RINGS IN WHICH ELEMENTS ARE SUM OF A CENTRAL ELEMENT AND AN ELEMENT IN THE JACOBSON RADICAL

Guanglin Ma, Yao Wang, Nanjing, André Leroy, Lens

(Received September 21, 2023)

Abstract. An element in a ring R is called CJ if it is of the form $c+j$, where c belongs to the center and j is an element from the Jacobson radical. A ring R is called CJ if each element of R is CJ. We establish the basic properties of CJ rings, give several characterizations of these rings, and connect this notion with many standard elementwise properties such as clean, uniquely clean, nil clean, CN, and CU. We study the behavior of this notion under various ring extensions. In particular, we show that the subring $C+J$ is always a CJ ring and that if $R[x]$ is a CJ ring then R satisfies the Koethe conjecture.

Keywords: CJ ring; center; Jacobson radical; clean ring

MSC 2020: 16U70, 16N20, 16N40

1. Introduction

Throughout this paper, all rings are associative with identity unless otherwise stated. A ring R is called *(uniquely) clean* if every element of R is (uniquely) the sum of an idempotent and a unit. A ring R is called *(uniquely) nil clean* if every element of R is (uniquely) the sum of an idempotent and a nilpotent. The concept of clean rings was first introduced by Nicholson (cf. [16]) in connection with lifting idempotents. This notion is at the heart of the development of elementwise properties in ring theory and was generalized in many directions. In particular, the notions of CN and CU rings have been introduced recently, cf. [11] and [12].

These two classes of rings are proper generalizations of commutative rings and uniquely clean rings. Inspired by these we introduce the notion of CJ rings.

The research has been supported by the China Scholarship Council (No. 202209040018) and the National Natural Science Foundation of China (No. 11071097).

If $C = C(R)$ and $J = J(R)$ denote the center and the Jacobson radical of a ring R, respectively, we consider the subring $C+J$. We say that R is a CJ ring if $C+J = R$.

In Section 2, we first establish the basic properties of CJ rings and give some examples. We relate the CJ property with other classical ones proving, in particular, that CJ rings are Dedekind finite and CU. On the other hand, uniquely clean rings and strongly nil clean are CJ rings. We also show that if $R[x]$ is a CJ ring, then R satisfies the Koethe conjecture. We give an example of a ring R such that R/J is commutative but R is not CJ and an example of a CJ ring with a non CJ subring.

Section 3 is devoted to studying the behavior of CJ rings under various ring extensions such as direct product, Dorroh extension, ideal extension, and $R\{D,C\}$ ring. Moreover, we prove that if R is a CJ ring, then some special subrings of full matrix rings over R are also CJ rings.

For a ring R, $U(R)$, $J(R)$, $Nil(R)$, $L(R)$, $Nil^*(R)$, and $C(R)$ denote the group of units, the Jacobson radical, the set of all nilpotent elements, the locally nilpotent radical, the upper nil radical, and the center of R, respectively. For a positive integer n, $\mathbb{Z}_n$ is the ring of integers modulo n and $M_n(R)$ denotes the full matrix ring over R, $T_n(R)$ stands for the subring of $M_n(R)$ consisting of all $n \times n$ upper triangular matrices and $GL_n(R)$ is the general linear group of $M_n(R)$. All other specific notations will be stated explicitly in the text.

2. Basic properties of CJ rings

In this section, we introduce CJ rings and study some basic properties of these rings.

Definition 2.1. Let R be a ring. An element $a \in R$ has a CJ decomposition if $a = c + j$ for some $c \in C(R)$ and $j \in J(R)$. We denote $C+J$ the set of elements that admit a CJ decomposition. The ring R is CJ if $R = C + J$

Example 2.1.

(1) Every commutative ring is CJ.
(2) Every radical ring ($J(R) = R$) is CJ.
(3) Every homomorphic image of a CJ ring is CJ.
(4) $C+J$ is a subring of R stable by automorphisms of R.
(5) $C(R[x]) + J(R[x]) = C(R)[x] + N'[x]$, where $N' = J(R[x]) \cap R$ is a nil ideal.
(6) $C(R[[x]]) + J(R[[x]]) = C(R) + (x)$.

There is no unicity of the decomposition of a CJ element as a sum of a central element and an element from the Jacobson radical. In the next proposition, we characterize the unique decomposition of CJ and CN rings.

Proposition 2.1. *Let R be a noncommutative ring. Then*

(1) *A CJ ring R is uniquely CJ if and only if $J(R) \cap C(R) = \{0\}$.*

(2) *A CN ring is uniquely CN if and only if $Nil(C) = 0$ (i.e., C is a reduced ring).*

Proof. (1) Suppose that R is noncommutative uniquely CJ and that $c \in C \cap J$. Let $0 \neq j \in J(R)$. We have $1 + j = (1 - c) + (c + j)$ and the unicity of the CJ decomposition gives that $c = 0$. Conversely if $C \cap J = \{0\}$ and $c + j = c' + j'$ are two CJ decompositions we have $c - c' = j' - j \in C \cap J = \{0\}$ and hence $c = c'$, $j = j'$, as required.

(2) Suppose R is uniquely CN. Let $c \in C$ be such that $c^l = 0$. For $n \in Nil(R) \setminus C(R)$, we have $1+n = (1-c)+(c+n)$ and the unicity of CN decomposition gives that $c = 0$, as required.

Conversely, suppose that R is CN and C is reduced. Then if $c + n = c' + n'$ are CN decompositions we get that $c - c' = n' - n$ and hence $n(n' - n) = n(c - c') = (c - c')n = (n' - n)n$. This gives that $n'n = nn'$ and so $c' - c = n - n' \in Nil(R)$. The fact that C is reduced leads to $c = c'$. This yields the proof. □

It is easy to see that if R is CJ, then R/J is commutative. But the following example (1) shows that the converse is not true.

Example 2.2.

(1) Let $\mathbb{C}$ be the complex field and $\sigma\colon \mathbb{C} \to \mathbb{C}$ given by $\sigma(a + b\mathrm{i}) = a - b\mathrm{i}$ be an automorphism of $\mathbb{C}$. Write $S = \mathbb{C}[x;\sigma]$ and $R = S/(x^2)$. Notice that the center of R is $\mathbb{R}$ which is reduced. So if R is CJ then it is uniquely CJ. Since $(x)/(x^2)$ is a nilpotent ideal, we have $(x)/(x^2) \subseteq J(R)$. In fact, since $(S/(x)^2)/((x)/(x)^2) \cong S/(x) \cong \mathbb{C}$, we have $J(R) = (x)/(x^2)$. So $R/J(R) \cong \mathbb{C}$ is commutative. Let $\bar{\mathrm{i}} + \bar{x} \in R$, $\bar{x} \in J(R)$. But $\bar{\mathrm{i}} \notin C(R)$ since $x\mathrm{i} + (x^2) = \sigma(\mathrm{i})x + (x^2) = -\mathrm{i}x + (x^2)$, $\mathrm{i} + (x^2) \notin C(R)$, and we quickly conclude that R is not a CJ ring.

