ALGEBRAIC AND F-INDEPENDENT SETS IN 2-FIRS

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Abstract. Let R denote a *2*-fir. The notions of F-independence and algebraic subsets of R are defined. The decomposition of an algebraic subset into similarity classes gives a simple way of translating the F-independence in terms of dimension of some vector spaces. In particular to each element $a \in R$ is attached a certain algebraic set of atoms and the above decomposition gives a lower bound of the length of the atomic decompositions of a in terms of dimensions of certain vector spaces. A notion of rank is introduced and fully reducible elements are studied in details.

1. Introduction and preliminaries

The main goal of this paper is to study factorizations in *2*-firs via a careful use of classical notions such as similarity and a systematic use of new notions such as algebraicity and F-independence. An attempt has been made to keep the paper relatively self-contained and examples have been given all along the paper to facilitate the reading.

Let us recall that a ring R is a *2*-fir if any right ideal of R generated by at most 2 elements is free of unique rank. Of course, a *2*-fir is a domain and it can be shown (Cf. [3]) that this definition is equivalent to the following one : A domain R is a *2*-fir if and only if

$$\forall a, b \in R,\ aR \cap bR \neq 0 \Rightarrow \exists c, d \in R \ :\ aR \cap bR = cR\ ;\ aR + bR = dR\,.$$

The lack of symmetry in this definition is only apparent and in the paper we will freely use the fact that it is in fact symmetric. For the convenience of the reader we include a proof of this fact. We first state a useful lemma (Cf. [3] and [4]).

Lemma 1.1. *Let R be a domain and a, a' be nonzero elements in R. Then, the following are equivalent :*

(i) $R/aR \cong R/a'R$.
(ii) $\exists\, b \in R$ *such that* $aR + bR = R$ *and* $aR \cap bR = ba'R$.
(iii) $\exists\, b' \in R$ *such that* $Ra' + Rb' = R$ *and* $Ra' \cap Rb' = Rab'$.

(iv) $R/Ra \cong R/Ra'$.

If $b \in R$ is as in ii) above, there exists $b' \in R$ satisfying the equalities in iii) and such that $ba' = ab'$. Moreover we then have $R/Rb \cong R/Rb'$.

Proof. $i) \iff ii)$ and $iii) \iff iv)$ are easy and left to the reader.
$ii) \Longrightarrow iii)$ Since $ba' \in aR$ and $aR+bR = R$, one can find $b', c', d' \in R$ such that $ba' = ab'$ and $ad' - bc' = 1$. This leads to $a(d'a-1) = bc'a \in aR \cap bR = ba'R = ab'R$. Hence there exists $c \in R$ such that $c'a = a'c$ and $d'a - 1 = b'c$. Similarly, we have $ad'b = b(c'b+1) \in aR \cap bR = ba'R$, hence there exists $d \in R$ such that $c'b + 1 = a'd$. We now get $a'cb' = c'ab' = c'ba' = (a'd-1)a' = a'(da'-1)$. This gives $cb' = da' - 1$ and thus $Rb' + Ra' = R$.
Now, if $x = pa' = qb' \in Ra' \cap Rb'$ we get $q(d'a-1) = qb'c = pa'c = pc'a \in Ra$. This shows that $q \in Ra$ and $x \in Rab' = Rba'$. We conclude $Ra' \cap Rb' = Rab'$.
$iii) \Longrightarrow ii)$ This is given by duality using the opposite ring R^{op}.
The last statement can be obtained by finding the right equations in the above proof and by using the following:

$$\frac{R}{Rb'} \cong \frac{Ra' + Rb'}{Rb'} \cong \frac{Ra'}{Ra' \cap Rb'} \cong \frac{Ra'}{Rab'} \cong \frac{Ra'}{Rba'} \cong \frac{R}{Rb}.$$

□

Let us mention that the equivalence $(i) \longleftrightarrow (iv)$ is due to Fitting (Cf [5]). We now get the desired left-right symmetry of the definition of a *2*-fir

Corollary 1.2. *Let R be a domain. The following are equivalent :*
i) $\forall a, b \in R,\ aR \cap bR \neq 0 \Rightarrow \exists c, d \in R\ :\ aR \cap bR = cR\ ;\ aR + bR = dR$.
ii) $\forall s, t \in R,\ Rs \cap Rt \neq 0 \Rightarrow \exists u, v \in R\ :\ Rs \cap Rt = Ru\ ;\ Rs + Rt = Rv$.

