \in R$. Note that $y \neq 0$. Then $(t-a)y \in Rt$ i.e. $\sigma(y)t+\delta(y)-ay \in Rt$ and so $\delta(y)-ay = 0$. Since $y \neq 0$ we obtain that $p = t - a = t - \delta(y)y^{-1}$.
(ii) ⇒ (iii). This is clear since, by (ii), every nonconstant polynomial, say $f \in R$, has a right factor of the form $t - 0^y$. Lemma 4.2 then implies that $f(\delta)(y) = 0$.
(iii) ⇒ (iv) is obvious.
(iv) ⇒ (i). Since any polynomial is a product of irreducible factors, we conclude that for any $f \in R$ there exists $y \neq 0$ such that $f(\delta)(y) = 0$. By Lemma 4.2 we get that $f(0^y) = 0$ and this shows that $t - 0^y$ divides f on the right. In particular, the irreducible polynomials are of the form $t - 0^y$. Since $R/R(t - 0^y) \cong R/Rt$ under the mapping $\overline{1} \mapsto \overline{y}$, we conclude that the left simple modules are all isomorphic to R/Rt. □

Corollary 5.2. *Suppose $R = K[t;\sigma,\delta]$ has a unique simple left module (up to isomorphisms), then $K = \Delta(0) = \{\delta(x)x^{-1} | x \in K \setminus \{0\}\}$.*

Proof. This is clear from the above proposition 5.1 (ii). □

Under the hypothesis that R admits a unique simple left R-module the following lemma gives a criterion for determining when R is a left V-domain.

LEMMA 5.3. *Suppose $R = K[t;\sigma,\delta]$ has a unique simple left R-module. Then the following are equivalent:*

(i) *R is a left V-domain.*
(ii) *δ is onto.*
(iii) *For every $c \in K$, $(t-c)(\delta)$ is onto.*

PROOF. (i)$\Rightarrow$(ii). If R is a left V-domain, Theorem 4.8 shows that δ is onto.
(ii)$\Rightarrow$(i). By hypothesis R admits a unique simple left module and δ is onto. Let us first show that $Rt+tR = R$. Since every $f \in R$ can be written as $f = qt+a$, $q \in R$, $a \in K$, it is enough to show that $K \subseteq Rt+tR$. Let a be an element in K. Since δ is onto, there exists $b \in K$ such that $\delta(b) = a$. So, we get $tb - \sigma(b)t = \delta(b) = a \in Rt+tR$. This yields that t divides the simple left R module R/Rt. Hence t divides all the simple modules. In other words $tR+Rp = R$ for any irreducible polynomial p (which we know is linear). This also shows that any irreducible p divides the right module R/tR i.e. $(R/tR)p = R/tR$. Let $q \in R$ be another irreducible polynomial. By hypothesis, we have $R/Rt \cong R/Rq$ and hence also $R/qR \cong R/tR$ (Cf. [$\mathbf{Co_1}$]). So p divides the R-module R/qR i.e. $(R/qR)p = R/qR$. This gives $Rp+qR = R$ for every irreducible polynomials $p, q \in R$. Theorem 3.2 shows that R is a left V-domain.
(ii)$\Leftrightarrow$(iii). By hypothesis, R has a unique simple left R-module (up to isomorphism) hence for any $c \in K$, $R/Rt \cong R/R(t-c)$. As noticed in the proof of (ii)$\Leftarrow$ (i) above, this implies that $R/tR \cong R/(t-c)R$ for all $c \in K$. Now, Lemma 4.5 shows that $(t-c)(\delta)$ is onto if and only if $R = (t-c)R + Rt$ which is equivalent to saying that t divides $R/(t-c)R \cong R/tR$. Using Lemma 4.5 again we conclude that δ is onto if and only if $(t-c)(\delta)$ is onto. $\square$

Let us recall that two matrices $A, B \in M_n(K)$ are (σ,δ) similar, denoted by $B = A^P$, if there exists an invertible matrix $P \in M_n(K)$ such that $B = \sigma(P)AP^{-1} + \delta(P)P^{-1}$.

THEOREM 5.4. *Let $R = K[t;\sigma,\delta]$ be an Ore extension over a division ring. Then the following are equivalent:*

(i) *R is a left V-domain with a unique simple left module.*
(ii) *Every monic polynomial is Wedderburn and $K = \Delta(0)$.*
(iii) *Every irreducible monic polynomial is of the form $t - \delta(y)y^{-1}$ for some $y \in K \setminus \{0\}$ and every quadratic polynomial is Wedderburn.*
(iv) *Every monic nonconstant linear differential equation has a nonzero solution and $(t-c)(\delta)$ is onto for every $c \in K$.*
(v) *Every square matrix over K is (σ,δ) similar to a diagonal matrix with coefficients in $\Delta(0)$.*
(vi) *Every square matrix over K is (σ,δ) similar to the zero matrix of the same size.*
(vii) *For every $n \geq 1$, for all $A, B \in M_n(K)$, A is (σ,δ) similar to B.*

PROOF. (i)$\Rightarrow$(ii). By Corollary 5.2 $K = \Delta(0)$. Let f be a monic polynomial. By Theorem 3.2 and Proposition 5.1 we can write $Rf = \cap_{i=1}^{n} R(t-a_i)$. This shows that f is Wedderburn.