(2) We now give an easy example of a CJ ring with a subring that is not CJ. Simply consider the twisted series ring $R = k[[x]][[t;\sigma]]$, where σ is the k-endomorphism of $k[[x]]$ defined by $\sigma(x) = x^2$. The center of R is k and the Jacobson radical of R is the ideal generated by x and t. Hence, R is CJ. The subring $S = k[x][t;\sigma]$ of R is obviously not CJ.

Proposition 2.2.

(1) *If R is a CJ ring, then $Nil(R) \subseteq J(R)$.*

(2) *If R is a CJ ring, then R is Dedekind finite.*

Proof. (1) Let R be a CJ ring and $a \in Nil(R)$. Then there exists $n \in \mathbb{N}$ such that $a^n = 0$. Let $\overline{R} = R/J$. Then $\bar{a}^n = 0$. Since R/J is commutative, $\bar{a} \in Nil(\overline{R}) = P(\overline{R}) \subseteq J(\overline{R})$. But $J(\overline{R}) = 0$. Thus, $\bar{a} = 0$ and hence $a \in J(R)$.

(2) Let R be a CJ ring and $ab = 1$. Put $\overline{R} = R/J$. Then $\bar{a}\bar{b} = \bar{1}$. Since R/J is commutative, we have $\bar{b}\bar{a} = \bar{1}$ and, from the fact that $1 + J(R) \subseteq U(R)$, we deduce that $ba \in U(R)$. Then $baba = ba$ and hence $ba(1 - ba) = 0$. But ba is a unit, so $ba = 1$. □

The next result is due to Amitsur. A ring is *locally nilpotent* if each of its finitely generated subrings (without unit) is nilpotent.

Lemma 2.1 ([1], Theorem 1). *We have $J(R[x]) = N[x]$ for a ring R, where $N = J(R[x]) \cap R$ is a nil ideal containing the locally nilpotent radical $L(R)$ of R.*

Proposition 2.3. *Let R be a ring such that $J(R)$ is locally nilpotent. Then R is CJ if and only if $R[x]$ is CJ.*

Proof. Assume that R is a CJ ring. Let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$. Then there exist $c_i \in C(R)$ and $j_i \in J(R)$ such that $a_i = c_i + j_i$ for any i. Let $c(x) = \sum_{i=0}^{n} c_i x^i$ and $j(x) = \sum_{i=0}^{n} j_i x^i$, we have $c(x) \in C(R[x])$ and $j(x) \in J(R)[x] \subseteq L(R)[x] \subseteq (J(R[x]) \cap R)[x] = J(R[x])$ by Lemma 2.1. It follows that $f(x) = c(x) + j(x)$ is a CJ decomposition of $f(x)$. The converse can be deduced from Example 2.1 (3). □

Let us remind the reader that the Koethe conjecture (cf. [14]) asks if the sum of two nil right ideals of ring is always nil. There are many equivalent forms of this conjecture and despite many efforts it is still open. One of these equivalent forms of the Koethe conjecture says that $J(R[x]) = Nil^*(R)[x]$. This conjecture is true, in particular, when the ring R is commutative.

Proposition 2.4. *If $R[x]$ is a CJ ring, then R satisfies the Koethe conjecture.*

Proposition 2.5. *Assume that $R[x]$ is CJ. Then $R[x]/J(R[x]) = R[x]/N[X] \cong R/N[x]$, where $N = J(R[x]) \cap R$ is a nil ideal. We have $0 = J(R[x]/J(R[x])) = J(R[x]/N[x]) = Nil^*(R/N)[x]$, where the last equality is due to the fact that $R/N[x]$ is commutative. Therefore, $Nil^*(R/N) = 0$ and hence, $Nil^*(R) \subseteq N$. Thus, $Nil^*(R) = N$.*

Proposition 2.6. *Let R be a ring. Then the following statements hold:*

(1) $U(R) \cap (C + J) \subseteq U(C + J)$.

(2) $J(R) \subseteq J(C + J)$.

P r o o f. (1) Let $a \in U(R) \cap (C+J)$. Then $a = c+j$ for a $c \in C(R)$ and a $j \in J(R)$ and there exists $b \in U(R)$ such that $ab = ba = 1$. We deduce that $c = a - j \in U(R)$ and hence $c \in C(R) \cap U(R) \subseteq U(C(R))$. Therefore, $ab = (c+j)b = cb + jb = 1$. Then $b = c^{-1}(1 - jb) \in C + J$. This shows that the inverse of a is indeed in $C + J$.

(2) Let $x \in J(R)$. Then $x \in C + J$ and $1 - xy \in U(R)$ for every $y \in R$. Thus, $1 - xy \in U(R)$ for every $y \in C + J$ and there exists $v \in U(R)$ such that $(1 - xy)v = v(1 - xy) = 1$. We thus obtain $v = 1 + xyv \in C + J$, and hence $1 - xy \in U(C + J)$ for every $y \in C + J$. This shows that $x \in J(C + J)$. □

Corollary 2.1. *Let R be a ring. Then $C + J$ is a CJ ring.*

Example 2.3. In general, $J(R) \subseteq J(C + J)$ may be a strict inclusion. Let k be a field of characteristic zero and $R = k[[x]][y][t; \sigma]$, where σ is the k-automorphism defined by $\sigma(x) = x$, $\sigma(y) = y + 1$. Let $\sum\limits_{i=0}^{m} \bigl(\sum s_{ji} y^j\bigr) t^i$, with $s_{ji} \in k[[x]]$, be a unit in R. Then there exists $\sum\limits_{k=0}^{n} \bigl(\sum t_{ji} y^j\bigr) t^k \in R$ such that

$$\sum_{i=0}^{m} \Bigl(\sum s_{ji} y^j\Bigr) t^i \sum_{k=0}^{n} \Bigl(\sum t_{ji} y^j\Bigr) t^k = 1.$$

If $m \geqslant 1$, we get a contradiction. So $m = 0$. Thus, $U(R) \subseteq U(k[[x]][y]) = k \setminus \{0\} + (x)$. Let $\sum\limits_{i=0}^{n} a_i t^i \in J(R) \setminus \{0\}$, where $a_i \in k[[x]][y]$. Then $1 - \Bigl(\sum\limits_{i=0}^{n} a_i t^i\Bigr)\Bigl(\sum\limits_{j=0}^{m} b_j t^j\Bigr) \in U(R)$ for every $\sum\limits_{j=0}^{m} b_j t^j \in R$. We have $\Bigl(\sum\limits_{i=0}^{n} a_i t^i\Bigr)\Bigl(\sum\limits_{j=0}^{m} b_j t^j\Bigr) \in k + (x)$, which is impossible. So we get $J(R) = 0$. But, since the characteristic of k is zero, $C(R) = k[[x]]$ and $J(C + J) = (x)$. In conclusion, $J(R) \subsetneq J(C + J)$.