Proof. Of course, we will only prove that $i)$ implies $ii)$. So let $s, t \in R$ be such that $Rs \cap Rt \neq 0$. We can find $a, b \in R$ such that $0 \neq as = bt$ and $i)$ shows that there exist $c, d \in R$ such that $aR \cap bR = cR$ and $aR + bR = dR$. Writing $c = ab' = ba'$, $a = dx$, and $b = dy$, we get $dxb' = ab' = ba' = dya'$. Since R is a domain this gives $xb' = ya'$ and we easily obtain that $xR \cap yR = xb'R = ya'R$ and $xR + yR = R$. Lemma 1.1 shows that $Ra' + Rb' = R$ and $Ra' \cap Rb' = Rxb' = Rya'$. Now, since $as = bt \in aR \cap bR = cR$ there exists $v \in R$ such that $as = bt = cv = ab'v = ba'v$ and so $s = b'v$ and $t = a'v$. We thus get the desired conclusions : $Rs + Rt = Rv$ and $Rs \cap Rt = Ru$ for $u = xb'v$. □

Lemma 1.1 and Corollary 1.2 will be used several times. For more details on *2*-fir we refer to P.M. Cohn's book "Free rings and their relations" ([3]). We assume now that R is a *2*-fir and we will analyze injectivity and surjectivity of some maps. For $a, a' \in R \setminus \{0\}$, a nonzero R-module homomorphism $\phi : R/Ra \longrightarrow R/Ra'$ is determined by an element $b' \in R \setminus Ra'$ such that $\phi(x + Ra) = xb' + Ra'$ for any $x \in R$. For the map ϕ to be well defined we must have $ab' \in Ra'$, and hence there exists $b \in R$ such that

$$0 \neq ba' = ab'.$$

In particular this implies that there exists a right R-module homomorphism : $\phi' : R/a'R \longrightarrow R/aR$ given by $\phi'(y + a'R) = by + aR$ for any $y \in R$. Notice that, since R is a domain, $b' \notin Ra'$ implies that $b \notin aR$; this shows that ϕ' is also nonzero. The next lemmas will establish a kind of duality between these two maps.

Lemma 1.3. *Let R be a 2-fir and $a, a' \in R \setminus \{0\}$. With the above notations the following are equivalent:*

(i) *ϕ is injective.*
(ii) *$xb' \in Ra' \Longrightarrow x \in Ra$.*
(iii) *$Ra' \cap Rb' = Rab' = Rba'$.*
(iv) *$aR + bR = R$.*
(v) *$\exists\, d' \in R$ such that $bd' - 1 \in aR$.*
(vi) *ϕ' is surjective.*

Proof. $(i) \Leftrightarrow (ii)$ This is obvious.
$(ii) \Leftrightarrow (iii)$ We always have $Rab' = Rba' \subseteq Ra' \cap Rb'$. On the other hand if $d = xb' \in Ra' \cap Rb'$ and (ii) holds, then $x \in Ra$ and $d \in Rab' = Rba'$. Conversely if (iii) holds and $xb' \in Ra'$, then $xb' \in Ra' \cap Rb'$. Since R is a domain we get $x \in Ra$.
$(iii) \Leftrightarrow (iv)$ Assume (iii) holds. We have $0 \neq ab' = ba' \in aR \cap bR$, and, since R is a 2-fir, we can write $aR + bR = dR$ for some $d \in R$. In particular, there exist $x, y \in R$ such that $a = dx$ and $b = dy$. So $dxb' = ab' = ba' = dya'$ and we get $xb' \in Ra' \cap Rb' = Rab'$. This gives $x \in Ra = Rdx$ and we conclude that d is a unit in R and $aR + bR = R$. Now, assume (iv) holds. Since $0 \neq ab' \in Ra' \cap Rb'$ we know that there exists $x \in R$ such that $Ra' \cap Rb' = Rx$. Let $r, s, t \in R$ be such that $x = sa' = tb'$ and $ab' = ba' = rx$. We then get $a = rt$ and $b = rs$. Using these equalities and (iv) we get that $1 = au + bv = rtu + rsv$. Hence r is a unit in R which implies $Rab' = Rrx = Rx = Ra' \cap Rb'$, as desired.
The other equivalences are easy and left to the reader. □