(ii)⇒(iii). Since Wedderburn polynomials split into linear factors, it is clear by (ii) that the monic irreducible polynomials are linear and hence they are of the form $t-\delta(y)y^{-1}$, for some $y \in K \setminus \{0\}$.
(iii)⇒(iv). Let $g(\delta)=0$ be a monic linear differential equation. The corresponding polynomial $g(t)$ has a right factor of the form $t-\delta(y)y^{-1}$, for some $y \in K \setminus \{0\}$. In this case, Lemma 4.2 shows that $g(\delta)(y)=0$. Let x be an element of K. To show that $(t-c)(\delta)$ is onto we produce an element a such that $(\delta)(a)-ca=x$. Consider the polynomial $f(t)=(t-\delta(x)x^{-1})(t-c)$. By hypothesis f is a Wedderburn polynomial and hence f has a right root d different from c. By invoking the product formula we get $0=f(d)=(d^{d-c}-\delta(x)x^{-1})(d-c)$.and so $d^{d-c}-\delta(x)x^{-1}=0$. Our hypothesis implies that there exists $y \in K \setminus \{0\}$ such that $d=0^y$. We then obtain $0^{\delta(y)-cy}=0^x$. This implies $(0^{\delta(y)-cy})^{x^{-1}}=(0^x)^{x^{-1}}=0^1=0$. This gives $0^{x^{-1}(\delta(y)-cy)}=0$. Hence, $x^{-1}(\delta(y)-cy) \in Ker\delta$ and so, $\delta(y)-cy=xb$ for some $b \in Ker\delta$. Since $c \neq 0^y$, b cannot be zero. Putting $a=yb^{-1}$ we get $\delta(a)-ca=x$. This shows that $(t-c)(\delta)$ is onto.
(iv)⇒(i). Proposition 5.1 shows that $R=K[t;\sigma,\delta]$ admits a unique simple left R-module (up to isomorphisms) and that the irreducible polynomials are linear. Lemma 5.3 implies that R is indeed a left V-domain, as desired.
(ii) ⇔(v). This equivalence is clear since it has been proved in [**LLO**] that a matrix is (σ,δ) diagonalizable if and only if the invariant factor of the highest degree is Wedderburn.
(v)⇔(vi). An easy computation shows that if $y_1,\dots,y_n$ are nonzero elements from K, then $diag(0^{y_1},\dots,0^{y_n})=(0_n)^P$ where $P=diag(y_1,\dots,y_n)$ and 0_n denotes the square zero matrix of size $n \times n$. This yields the equivalence.
(vi)⇔(vii). This is obvious because of the transitivity of the (σ,δ)-similarity.

□

Remarks 5.5. a) The conditions (iii) and (iv) in the above theorem do not explicitly refer to σ. This may, mistakenly be looked as a condition that is independent of σ. But we remark that δ is a σ derivation (with a unique σ).
b) Let us briefly describe Resco's result (Cf.[**R**]) and indicate how it can be imitated in order to give another proof of the equivalence (i) ⇔ (vii) in Theorem 5.4 above. Let K be a division ring with center k and $T=K[t](k[t])^{-1}=K \otimes k(t)$ be the central localization of $K[t]$. A square matrix $A \in M_n(K)$ determines a structure of left $K[t]$-module on the space of rows ${}^nK$ via $t.(v_1,\dots,v_n)=(v_1,\dots,v_n)A$ and $\alpha.(v_1,\dots,v_n)=(\alpha v_1,\dots,\alpha v_n)$. This action can be extended into a left T-module structure for ${}^nK$ if and only if for any $f(t) \in k[t]$, $f(A)$ is invertible. Such a matrix is called totally transcendental. Resco (Cf.[**R**] or P.M.Cohn [**Co**$_2$] Theorem 8.4.9 p. 391) showed that T is a left (and right) V-domain if and only if any two totally transcendental square matrices of the same size are similar (with Cohn's terminology K is said to be matrix homogeneous). Resco's result was used by Faith and Menal [**FM**] to construct an example of a left noetherian left annihilator ring which is not left artinian. In fact we can use similar methods as the ones used by Resco to give another proof of a part of the above theorem. In our case if $R=K[t;\sigma,\delta]$ is simple we don't need localization and for any matrix $A \in M_n(K)$, we can define a left R-module structure on the space ${}^nK$ via:

$$t.(v_1,\dots,v_n)=(\sigma(v_1),\dots,\sigma(v_n))A+(\delta(v_1),\dots,\delta(v_n)).$$