In Proposition 2.7 and Example 2.4, we will show that all CJ rings are CU, but there exists a CU ring that is not CJ. Thus, the class of CJ rings is a proper subclass of the class of CU rings.

Proposition 2.7. *Every CJ ring is CU.*

P r o o f. Let R be a CJ ring and $a \in R$. By assumption, we have $a - 1 = c + j$ for some $c \in C(R)$ and $j \in J(R)$. Hence, $a = c + (1 + j)$ is a CU decomposition of a. □

Let $D_n(R) = \{(a_{ij}) \in T_n(R)\colon$ all main diagonal entries of (a_{ij}) are equal$\}$.

Example 2.4. Let K be a division ring and consider the ring $D_2(K)$. The ring $D_2(K)$ is a noncommutative local ring, and so it is a CU ring. But it is not a CJ ring.

Remark 2.1. If a ring R is such that $U(R) = 1 + J(R)$ (such a ring is called a JU ring in [5]) then R is CJ if and only if it is CU.

Let us recall a few classical definitions, cf. [7], [8], [9], [15], [17].

Definition 2.2. An element a of a ring R is called *uniquely clean* (or *uniquely nil clean*) if $a = e+u$, where $e^2 = e$ and $u \in U(R)$ (or $u \in Nil(R)$, respectively), and also this representation is unique. A ring R is called a *uniquely clean ring* (or *uniquely nil clean*) if every element is uniquely clean (or *uniquely nil clean*, respectively). Assuming e and u commute in the above definition leads to the notion of *uniquely strongly clean* (or *uniquely strongly nil clean*) elements and rings. When an involution $*$ is defined on a ring R we have the analogous $*$ definitions by replacing the idempotent by a projection (i.e., an idempotent stable by the involution $*$).

Proposition 2.8. *Every uniquely clean ring is CJ.*

Proof. Let R be a uniquely clean ring. Due to [17], Theorem 20, for all $a \in R$, there exists a unique idempotent $e \in R$ such that $e - a \in J(R)$. In addition, every idempotent in a uniquely clean ring is central by [17], Lemma 4. So every uniquely clean ring is CJ. This finishes the proof. □

The ring of integers $\mathbb{Z}$ is a simple example of a commutative ring that is not even clean.

We now characterize when a CJ ring is a uniquely clean ring.

Proposition 2.9. *The following conditions are equivalent for a ring R:*

(1) *R is uniquely clean.*

(2) *R is CJ with $C(R)$ is uniquely clean and all idempotents of R are in $C(R)$.*

Proof. (1) $\Rightarrow$ (2) This claim can be shown by Proposition 2.8 and it is clear that the center $C(R)$ of every uniquely clean ring R is again uniquely clean.

(2) $\Rightarrow$ (1) Given $r \in R$, say $r = c_0 + j$, $c_0 \in C(R)$, $j \in J(R)$, write $c_0 = e + u$, where $e^2 = e \in C(R)$ and u is a unit in C and so in R. Hence, $r = e + (u + j)$ and $u + j$ is a unit in R. Thus, R is clean. To show that R is uniquely clean, let $e + a = f + b$ in R, where $e^2 = e$, $f^2 = f \in R$, and $a, b \in U(R)$. Then write $a = c + v$ and $b = d + w$, where $c, d \in C(R)$ and $v, w \in J(R)$. Hence, c and d are units in R and $e, f \in C(R)$. Then $e + a = f + b$ becomes $e + c + v = f + d + w$. Hence, $v - w \in C(R)$ and so $e + (c + v - w) = f + d$ is a clean decomposition in $C(R)$, where $c + v - w$ and d are units of R. Since $C(R)$ is uniquely clean, we obtain $e = f$. This completes the proof of (1). □

It is known that uniquely nil clean rings are uniquely clean (cf. [7], Theorem 5.9) and hence uniquely nil clean rings are CJ. As a direct consequence of Proposition 2.8 we get the following corollary.

Corollary 2.2. *Strongly nil clean rings, uniquely strongly nil clean rings, strongly nil $*$-clean rings, and uniquely strongly nil $*$-clean rings are all CJ rings.*

P r o o f. These classes of rings are uniquely nil clean. Hence, we get the result by Proposition 2.8. □

Recall from [2] and [6] that a ring R is called a UU ring if $U(R) = 1 + Nil(R)$.

Proposition 2.10. *Let R be a UU ring. Then R is a CN ring if and only if R is a CU ring.*

Recall that an element r of a ring R is called *regular* (in the sense of von Neumann) if there exists $a \in R$ such that $r = rar$. We use $\mathrm{Reg}(R)$ to denote the set of regular elements. In [10], a ring is called *NR-clean* if every element is the sum of a nilpotent and a regular. The following constructions illustrate that the CJ and CN concepts are independent.

▷ There is a CN ring which is not a CJ ring.

If R is semiprimitive and NR-clean with $\mathrm{Reg}(R) \subseteq C(R)$, then R is CN, but R is not a CJ ring unless it is commutative.

▷ The ring constructed in Example 2.2 (2) is a CJ ring which is not a CN ring.

The proof of the following lemma is a direct consequence of the definitions.

Lemma 2.2.

(1) *If R is JU and CN, then it is CJ.*

(2) *If R is UU and CJ, then it is CN.*

A ring R is an *exchange* ring if, for every $a \in R$, there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$.

In the next theorem, we give some classes of rings such that CJ and CN notions coincide.

Proposition 2.11. *If a ring R satisfies one of the following conditions then R is CJ if and only if R is CN.*

(1) *R is Artinian JU.*

(2) *R is exchange UU.*

(3) *R is finite UU.*

P r o o f. (1) This is a direct consequence of the fact that in an Artinian ring, the Jacobson radical is nilpotent.