As we have seen the notion of a 2-fir is left-right symmetric, hence using similar arguments as the ones used in the above proof we also get the following:

Lemma 1.4. *With the notations of the previous lemma, the following are equivalent:*

(i) *ϕ is surjective.*
(ii) *$\exists\, d \in R$ such that $db' - 1 \in Ra'$.*
(iii) *$Ra' + Rb' = R$.*
(iv) *$aR \cap bR = ab'R = ba'R$.*
(v) *$bx \in aR \Longrightarrow x \in a'R$.*

(vi) *ϕ' is injective.*

Proof. This is left to the reader. $\square$

One of our aims in this paper is to analyze atomic factorizations of elements of a *2*-fir R using dimensions of some vector spaces over division rings of the form $End_R(R/Rp)$ where p is an atom of R [1]. If R is left principal $\frac{R}{Rp}$ is simple and Schur's lemma implies that $End_R(R/Rp)$ is a division ring. For an atom p in a *2*-fir R it is not true, in general, that R/Rp is a simple module (Cf. 1.7 d), below), nevertheless, as is well known, the analogue of Schur's lemma is true. We include a short proof for completeness:

Corollary 1.5. *Let p be an atom in a 2-fir R. Then $End_R(R/Rp)$ is a division ring.*

Proof. Let $\phi \in End_R(\frac{R}{Rp}) \setminus \{0\}$ and put $\phi(1 + Rp) = b' + Rp$. There exist $b, d \in R$ such that $0 \neq pb' = bp \in pR \cap bR$ and $pR + bR = dR$. In particular, there exists $d' \in R$ such that $p = dd'$. Assume d' is a unit then $b \in dR = pR$. We then get $pb' = bp \in pRp$, so that $b' \in Rp$. But this contradicts the fact that $\phi \neq 0$ and so d' cannot be a unit. Since $p = dd'$ is an atom we must have that d is a unit and $pR + bR = R$. Lemma 1.3 implies that ϕ is injective. A similar argument shows that $\phi' : R/pR \longrightarrow R/pR$ defined by $\phi'(1 + pR) = b + pR$ is also injective and so Lemma 1.4 implies that ϕ is surjective. $\square$

Remark 1.6. A complete characterization of elements $a \in R$ which are such that $End_R(R/Ra)$ is a division ring has been obtained by the second author ([13]).

We close this section with some examples :

Examples 1.7. a) A *2*-fir is a domain and, since in a commutative domain R two nonzero elements $a, b \in R$ are always such that $0 \neq ab \in aR \cap bR$, we see that a commutative *2*-fir is simply a domain in which every finitely generated ideal is principal. These rings are called Bézout domains in the literature.

b) In the same spirit, if R is a right noetherian domain then, as is well-known and easy to check, $0 \neq aR \cap bR$ for any nonzero elements $a, b \in R$. Hence a right noetherian *2*-fir is such that every finitely generated right ideal is principal. Of course, such a ring need not be right principal

[1] Let us recall that a nonzero element in a ring R is an atom if it is not a unit and cannot be written as a product of two non units

(Consider for example the commutative ring $R = \mathbb{Z} + x\mathbb{Q}[[x]]$ discussed in the next section, Cf example 2.3).

c) One good source of inspiration for our purpose is the case of an Ore extension : $R = K[t; S, D]$ where K is a division ring, S an endomorphism of K and D an S-derivation (Let us recall that the elements of R are polynomials $\sum_{i=0}^{n} a_i t^i$ with coefficients $a_i \in K$ written on the left and the commutation rule is given by $ta = S(a)t + D(a)$) . Since R is always a left principal ideal domain but R is right principal if and only if S is onto, the ring R is a *2*-fir which is not necessarily a right PID. This ring and factorization of its elements have been extensively studied in [8],[9] and [10]. These papers were starting points for our reflections.