Let us denote this left R-module by $({}^nK, A)$. As in Resco's paper one can easily show that every left R-module of finite length is induced by a square matrix (the proof is even simpler since we don't have to check that the matrix is totally transcendental). Adapting the Resco's arguments we then find back the equivalence (i) $\Leftrightarrow$ (vii) in Theorem 5.4 above. Of course, in this circumstances, the unique simple left $R = K[t; \sigma, \delta]$-module is K with the R-module structure given by $f(t).x = f(\delta)(x)$, for $x \in K$.

6. $K[t; \sigma, \delta]$ as Both Left and Right V-domain

We will now examine the right structure of the Ore extension $R = K[t; \sigma, \delta]$. If σ is an automorphism then R is a right and left principal ideal domain. Hence, in this case, R is a right V-domain if and only if it is a left V-domain. So we will assume from now on that σ is not onto and study conditions that make R a left and right V-domain. Let us recall that R is a 2-fir. This implies, in particular, that for any atom $p \in R$, the ring $End_R(R/pR)$ is a division ring and hence R/pR is indecomposable. Recall that, for a 2-fir R, if $fR \cap gR \neq 0$, $f, g \in R$ then there exists $h \in R$ such that $fR + gR = hR$. We remark that R/Rf is divisible for all $f \neq 0$ if and only if R/fR is divisible for all $f \neq 0$ as both statements are equivalent to $Rf + gR = R$ for every $f, g \in R$. Let $\{a_i\}_{i \in I}$ be a basis of K over $\sigma(K)$ i.e. $K = \oplus_{i \in I} a_i \sigma(K)$. We may assume that $1 \in \{a_i \mid i \in I\}$. Let us put $a_1 = 1$. We say that a right ideal J of R is closed if for any right ideal I such that J is essential in I we have $J = I$.

Proposition 6.1. *Let $R = K[t; \sigma, \delta]$ be an Ore extension over a division ring K such that σ is not onto. Then*

(1) *For every $f \in R$, R/fR is not uniform.*
(2) *For every $0 \neq f \in R$, R/fR is not injective.*
(3) *For every $a \in R$, aR is closed.*

Proof. As above let us fix a basis $\{a_i\}_{i \in I}$ of K considered as a right vector space over $\sigma(K)$.
(1). If $f = 0$, then $R/fR = R$ contains the direct sum $\oplus_{i \in I} a_i t R$ and hence R is not uniform. If f is monic and $a_i \neq 1$ it is easy to check that the sum $fR + a_i fR$ is direct. Thus $fR \oplus a_i fR \subseteq R$ and $R \cong a_i fR$ embeds in R/fR. Thus R/fR is not uniform because R is not uniform. If $0 \neq f$ is not monic, write $f = ag$ with g monic, $a \in K$. We then get $R/fR = aR/(agR) \cong R/gR$ which is not uniform by the previous case.
(2). We proceed by induction on the numbers of factors in an atomic factorization of f. If f is an atom we know, by the remark made at the beginning of this section, that R/fR is indecomposable. If we assume that R/fR is injective we conclude that it must be uniform (Cf.[**La**$_4$], Theorem 3.52). This contradicts (1).

Assume now that f is not an atom and that R/fR is injective. If R/fR is indecomposable it will be uniform which again contradicts (1). Thus we can write $R/fR = \frac{x_1R + fR}{fR} \oplus \frac{x_2R + fR}{fR}$ for some $x_1, x_2 \in R$ such that each direct summand is not zero. If either $x_1R \cap fR = 0$ or $x_2R \cap fR = 0$, then the corresponding summand is isomorphic to R_R and, as a direct summand of the injective module R/fR, is injective itself. This contradiction shows that $x_iR \cap fR \neq 0$, $i = 1, 2$. By the property of 2-firs recalled at the beginning of this section, we get $x_iR + fR = y_iR$, for some $y_1, y_2 \in R$. Let us write $f = y_i z_i$ for $i = 1, 2$. we obtain $R/fR =$