(2) According to [6], Corollary 4.5, $J(R) = Nil(R)$ when R is exchange UU. Hence, the rest of the proof is routine.

(3) Since a finite ring is semiperfect and hence an exchange ring, the assertion (2) yields the conclusion. □

Proposition 2.12. *Let R be a commutative ring. Then $T_2(R)$ is a CJ ring if and only if for any $a, b \in R$, there exists $c \in R$ such that $a - c, b - c \in J(R)$.*

In Proposition 2.12, one may suspect that $T_n(R)$ is CJ when R is CJ. The next example eliminates this possibility.

Example 2.5.

(1) Let $R = \mathbb{Z}$ and $A = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right) \in T_2(R)$. Then there is no $c \in \mathbb{Z}$ such that $2 - c$ and $3 - c$ belong to the Jacobson radical. By Proposition 2.12, we deduce that $T_2(\mathbb{Z})$ is not CJ.

(2) Since $M_n(R)/J(M_n(R)) \cong M_n(R/J(R))$, we remark that $M_n(R)$ is never a CJ ring if $n \geqslant 2$.

For any positive integer n, in spite of the fact that $T_n(R)$ need not be CJ if R is CJ, there are CJ subrings of $T_n(R)$. We denote by e_{ij} the $n \times n$ matrix whose (i, j)-entry is 1 and all other entries are 0. Let

$$V_n(R) = \Bigl\{ \sum_{i=j}^{n} \sum_{j=1}^{n} a_j e_{(i-j+1)i} \colon \ a_j \in R \Bigr\}.$$

Proposition 2.13. *For a ring R and an integer $n \geqslant 1$, the following are equivalent:*

(1) *R is CJ.*

(2) *$D_n(R)$ is CJ.*

(3) *$V_n(R)$ is CJ.*

Proof. (1) $\Rightarrow$ (2) We show this implication in the case $n = 4$. Let

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_1 & a_5 & a_6 \\ 0 & 0 & a_1 & a_7 \\ 0 & 0 & 0 & a_1 \end{pmatrix} \in D_4(R).$$

By (1), there exist $c \in C(R)$ and $j \in J(R)$ such that $a_1 = c + j$. Let

$$C = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} j & a_2 & a_3 & a_4 \\ 0 & j & a_5 & a_6 \\ 0 & 0 & j & a_7 \\ 0 & 0 & 0 & j \end{pmatrix}.$$

Then $C \in C(D_4(R))$ and $J \in J(D_4(R))$.

(2) $\Rightarrow$ (1) We consider the case $n = 4$. Let $a \in R$. By (2),

$$A = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \in D_4(R)$$

has a CJ decomposition $A = C + J$, where

$$C = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \in C(D_4(R)) \quad \text{and} \quad J = \begin{pmatrix} j & * & * & * \\ 0 & j & * & * \\ 0 & 0 & j & * \\ 0 & 0 & 0 & j \end{pmatrix} \in J(D_4(R)).$$

Then $a = c + j$ with $c \in C(R)$ and $j \in J(R)$.

The equivalence (1) $\Leftrightarrow$ (3) is proved similarly. □

3. Extensions of CJ rings

In this section, we investigate several kinds of ring extensions of CJ rings which can be helpful to related studies.

Proposition 3.1. *Let $R = \prod_{i \in I} R_i$ be a direct product of rings. Then R is CJ if and only if R_i is CJ for each $i \in I$.*

Proof. This claim is clear. □

Let R be a ring and T be a ring (possibly with no unity) which is an (R, R)-bimodule such that $(st)r = s(tr)$, $(sr)t = s(rt)$ and $(rs)t = r(st)$ hold for all $s, t \in T$ and $r \in R$. Then the ideal extension $I(R; T)$ of R by T is defined to be the additive Abelian group $I(R; T) = R \oplus T$ with multiplication $(r, t)(s, w) = (rs, rw + ts + tw)$. Notice that $(1, 0)$ is the unity of S.

Lemma 3.1. *Let $S = I(R; T)$ be as in the above definition and suppose that $J(T)$ is an $R - R$ bimodule of T such that $J(R)T \subseteq J(T)$. Then*
(1) $(0, J(T)) \subseteq J(S)$,
(2) $(J(R), 0) \subseteq J(S)$,
(3) $J(S) = (J(R), J(T))$.

Proof. (1) By hypothesis, $J(T)$ is stable by left and right multiplication by elements of R and this easily leads to the fact that $(0, J(T))$ is a two sided ideal of S. Moreover, we easily check that elements of $(0, J(T))$ are quasi-regular in S. This yields the proof of statement (1).

(2) Let $(J(R), 0)$ be an ideal of S. Now, if $j \in J(R)$ and $(r, w) \in S$ we compute $(1, 0) - (j, 0)(r, w) = (1 - jr, -jw)$ and, since $j \in J(R)$, we have $1 - jr \in U(R)$. We can thus write $(1, 0) - (j, 0)(r, w) = (1 - jr)(1, 0) - (0, (1 - jr)^{-1}jw) \in U(S)$, since $J(T)$ is stable by left and right multiplication by elements of R and, by the first statement, $(J(R), 0) \subseteq J(S)$.

(3) Thanks to the statements (1) and (2), it is enough to prove that $J(S) \subseteq (J(R), J(T))$. So let $(j,t) \in J(S)$. For any $(r,s) \in S$ we have that $(1,0) - (j,t)(r,s) \in U(S)$ and so $1 - jr \in U(R)$. Since this is true for every $r \in R$, we have that $j \in J(R)$. By the first statement we thus have $(j,0) \in J(S)$ and we conclude that $(0,t) \in J(S)$. Hence, there exists an element $(r,w) \in S$ such that $(0,t) + (r,w) + (0,t)(r,w) = (0,0)$ and we conclude that $r = 0$ and $t + w + tw = 0$, this shows that $t \in J(T)$, as requested. □

Proposition 3.2. *Let R, S, T be as in the above lemma and suppose that $ct = tc$ for any $c \in C(R)$ and for any $t \in T$. Then S is CJ if and only if R and T are CJ.*

P r o o f. The above lemma shows that $J(S) = (J(R), J(T))$ and the additional hypothesis gives that $C(S) = (C(R), C(T))$. This yields the conclusion. □

An important case of the idealization construction is the Dorroh extension: Let R be an algebra over a commutative ring S. Recall that the *Dorroh extension* of R by S is the ring $D(S,R)$ with operations $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1 r_2 + s_1 r_2 + s_2 r_1, s_1 s_2)$, where $r_i \in R$ and $s_i \in S$. It is clear that $C(D(S,R)) = S \oplus C(R)$. The identity of $D(S,R)$ is (1,0).