d) Let k be a field and $T = k(x)[t; S]$ be the Ore extension where the k-endomorphism S is defined by $S(x) = x^2$. Consider $R = T^{op}$ the opposite ring of T. R is a *2*-fir, $t - x$ is an atom of R but we claim that $\frac{R}{R(t-x)}$ is not a simple R-module. Equivalently we must show that $\frac{T}{(t-x)T}$ is not a simple right T-module. For any $a \in k(x)$ there exist a_0 , $a_1 \in k(x)$, uniquely determined, such that $a = S(a_0) + xS(a_1)$, and we have $at = (t - x)a_0 + xta_1 + xa_0$. From this it is easy to check that $T = (t-x)T \bigoplus xtT \bigoplus k(x)$ (using induction on the degree to get a decomposition of any polynomial and the fact that $x \notin S(k(x))$ in order to prove that the sum is direct). We get that $\frac{T}{(t-x)T} = xtT \bigoplus k(x)$ (notice that the right T-module structure of $k(x)$ is given by $a.t = xa_0$). This module is obviously not simple and in fact it is not even semisimple since it is finitely generated but not artinian.

e) If k is a field, the free k-algebra $k < x, y >$ is a *2*-fir. We refer to P.M.Cohn's book for a proof of this fact (Cf. [3]).

2. Length and similarity

Let us start with a definition which is crucial while dealing with factorization in a non-commutative setting.

Definition 2.1. Two nonzero elements a, a' in a domain R are **similar** if $R/Ra \cong R/Ra'$. We will then write $a \sim a'$.

Lemma 1.1 shows that this notion is left-right symmetric and provides other characterizations of this definition. In fact this notion defines an equivalence relation on the set R. The decomposition into similarity classes will play an important role in our considerations.

Examples 2.2. a) Two elements $a, a' \in R\backslash\{0\}$ are associate (resp. right associate or left associate) if there exist invertible elements $u, v \in R$ such that $a' = uav$ (resp. take resp. u=1 or v=1). We leave to the reader to check that associate elements are in fact similar.

b) In the case of an Ore extensions $R = K[t; S, D]$ where K is a division ring and D is an S-derivation, two elements $t-a$ and $t-a'$ are similar if and only if there exists a nonzero $c \in K$ such that $a'c = S(c)a + D(a)$. This plays an important role in the evaluation and in the factorization theories in Ore extensions [8], [9] and [10].

In this section we want to investigate the relations between similarity and length. We will try to avoid assuming that our *2*-fir is atomic. Let us first give an example of a non atomic *2*-fir. This classical example ([1]) will also be used later in the paper.

Example 2.3. We remarked in 1.7 that a commutative ring is a *2*-fir if and only if it is a Bézout domain. We will show that the ring $R = \mathbb{Z} + x\mathbb{Q}[[x]]$ is a *2*-fir but is not atomic. For a nonzero series $f = \sum_{i=0}^{\infty} a_i x^i$ we define $o(f) = min\{i \in \mathbb{N} | a_i \neq 0\}$. This series is invertible if and only if $a_0 \in \{+1, -1\}$. Let us remark that any element f of R can be written in the form $a_m x^m u$ where $a_m \in \mathbb{Q}, m = o(f), u \in U(R)$, the units of R (obviously if $m = 0$, $a_0 \in \mathbb{Z}$). This implies that the nonzero principal ideals of R are of the form $a_m x^m R$ where $a_m \in \mathbb{Q}$ if $m > 0$ and $a_m \in \mathbb{N}$ if $m = 0$. Let $A = a_m x^m R$ and $B = b_l x^l R$ be two nonzero principal ideals. In order to show that R is a *2*-fir we must prove that $A + B$ is again principal. Without loss of generality we may assume that $m \leq l$. We consider three cases:

case 1 If $m = l = 0$ then $A = a_0 R,\ B = b_0 R$ and $A + B = dR$ where d is a greatest common divisor of a_0 and b_0 in $\mathbb{Z}$

case 2 If $0 \leq m < l$ then $b_l x^l R = a_m x^m b_l a_m^{-1} x^{\ell - m} R \subset a_m x^m R$ hence $A + B = a_m x^m R$

case 3 If $0 < l = m$. Let us write $a_m = ce^{-1},\ b_m = de^{-1}$ with $c, d, e \in \mathbb{Z}$ and $e \neq 0$. Define f to be the least common multiple of c and d. Now we have $a_m x^m R \cap b_m x^m R = (e^{-1} x^m)(cR \cap dR) = e^{-1} x^m f R$. We conclude that R is indeed a *2*-fir. Now, $2 \in R$ is obviously an atom and for all $n \in \mathbb{N}$ we can write $x = 2^n(x2^{-n})$. This shows that x is divisible by elements of arbitrary length. Hence R cannot be atomic.