$y_1R/fR \oplus y_2R/fR \cong R/z_1R \oplus R/z_2R$. Since the direct summands y_1R/fR and y_2R/fR are both nonzero, y_1 and y_2 are both nonunits, so z_1 and z_2 have shorter factorizations than f. As direct summands of the injective module R/fR, the modules R/z_iR are injective. Our induction hypothesis then gives the desired contradiction.
(3) If aR is not closed then aR is essential in some right ideal $I \neq aR$. We claim that there exists $b \in R$ such that aR is essential in bR. For $i \in I \backslash aR$ the essentiality of aR in I shows that $iR \cap aR \neq 0$. By the 2-fir property of R, $iR + aR = bR$, for some $b \in R$. Then $aR \subset bR \subseteq_e I$, proving our claim.
Let us write $a = bc$, where c not a unit. Then $bcR \subseteq_e bR$ and so $cR \subseteq_e R$. If p is an atomic left factor of c we have $cR \subseteq pR \subseteq R$. So, $pR \subseteq_e R$. Choose $q \notin pR$. By essentiality of pR in R, $pR \cap qR \neq 0$ and then, by the property of 2-fir, there exists $h \in R$ such that $pR + qR = hR$. So, $pR \subseteq hR$; write $p = hr$, for some $r \in R$. Since $q \notin pR$, pR is strictly contained in hR, i.e. r is not a unit. Thus h is a unit and we have $pR + qR = R$, for any $q \notin pR$. This shows that pR is maximal. Thus R/pR is simple and hence uniform. Part (1) then yields that σ must be onto. This contradiction shows that aR is closed. □

Theorem 6.2. *Let $R = K[t;\sigma,\delta]$ be a left V-domain. Then the following statements are equivalent:*

(i) *There exists an element $f \in R$ such that R/fR is injective.*
(ii) *σ is onto.*
(iii) *R is a right principal Ore V-domain.*
(iv) *for every nonzero $f \in R$, R/fR is injective.*
(v) *There exists an irreducible polynomial $f \in R$ such that R/fR is uniform.*
(vi) *There exists an irreducible polynomial $f \in R$ such that R/fR is CS.*
(vii) *R is a right PCI domain.*
(viii) *$Q^r_{max}(R)$ is directly finite.*

Proof. (i)⇒(ii) This implication follows by Proposition 6.1.
(ii)⇒(iii) Since R is both a right and left principal Ore domain when σ is onto, the implication (ii)⇒(iii) is due to the symmetry of some of the statements in Theorem 3.2.
(iii)⇒(iv). Since R is a right principal ideal domain, a right R-module is injective if and only it is divisible. Since R is a left V-domain, we have $Rf + gR = R$ for all $f, g \in R$ (Cf. Theorem 3.2) and thus R/fR is divisible for all $f \in R$.
The implication (iv)⇒(i) is obvious.
(iv)⇒(v). This is clear since injective indecomposable modules are uniform.
The equivalence (v)⇔(vi) is true because an indecomposable module is CS if and only if it is uniform.
(v)⇒(vii). By Proposition 6.1, σ is onto. Theorem 3.2 shows that for all $f, g \in R \setminus \{0\}$, $Rf + gR = R$ i.e. R/gR is divisible for all nonzero $g \in R$. Since σ is onto, R is a right principal ideal domain. Thus, R/gR is injective for all $0 \neq g \in R$, as required.
(vii)⇒(viii). If R is a right PCI-domain, $Q^r_{max}(R)$ is a division ring.
(viii)⇒(ii). Let us first show that nonzero elements of R are left invertible in $Q := Q^r_{max}(R)$. This is part of folklore; If $0 \neq a \in R$, there exists $q \in Q$ such that $a = aqa$. If $qa - 1 = 0$, we are done. If not then the nonzero element

$0 \neq qa - 1 \in rann_Q a := \{x \in Q \mid ax = 0\}$. Since $R \subset_e Q$, we get that $rann_R a \neq 0$. This is impossible since R is a domain.
Now, if $x \in Q$, there exists an essential right ideal E such that $0 \neq xE \subseteq R$. In particular there exists an element $e \in R$ such that $0 \neq xe \in R$. By the above xe is left invertible in Q, i.e. there exists $q \in Q$ such that $qxe = 1$. Direct finiteness of Q implies that $xeq = 1$. This shows that every nonzero element of Q is right invertible and hence Q is a division ring. This shows that R is left and right Ore. Hence σ is onto. $\square$

REMARK 6.3. Damiano showed that a right PCI-domain R which is left coherent is a left PCI-domain([**D**]). However his proof contains an error. He claims, by invoking a result in Stenstrom's book ([**S**] XI, Corollary 3.2) that the right maximal quotient ring of R is flat as right R-module. In fact the right maximal quotient ring is flat as a left R-module. This leads Damiano to the wrong conclusion that the right and left maximal quotient rings of R are isomorphic and R must be a left Ore domain.

Since a polynomial ring $R = K[t; \sigma, \delta]$ is always a semifir, it is both left and right coherent. If correct, Damiano's theorem would imply that if $R = K[t; \sigma, \delta]$ is a left Ore V-domain then it is also a right Ore domain. Consequently σ would be onto. We do not have any example of an Ore extension $K[t; \sigma, \delta]$ is a left V-domain with σ not onto.

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