Proposition 3.3. *Let R be an algebra over a field F. Then R is CJ if and only if $D(F,R)$ is CJ.*

P r o o f. It is enough to check that all the hypotheses of the proposition are satisfied in the case of a Dorroh extension. We leave this check to the reader. □

Let B be a subring of the ring A with $1_A \in B$. We set

$$\begin{aligned} R\{A,B\} &= \{(a_1, \dots, a_n, b_{n+1}, b_{n+2}, \dots)\colon\ b_i \in B,\ a_j \in A,\ n \geqslant 1\},\\ R(A,B) &= \{(a_1, \dots, a_n,\ b_{n+1}, b_{n+2}, \dots) \in R\{A,B\}\colon \text{ only a finite number of } b_j \\ &\qquad \text{are not zero}\},\\ R[A,B] &= \{(a_1, \dots, a_n, b, b, \dots)\colon\ a_i \in A,\ b \in B,\ n \geqslant 1\}. \end{aligned}$$

On the above three sets, we define addition and multiplication componentwise, it is easy to see that they are all rings. Also, $J(R\{A,B\}) = R\{J(A), J(A) \cap J(B)\}$, see [18]. Note that $C(R\{A,B\}) = R\{C(A), C(A) \cap C(B)\}$.

Proposition 3.4. *Let B be a subring of the ring A. If $R\{A,B\}$ is CJ, then A and B are CJ. The converse holds if $C(B) \subseteq C(A)$ and $J(B) \subseteq J(A)$.*

P r o o f. Assume that $R\{A,B\}$ is a CJ ring, since both A and B are homomorphic images of $R\{A,B\}$, it is clear that both A and B are CJ.

Conversely, let A and B be CJ rings and $Y=(a_1,a_2,\ldots,a_m,s_{m+1},s_{m+2},\ldots)$ be an arbitrary element in $R\{A,B\}$. Then there exist $c_i\in C(A)$, $b_i\in J(A)$, $1\leqslant i\leqslant m$, and $c_j\in C(B)$, $t_j\in J(B)$, $j>m$ such that $a_i=c_i+b_i$ for all $1\leqslant i\leqslant m$ and $s_j=c_j+t_j$ for any $j>m$. Let $C=(c_1,c_2,\ldots,c_m,c_{m+1},c_{m+2},\ldots)$ and $J=(b_1,b_2,\ldots,b_m,t_{m+1},t_{m+2},\ldots)$. Then $C\in C(R\{A,B\})$ and $J\in J(R\{A,B\})$ since $C(B)\subseteq C(A)$ and $J(B)\subseteq J(A)$. Hence, $Y=C+J$ is a CJ decomposition of Y. This finishes the proof. □

Corollary 3.1. *Let B be a subring of a ring A. If $C(B)\subseteq C(A)$ and $J(B)\subseteq J(A)$, then the following statements are equivalent:*
(1) *A and B are CJ.*
(2) *$R(A,B)$ is CJ.*
(3) *$R[A,B]$ is CJ.*

P r o o f. The proof is similar to the proof of Proposition 3.4. □

We noticed earlier that if $n\geqslant 2$, then $M_n(R)$ is not CJ. We now exhibit some subrings of $M_3(R)$ that are CJ when R is CJ. Let R be a ring, and $s,t\in C(R)$. Write

$$L_{(s,t)}(R)=\left\{\begin{pmatrix} a & 0 & 0\\ sc & d & te\\ 0 & 0 & f\end{pmatrix}\in M_3(R)\colon\ a,c,d,e,f\in R\right\},$$

where the operations are defined as those in $M_3(R)$. Then $L_{(s,t)}(R)$ is a subring of $M_3(R)$. Let I_n denote the $n\times n$ matrix unit.

Lemma 3.2. *Let R be a ring, and let $s,t\in C(R)$. Then the following statements hold:*
(1) *The set of all unit elements of $L_{(s,t)}(R)$ is*

$$U(L_{(s,t)}(R))=\left\{\begin{pmatrix} a & 0 & 0\\ sc & d & te\\ 0 & 0 & f\end{pmatrix}\in L_{(s,t)}(R)\colon\ a,d,f\in U(R),\ c,e\in R\right\}.$$

(2) *The Jacobson radical of $L_{(s,t)}(R)$ is*

$$J(L_{(s,t)}(R))=\left\{\begin{pmatrix} a & 0 & 0\\ sc & d & te\\ 0 & 0 & f\end{pmatrix}\in L_{(s,t)}(R)\colon\ a,d,f\in J(R),\ c,e\in R\right\}.$$

(3) *The set of all central elements of* $L_{(s,t)}(R)$ *is*

$$C(L_{(s,t)}(R)) = \left\{ \begin{pmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{pmatrix} \in L_{(s,t)}(R) \colon \ sa = sd, \ td = t\,f, a, d, f \in C(R) \right\}.$$

P r o o f. (1) Assume that $a, d, f \in U(R)$ and let a^{-1}, d^{-1} and f^{-1} denote the inverses of a, d and f, respectively. Let

$$A = \begin{pmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{pmatrix} \in L_{(s,t)}(R) \text{ and } B = \begin{pmatrix} a^{-1} & 0 & 0 \\ -d^{-1}sca^{-1} & d^{-1} & -d^{-1}tef^{-1} \\ 0 & 0 & f^{-1} \end{pmatrix} \in L_{(s,t)}(R).$$

Then $B = A^{-1} \in L_{(s,t)}(R)$, and hence $A \in U(L_{(s,t)}(R))$.

Conversely, suppose that

$$A = \begin{pmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{pmatrix} \in U(L_{(s,t)}(R)) \quad \text{with the inverse} \quad B = \begin{pmatrix} x & 0 & 0 \\ y & z & u \\ 0 & 0 & v \end{pmatrix}.$$

Then $AB = BA = I_3$. Comparing entries we reach $ax = xa = 1$, $dz = zd = 1$ and $fv = vf = 1$. Hence, $a, d, f \in U(R)$.