The following lemma is the the starting point for establishing the relations between atomic factorizations and lengths. A proof of it is given for instance in [8].

Lemma 2.4. *Let p be an atom in a 2-fir R. If $q \in R$ is similar to p, then q is also an atom.*

Let us mention that the result is not true if R is a domain which is not a *2*-fir, once again an example can be found in [8].

Lemma 2.5. *Let a and a' be two similar elements in a 2-fir. If $a = bc$ then there exist $b', c' \in R$ such that $b \sim b', c \sim c'$ and $a' = b'c'$.*

Proof. Let $\phi : R/a'R \longrightarrow R/aR$ be an isomorphism determined by $\phi(1+a'R) = x + aR$. Lemma 1.1 shows that there exists $x' \in R$ such that $xR + aR = R$, $Rx' + Ra' = R$ and $ax' = xa'$ i.e. $bcx' = xa'$. Since R is a *2*-fir we have $Rcx' + Ra' = Rc'$, for some element $c' \in R$. In particular there exist $r, b' \in R$ such that $cx' = rc', a' = b'c'$. We claim that $b \sim b'$. Let us put $\psi : R/b'R \longrightarrow R/bR : y + b'R \mapsto xy + bR$. This map is well defined since $xb'c' = xa' = ax' = bcx' = brc'$ which shows that $xb' = br \in bR\backslash\{0\}$. Moreover we have $R = xR + aR \subseteq xR + bR$, hence $xR + bR = R$. Now Lemma 1.3 implies that ψ is surjective. On the other hand, $0 \neq xb' = br \in Rr \cap Rb'$ and, since R is a *2*-fir, we have $Rr + Rb' = Rd$ for some $d \in R$. But then, $Rc' = Rcx' + Ra' = Rrc' + Rb'c' = Rdc'$ which shows that d is a unit. Hence $Rr + Rb' = R$ and the lemma 1.4 implies that ψ is injective and so, we conclude that $b \sim b'$. This proves the claim. As $a' = b'c'$, it remains to show that $c \sim c'$. Define a right R-morphism $\Gamma : \frac{b'R}{a'R} \longrightarrow \frac{bR}{aR} : b'y + a'R \mapsto xb'y + aR$. This map is well defined since $xa' \in aR$. We claim that Γ is an isomorphism. If $\Gamma(b'y + a'R) = 0$ we get $xb'y \in xR \cap aR = xa'R$, where the last equality is due to the injectivity of ϕ. Since R is a domain, this gives $b'y \in a'R$. This shows that Γ is injective. Let us now show that it is also surjective: it is enough to show that $b + aR \in \Gamma(\frac{b'R}{a'R})$. Since $xR + aR = R$, we know that there exist $u, v \in R$ such that $xu + av = 1$. Consider $\psi(ub + b'R) = xub + bR = (1-av)b + bR = b(1-cvb) + bR = bR$. Since ψ is an isomorphism, we conclude that $ub \in b'R$ and $\Gamma(ub + a'R) = xub + aR = b + aR$, as desired. This means that Γ is an isomorphism. We conclude

$$\frac{R}{c'R} \cong \frac{b'R}{a'R} \overset{\Gamma}{\cong} \frac{bR}{aR} \cong \frac{R}{cR},$$

as required. □

Remark 2.6. Let us notice that we can have $b \sim b'$ but $bc \not\sim b'c$. Let $\mathbb{H}$ denote the division ring of real quaternions and let $R = \mathbb{H}[t]$. Since $(j-i)j = -i(j-i)$ we have that $t-j \sim t+i$ (Cf. example b) in 2.2). It can be shown that the only non trivial monic right factor of the polynomial $f(t) = (t-j)(t-i)$ is $t-i$ hence the module R/Rf cannot be semisimple but for $g(t) = (t+i)(t-i) = t^2+1$ the left R-module $R/Rg \cong R/R(t-i) \bigoplus R/R(t-j)$ is semisimple. This shows that $f(t)$ is not similar to $g(t)$.