(2) Assume that $a, d, f \in J(R)$. Let

$$A = \begin{pmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{pmatrix} \in L_{(s,t)}(R).$$

For any element

$$B = \begin{pmatrix} x & 0 & 0 \\ sy & z & tu \\ 0 & 0 & v \end{pmatrix} \in L_{(s,t)}(R),$$

we have

$$I_3 - AB = \begin{pmatrix} 1 - ax & 0 & 0 \\ s(cx + dy) & 1 - dz & t(du + ev) \\ 0 & 0 & 1 - fv \end{pmatrix}.$$

By our hypothesis, $1-ax, 1-dz, 1-fv \in U(R)$. Then by (1), $I_3 - AB \in U(L_{(s,t)}(R))$. Thus, $A \in J(L_{(s,t)}(R))$.

Conversely, suppose that $a, d, f \in R$ and

$$A = \begin{pmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{pmatrix} \in J(L_{(s,t)}(R)).$$

For any $x, y, z, v, u \in R$, let

$$B = \begin{pmatrix} x & 0 & 0 \\ sy & z & tu \\ 0 & 0 & v \end{pmatrix} \in L_{(s,t)}(R),$$

we have $I_3 - AB \in U(L_{(s,t)}(R))$, i.e.,

$$I_3 - AB = \begin{pmatrix} 1-ax & 0 & 0 \\ s(cx+dy) & 1-dz & t(du+ev) \\ 0 & 0 & 1-fv \end{pmatrix} \in U(L_{(s,t)}(R)).$$

The statement (1) then gives $1-ax, 1-dz, 1-fv \in U(R)$. This shows that $a, d, f \in J(R)$.

(3) This claim was shown in [12], Lemma 3.1. □

Consider the following subring of $L_{(s,t)}(R)$,

$$V_2(L_{(s,t)}(R)) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & te \\ 0 & 0 & a \end{pmatrix} \in L_{(s,t)}(R) \colon \ a, e \in R \right\}.$$

Proposition 3.5. *Let R be a ring. Then the following statements hold:*

(1) *R is a CJ ring if and only if $V_2(L_{(s,t)}(R))$ is a CJ ring.*

(2) *Assume that R is a CJ ring. Suppose that any $\{a, d, f\} \subseteq R$ having a CJ decomposition $a = x + p, d = y + q$ and $f = z + r$ with $\{x, y, z\} \subseteq C(R)$ and $\{p, q, r\} \subseteq \mathrm{J}(R)$ satisfy $sx = sy$ and $ty = tz$, then $L_{(s,t)}(R)$ is a CJ ring.*

P r o o f. (1) Assume that R is a CJ ring. Let

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & a & te \\ 0 & 0 & a \end{pmatrix} \in V_2(L_{(s,t)}(R).$$

Then there exist $c \in C(R)$ and $j \in J(R)$ such that $a = c + j$. Hence,

$$C = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \in C(L_{(s,t)}(R)) \quad \text{and} \quad J = \begin{pmatrix} j & 0 & 0 \\ 0 & j & te \\ 0 & 0 & j \end{pmatrix} \in J(V_2(L_{(s,t)}(R))),$$

and $A = C + J$ is the CJ decomposition of A in $V_2(L_{(s,t)}(R))$.

For the converse implication, let $r \in R$ and consider

$$A = \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{pmatrix} \in V_2(L_{(s,t)}(R)).$$

Then there exist

$$C = \begin{pmatrix} a & 0 & 0 \\ 0 & a & te \\ 0 & 0 & a \end{pmatrix} \in C(V_2(L_{(s,t)}(R))) \quad \text{and} \quad J = \begin{pmatrix} p & 0 & 0 \\ 0 & p & tu \\ 0 & 0 & p \end{pmatrix} \in J(V_2(L_{(s,t)}(R))).$$

Thus, $a \in C(R)$ and $p \in J(R)$, and $r = a+p$ is the CJ decomposition of r. Hence, R is a CJ ring.

(2) Suppose that

$$A = \begin{pmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{pmatrix} \in L_{(s,t)}(R).$$

Let $a = x + p$, $d = y + q$ and $f = z + r$ denote the CJ decompositions of a, d and f, respectiviely. By our hypothesis $sx = sy$ and $ty = tz$. By Lemma 3.2, we deduce that A has a CJ decomposition in $L_{(s,t)}(R)$ as $A = C + J$, where

$$C = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} \in C(L_{(s,t)}(R)) \quad \text{and} \quad J = \begin{pmatrix} p & 0 & 0 \\ sc & q & te \\ 0 & 0 & r \end{pmatrix} \in J(L_{(s,t)}(R)).$$

□

Corollary 3.2. *Let R be a ring. If $L_{(s,t)}(R)$ is a CJ ring, then R is a CJ ring.*

Proof. Assume that $L_{(s,t)}(R)$ is a CJ ring and let $a \in R$ and

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \in L_{(s,t)}(R).$$

By our hypothesis A has a CJ decomposition in $L_{(s,t)}(R)$. This easily leads to the fact that a admits a CJ decomposition in R. □

The next example shows that there are CJ rings such that $L_{(s,t)}(R)$ need not be a CJ ring.

Example 3.1. Let $R = \mathbb{Z}$ and

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix} \in L_{(1,1)}(R).$$

Assume that $A = C + J$ is a CJ decomposition of A. Since A is neither central nor an element in the Jacobson radical, by Lemma 3.2, we should get A had a CJ decomposition as $A = C + J$, where

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in C(L_{(1,1)}(R)) \quad \text{and} \quad J = \begin{pmatrix} x & 0 & 0 \\ c & y & e \\ 0 & 0 & z \end{pmatrix} \in J(L_{(1,1)}(R)),$$

where $\{x, y, z\} \subseteq J(\mathbb{Z})$. This leads to a contradiction.

Proposition 3.6. *A ring R is CJ ring if and only if so is $L_{(0,0)}(R)$.*

P r o o f. Note that $L_{(0,0)}(R)$ is isomorphic to the ring $R \times R \times R$. By Proposition 3.1, this ring is CJ if and only if each R_i is a CJ ring for each $i \in I$. □

Let R be a ring and $s, t \in C(R)$. Let $H_{(s,t)}(R)$ be the subring of $M_3(R)$ given by

$$H_{(s,t)}(R) = \left\{ \begin{pmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{pmatrix} \in M_3(R) \colon\ a, c, d, e, f \in R,\ a - d = sc,\ d - f = te \right\}.$$

Any element A of $H_{(s,t)}(R)$ has he form

$$\begin{pmatrix} sc + te + f & 0 & 0 \\ c & te + f & e \\ 0 & 0 & f \end{pmatrix}.$$

Lemma 3.3. *Let R be a ring, and let $s, t \in C(R)$. Then the following statements hold:*

(1) *The set of all central elements of $H_{(s,t)}(R)$ is*

$$C(H_{(s,t)}(R)) = \left\{ \begin{pmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{pmatrix} \in H_{(s,t)}(R) \colon\ c, e, f \in C(R) \right\}.$$

(2) *The set of all unit elements of $H_{(s,t)}(R)$ is*

$$U(H_{(s,t)}(R)) = \left\{ \begin{pmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{pmatrix} \in H_{(s,t)}(R) \colon\ a, d, f \in U(R),\ c, e \in R \right\}.$$

(3) *The Jacobson radical of $H_{(s,t)}(R)$ is*

$$J(H_{(s,t)}(R)) = \left\{ \begin{pmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{pmatrix} \in H_{(s,t)}(R) \colon\ a, d, f \in J(R),\ c, e \in R \right\}.$$

P r o o f. (1) and (2) are proved in [11], Lemmas 3.3 and 3.4, respectively.