Definition 2.7. Let a be a nonzero element in a *2*-fir R. An element a in R is of finite length n if it can be factorized into a product of n atoms but does not admit a factorization into a product of less than n atoms . If an element in R cannot be factorized into a finite product of atoms we will say that it is of infinite length. The invertible elements of R are of length 0.

We denote the subset of elements that can be written as products of atoms by $\mathcal{F} = \{x \in R | \ell(x) < \infty\}$. Notice in particular that units of R are included in $\mathcal{F}$, but $0 \notin \mathcal{F}$.

Theorem 2.8. *In a 2-fir, similar elements have the same length.*

Proof. Let a, a' be nonzero similar elements in R. We proceed by induction on the length of a. The claim is obvious if $\ell(a) = 0$. The above lemma 2.4 shows that the theorem is true for $\ell(a) = 1$. Now, let $n \geq 2$ and assume that the theorem is true for elements of length $\leq n-1$ and let $a \in R$ be such that $\ell(a) = n$. Obviously we must have $\ell(a') \geq n$. Let us write $a = bq$ where $\ell(b) = n-1$. So q is an atom. Since a is similar to a', lemma 2.5 shows that there exist b', q' such that $a' = b'q'$ where $q' \sim q$ and $b' \sim b$. The induction hypothesis implies that $\ell(q') = 1$ and $\ell(b') = n-1$. We thus conclude that $\ell(a') = n = \ell(a)$. This proves the theorem. $\square$

For two elements a, b in a *2*-fir R such that $Ra \cap Rb \neq 0$, we will denote by a^b an element in R such that $Ra \cap Rb = Ra^b b$. Notice first that a^b is defined up to a left multiple by a unit. We will also use $Ra \cap Rb = R[a, b]_\ell$ with $[a, b]_\ell$ in R. In the next lemma we briefly study some properties of a^b. Recall that for $(a, b) \in R^2$, $(Ra)b^{-1} = \{x \in R | xb \in Ra\}$.

Lemma 2.9.
(a) *Let a be an atom in R and $b \in R \backslash Ra$ such that $Ra \cap Rb \neq 0$. Write $Ra \cap Rb = Ra^b b$. Then a^b is an atom similar to a. Conversely : if a is an atom and $a' \in R$ is such that $a' \sim a$ then there exists $b \in R \setminus Ra$ such that $Ra' = Ra^b$. Moreover we have $Ra^b = (Ra)b^{-1}$.*
(b) *If a, b, c are elements of R such that $Ra \cap Rcb \neq 0$ then we have $Ra^{cb} = R(a^b)^c$.*
(c) *If $Ra \cap Rb \cap Rc \neq 0$ then we have $R[a, b]_\ell^c = R[a^c, b^c]_\ell$.*

Proof. (a) As $b \notin Ra$, a^b is not a unit. Let us remark that $Ra \cap Rb \neq 0$ implies that $Ra + Rb = Rd$ for some $d \in R$. As a is an atom and $b \notin Ra$, d is a unit in R. So, by lemma 1.1 $a^b \sim a$. The converse and the additional statement are left to the reader.
(b) We have $Ra \cap Rb = Ra^b b$ and $Ra^b \cap Rc = R(a^b)^c c$ (both of these intersections are nonzero thanks to our assumption in (b)). Multiplying the last relation by b on the right we get $R(a^b)^c cb = Ra^b b \cap Rcb$. Using the first relation this leads to $R(a^b)^c cb = Ra \cap Rb \cap Rcb = Ra \cap Rcb = Ra^{cb} cb$ and this gives the desired relation.

(c) $R[a,b]_\ell^c c = R[a,b]_\ell \cap Rc = Ra \cap Rb \cap Rc = Ra \cap Rc \cap Rb \cap Rc = Ra^c c \cap Rb^c c = R[a^c, b^c]_\ell c$. This leads to the desired equality.