(3) Assume that $a, d, f \in J(R)$, let

$$A = \begin{pmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{pmatrix} \in H_{(s,t)}(R)$$

for any $x, y, z, u, v \in R$, let

$$B = \begin{pmatrix} x & 0 & 0 \\ y & z & u \\ 0 & 0 & v \end{pmatrix}$$

Then

$$E_3 - AB = \begin{pmatrix} 1-ax & 0 & 0 \\ y & 1-dz & u \\ 0 & 0 & 1-fv \end{pmatrix} \in H_{(s,t)}(R).$$

Since $1-ax, 1-dz, 1-fv \in U(R)$, due to (2), we get $E_3 - AB \in U(H_{(s,t)}(R))$. This implies that $A \in J(H_{(s,t)}(R))$.

Conversely, suppose that

$$A = \begin{pmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{pmatrix} \in J(H_{(s,t)}(R)).$$

For any $x, y, z, u, v \in R$, let

$$B = \begin{pmatrix} x & 0 & 0 \\ y & z & u \\ 0 & 0 & v \end{pmatrix}.$$

Then

$$E_3 - AB = \begin{pmatrix} 1-ax & 0 & 0 \\ y & 1-dz & u \\ 0 & 0 & 1-fv \end{pmatrix} \in U(H_{(s,t)}(R)).$$

According to (2), we have $1-ax, 1-dz, 1-fv \in U(R)$. Thus, we obtain $a, d, f \in J(R)$. □

Proposition 3.7. *A ring R is a CJ ring if and only if $H_{(s,t)}(R)$ is a CJ ring.*

P r o o f. Assume that R is CJ. Let

$$A = \begin{pmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{pmatrix} \in H_{(s,t)}(R).$$

Let $f = f_1 + f_2$, $e = e_1 + e_2$ and $c = c_1 + c_2$ denote the CJ decompositions of f, e and c, where $f_1, e_1, c_1 \in J(R)$ and $f_2, e_2, c_2 \in C(R)$. Choose $d_2 = f_2 + te_2$ and

$a_2 = d_2 + sc_2$. We can conclude from Lemma 3.3 that

$$C = \begin{pmatrix} a_2 & 0 & 0 \\ c_2 & d_2 & e_2 \\ 0 & 0 & f_2 \end{pmatrix} \in C(H_{(s,t)}(R)).$$

Moreover, $d - d_2 = d - f_2 - te_2 = d - f + f_1 - te_2 = f_1 - te - te_2 \in J(R)$ since $f_1 \in J(R)$ and $te - te_2 \in J(R)$. Similarly, $a - a_2 = d - d_2 + sc - sc_2 \in J(R)$. Let

$$J = \begin{pmatrix} a_1 & 0 & 0 \\ c_1 & d_1 & e_1 \\ 0 & 0 & f_1 \end{pmatrix}$$

with $a_1 = a - a_2$ and $d_1 = d - d_2$. By Lemma 3.3, we get $J \in J(H_{(s,t)}(R))$. Hence, $A = C + J$ is a CJ decomposition of A.

Conversely, suppose that $H_{(s,t)}(R)$ is CJ and $a \in R$. Let

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \in H_{(s,t)}(R).$$

Then there exist

$$C = \begin{pmatrix} x' & 0 & 0 \\ y' & z' & t' \\ 0 & 0 & u' \end{pmatrix} \in C(H_{(s,t)}(R)) \quad \text{and} \quad J = \begin{pmatrix} x & 0 & 0 \\ y & z & t \\ 0 & 0 & u \end{pmatrix} \in J(H_{(s,t)}(R))$$

such that $A = C + J$. Hence, $u' \in C(R)$ and $u \in J(R)$. Therefore, $a = u' + u$ is a CJ decomposition of a. □

Since Jacobson radical has no nonzero idempotents, one might reasonably expect that CJ rings are abelian. This is not the case as the following example shows.

Example 3.2. Let $R = H_{(1,t)}(\mathbb{Z})$. Then R is a CJ ring by Proposition 3.7. But

$$\begin{pmatrix} 1 & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R$$

is a noncentral idempotent if x is not central in R. Thus R is not abelian.

Let R be a ring and s a central element of R. Then $\begin{pmatrix} R & R \\ R & R \end{pmatrix}$ becomes a ring denoted by $K_s(R)$ with addition defined componentwise and with multiplication defined in [13] by

$$\begin{pmatrix} a_1 & x_1 \\ y_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & x_2 \\ y_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + sx_1y_2 & a_1x_2 + x_1b_2 \\ y_1a_2 + b_1y_2 & sy_1x_2 + b_1b_2 \end{pmatrix}.$$

In [13], $K_s(R)$ is called a *generalized matrix ring* over R.

Lemma 3.4. *Let R be a commutative ring. Then the following statements hold.*
(1) $J(K_0(R)) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(R)\colon\ a, b \in J(R) \right\}$.
(2) $C(K_0(R))$ *consists of all scalar matrices.*

P r o o f. (1) and (2) are proved in [13]. □

Lemma 3.5. *The ring R is a CJ ring if and only if $D_n(K_0(R))$ is a CJ ring.*

P r o o f. Let R be a CJ ring, we assume that $n = 2$. Let $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in D_2(K_0(R))$. By the assumption, $a = c_1 + j_1$, where $c_1 \in C(K_0(R))$ and $j_1 \in J(K_0(R))$. Let $C = \begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix} \in C(D_2(K_0(R)))$ and $J = \begin{pmatrix} j_1 & b \\ 0 & j_1 \end{pmatrix} \in J(D_2(K_0(R)))$. Then $A = C + J$ is the CJ decomposition of A.