$\square$

Lemma 2.10. *Let a, b be nonzero elements of R and p an atom in R such that $ab \in Rp$ but $b \notin Rp$. Then $a \in Rp^b$.*

Proof. Since $0 \neq ab \in Rp \cap Rb$, we know that $Rp \cap Rb = Rp^b b$. In particular there exists $c \in R$ such that $ab = cp^b b$ and we get $a = cp^b$. $\square$

Since atomic factorizations of two elements b and c yield an atomic factorization of their product bc, we have $\ell(bc) \leq \ell(b) + \ell(c)$. The reverse inequality is not completely clear and we offer a short proof of it in the next lemma.

Lemma 2.11. *Let R be 2-fir. Then*

$$\ell(bc) = \ell(b) + \ell(c) \quad \forall \;\, b, c \in R$$

Proof. The case when $a := bc$ has infinite length is clear and we may assume that $\ell(a) = n < \infty$. We must show that $n = \ell(b) + \ell(c)$. We proceed by induction on n. The claim is obvious for $n = 0$. If $n = 1$, a is an atom and the result is clear. Assume $n > 1$, obviously we may assume that neither b nor c are invertible. Write $a = p_1 p_2 \cdots p_n = bc$. If $c \in Rp_n$ then there exists $c' \in R$ such that $c = c' p_n$ and we get $bc' = p_1 p_2 \cdots p_{n-1}$ and the induction hypothesis allows us to conclude easily. We may thus assume that $c \notin Rp_n$. We have

$$a = p_1 p_2 \cdots p_n = bc \in Rc \cap Rp_n = Rc' p_n = Rp'_n c$$

where $p_1, p_2, \ldots, p_n$ are given atoms and c', p'_n are elements in R. Notice also that by Theorem 2.8 and Lemma 2.9 we have $\ell(p'_n) = \ell(p_n) = 1$ and $\ell(c') = \ell(c) \geq 1$. The above displayed equality shows that there exist $r \in R$ and α a unit in R such that $bc = a = rp'_n c$ and $p'_n c = \alpha c' p_n$ hence $b = rp'_n$ and $p_1 p_2 \cdots p_{n-1} = r\alpha c'$. The induction hypothesis then shows that we have $n - 1 = \ell(r) + \ell(c')$. Since $b = rp'_n$ and p'_n is an atom we get $\ell(b) \leq \ell(r) + 1 = n - 1 - \ell(c') + 1 \leq n - 1$. Hence we can again apply our induction hypothesis and we obtain $\ell(b) = \ell(rp'_n) = \ell(r) + \ell(p'_n) = \ell(r) + 1$ This gives $n = \ell(r) + \ell(c') + 1 = \ell(b) + \ell(c)$, as desired

$\square$

The next theorem is part of folklore.

Theorem 2.12. *Let R be a 2-fir and let $a, b \in R \setminus \{0\}$ such that $Ra \cap Rb \neq 0$. Write $Ra \cap Rb = R[a,b]_\ell$ and $Ra + Rb = R(a,b)_r$. Then*

$$\ell([a,b]_\ell) + \ell((a,b)_r) = \ell(a) + \ell(b)$$

Proof. If $\ell(a)$ or $\ell(b)$ are infinite then the equality is obvious. The Noether's Isomorphism Theorem gives an isomorphism of R-modules $(Ra+Rb)/Ra \cong Rb/Ra\cap Rb$. Let us write $a = a''(a,b)_r$ for some $a'' \in R$ and $[a,b]_\ell = a'b$ with $a' \in R$. We get

$$R/Ra'' \cong R/Ra' \text{ i.e. } a'' \sim a'$$

So, by theorem 2.8, $\ell(a'') = \ell(a')$. Now we have $\ell([a,b]_\ell) = \ell(a'b) = \ell(a') + \ell(b) = \ell(a'') + \ell(b) = \ell(a) - \ell((a,b)_r) + \ell(b)$. This completes the proof. □

Corollary 2.13. *Let a, b be nonzero and non unit in R such that $Ra\cap Rb \neq 0$. Write $Ra \cap Rb = Ra^b b$. Then $\ell(a^b) \leq \ell(a)$.*

Proof. The previous theorem gives $\ell(a^b) + \ell(b) = \ell(a^b b) \leq \ell(a) + \ell(b)$. □

Let us now offer a few easy but important facts about the subset $\mathcal{F} = \{x \in R | \ell(x) < \infty\}$ which was introduced in the paragraph before 2.8.