To show the converse, let $a \in R$, and $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in D_2(K_0(R))$ has a CJ decomposition $A = C + J$ with $C = \begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix} \in C(D_2(K_0(R)))$ and $J = \begin{pmatrix} j_1 & b_1 \\ 0 & j_1 \end{pmatrix} \in J(D_2(K_0(R)))$, where $c_1 \in C(R)$ and $j_1 \in J(R)$. By comparing components of matrices we get $a = c_1 + j_1$. It is a CJ decomposition of a. □

Note that $K_0(R)$ need not be a CJ ring.

Example 3.3. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in K_0(\mathbb{Z})$ have a CJ decomposition as $A = C + J$, where $C \in C(K_0(\mathbb{Z}))$ and $J \in J(K_0(\mathbb{Z}))$. Then we should have $C = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ and $J = \begin{pmatrix} 1-x & 0 \\ 0 & -x \end{pmatrix}$. These imply $x = 1$ or $x \in J(R)$. This leads to a contradiction.

Given a ring R and a group G, we denote the *group ring* of R over G by $R[G]$. An arbitrary element α of $R[G]$ is of the form $\alpha = \sum\limits_{g \in G} r_g g$, where the support of α, $\operatorname{supp}(\alpha) = \{g \in G\colon\ r_g \neq 0\}$, is assumed to be finite. If we define $\omega\colon R[G] \to R$ by $\omega\Bigl(\sum\limits_{g \in G} r_g g\Bigr) = \sum\limits_{g \in G} r_g$, then ω is a ring epimorphism. The kernel of this epimorphism $\ker(\omega)$ (denoted by $\omega(R[G])$) is called the *augmentation ideal* of $R[G]$ and it is well known that $\omega(R[G])$ is generated by the set $\{1 - g\colon\ g \in G\}$. Recall that a group G is called *locally finite* if every finitely generated subgroup of G is finite.

Lemma 3.6 ([3]). *If R is any ring and G is a locally finite group, then $J(R)G \subseteq J(RG)$.*

In fact, Connell [4] shows that $J(R) = J(RG) \cap R$ for any ring R and locally finite group G. However, a simple proof of Lemma 3.6 is given in [3].

Proposition 3.8. *If R is a CJ ring and G is a locally finite Abelian group, then RG is CJ if and only if R is CJ.*

Proof. It is easy to see that $C(RG) = C(R)G$. Assume R is CJ and let $w = \sum a_i g_i \in RG$. Then $a_i = c_i + j_i$, where $c_i \in C(R)$ and $j_i \in J(R)$. Hence, we have $w = \sum(c_i + j_i)g_i = \sum c_i g_i + \sum j_i g_i \in C(R)G + J(R)G \subseteq C(RG) + J(RG)$ by the hypothesis and Lemma 3.6. The converse is clear since $RG/\omega(RG) \cong R$. $\square$

Acknowledgements. The authors would like to express their special deep thanks to Dr. Mehrdad Nasernejad for his helpful discussions. They are also grateful to the referee for his valuable comments.

References

[1] *S. A. Amitsur*: Radicals of polynomial rings. Can. J. Math. *8* (1956), 355–361. zbl MR doi
[2] *G. Călugăreanu*: UU rings. Carpathian J. Math. *31* (2015), 157–163. zbl MR
[3] *J. Chen, W. K. Nicholson, Y. Zhou*: Group rings in which every element is uniquely the sum of a unit and an idempotent. J. Algebra *306* (2006), 453–460. zbl MR doi
[4] *I. G. Connell*: On the group ring. Can. J. Math. *15* (1963), 650–685. zbl MR doi
[5] *P. V. Danchev*: Rings with Jacobson units. Toyama Math. J. *38* (2016), 61–74. zbl MR
[6] *P. V. Danchev, T.-Y. Lam*: Rings with unipotent units. Publ. Math. Debr. *88* (2016), 449–466. zbl MR doi
[7] *A. J. Diesl*: Nil clean rings. J. Algebra *383* (2013), 197–211. zbl MR doi
[8] *A. Harmanci, H. Chen, A. Ç. Özcan*: Strongly nil $*$-clean rings. J. Algebra Comb. Discrete Struct. Appl. *4* (2017), 155–164. zbl MR doi
[9] *Y. Hirano, H. Tominaga, A. Yaqub*: On rings in which every element is uniquely expressable as a sum of a nilpotent element and a certain potent element. Math. J. Okayama Univ. *30* (1988), 33–40. zbl MR
[10] *A. H. Khashan*: NR-clean rings. Vietnam J. Math. *44* (2016), 749–459. zbl MR doi
[11] *Y. Kurtulmaz, S. Halicioglu, A. Harmanci, H. Chen*: Rings in which elements are a sum of a central and a unit element. Bull. Belg. Math. Soc. - Simon Stevin *26* (2019), 619–631. zbl MR doi
[12] *Y. Kurtulmaz, A. Harmanci*: Rings in which elements are a sum of a central and a nilpotent element. Available at `https://arxiv.org/abs/2005.12575` (2020), 15 pages. doi
[13] *P. A. Krylov*: Isomorphism of generalized matrix rings. Algebra Logic *47* (2008), 258–262. zbl MR doi
[14] *T. Y. Lam*: A First Course in Noncommutative Rings. Graduate Text in Mathematics 131. Springer, New York. zbl MR doi
[15] *C. Li, Y. Zhou*: On strongly $*$-clean rings. J. Algebra Appl. *10* (2011), 1363–1370. zbl MR doi
[16] *W. K. Nicholson*: Lifting idempotents and exchange rings. Trans. Am. Math. Soc. *229* (1977), 269–278. zbl MR doi
[17] *W. K. Nicholson, Y. Zhou*: Rings in which elements are uniquely the sum of an idempotent and a unit. Glasg. Math. J. *46* (2004), 227–236. zbl MR doi
[18] *Y. Wang, H. Zhou, Y. Ren*: Some properties of ring $R\{D, C\}$. Math. Pract. Theory *50* (2020), 173–181. zbl

Authors' addresses: Guanglin Ma (corresponding author), Yao Wang, School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Shangxian Building No.725, Nanjing, P. R. China, e-mail: `linlinguangma@163.com`, `wangyao@nuist.edu.cn`; André Leroy, Laboratoire Mathématiques Lens (LML), University of Artois, U. R. 2462, Rue Jean Souvraz SP18, 62307 Lens, France, e-mail: `andre.leroy@univ-artois.fr`