Proposition 2.14. *Let a, b be elements in $\mathcal{F}$. Then :*

a) *$ab \in \mathcal{F}$.*
b) *If $a \sim a'$, then $a' \in \mathcal{F}$.*
c) *If $Ra + Rb = Rc$, then $c \in \mathcal{F}$.*
d) *If $Ra \bigcap Rb = Rd \neq 0$, then $d \in \mathcal{F}$.*
e) *If Γ is a finite subset of $\mathcal{F}$ such that $\bigcap_{\{\gamma\in\Gamma\}} R\gamma \neq 0$, then there exists $h \in \mathcal{F}$ such that $\bigcap_{\{\gamma\in\Gamma\}} R\gamma = Rh$.*

Proof. a) This is clear from 2.11.
b) This comes from 2.8.
c) Since $a \in Rc$ this also follows from 2.11.
d) This follows from 2.12.
e) This is obtained by repeated applications of the point e) above.

□

We end this section with the following remark:

Remark 2.15. If $\Delta \subseteq R$ is such that $\bigcap_{\delta\in\Delta} R\delta \bigcap \mathcal{F} \neq \emptyset$ then $\Delta \subseteq \mathcal{F}$

Proof. Let $x \in (\bigcap_{\delta\in\Delta} R\delta) \bigcap \mathcal{F}$. Then $\ell(x) < \infty$ and, since $x \in R\delta$ for any $\delta \in \Delta$, we have that $\delta \in \mathcal{F}$ for all $\delta \in \Delta$. □

3. F-independence on 2-fir

We now introduce some central definitions. In this section R will denote a *2*-fir. Due partly to the last remark and to make life easier, we will only consider, in the next definition, subsets of $\mathcal{F}$.

Definition 3.1. Recall that $U(R)$ stands for the set of invertible elements of R. A subset Δ of $\mathcal{F} \setminus U(R)$ is said to be ***F*-algebraic** if $\bigcap_{\delta \in \Delta} R\delta \neq 0$.

Let us first remark that, according to the above definition, the empty set is algebraic (we follow the convention that the intersection of an empty family of subsets of R is R itself). We now make some more remarks and introduce some notations in the following:

Notations 3.2. If Δ is an F-algebraic subset of R we will denote by Δ_ℓ an element (when there exists one) such that $\bigcap_{\delta \in \Delta} R\delta = R\Delta_\ell$. For convenience and in accordance with future notations, we will put $\emptyset_\ell = 1$. If it exists Δ_ℓ is not unique but all such elements are left associates and have the same length. We will sometimes use the word "algebraic" meaning in fact "F-algebraic".

Remarks 3.3. a) We have excluded the invertible elements from algebraic sets. The first reason is that algebraic sets are in fact a tool for the study of factorization the second reason is that if one admits invertible elements in algebraic sets this creates technical problems and more complicated statements.
b) Notice that if $\Delta \subseteq \mathcal{F}$ is algebraic and finite, then, since R is a *2*-fir, Δ_ℓ always exists and is nonzero. Moreover, Lemma 2.14 e) shows that in this case $\Delta_\ell \in \mathcal{F}$. In particular, for any finite subset Γ contained in an algebraic set Δ there exists $\Gamma_\ell \in \mathcal{F}$ such that $\bigcap_{\gamma \in \Gamma} R\gamma = R\Gamma_\ell$.
c) Remark also that if Δ is algebraic and Δ' is a subset of R consisting of right non invertible divisors of elements of Δ then Δ' is also algebraic.
d) Let us mention that although Δ is a subset of $\mathcal{F}$, the element Δ_ℓ, when it exists, might be of infinite length (cf example 3.11, e).
e) In [11] (S, D)-algebraic sets are defined in the context of an Ore extension $R := K[t; S, D]$ over a division ring K. The relation between this notion and the notion of F-algebraic sets introduced above is as follows : a subset $\Delta \subseteq K$ is (S, D)-algebraic if and only if the set $\{\, t - \delta \mid \delta \in \Delta \,\} \subset R$ is F-algebraic in the sense defined in 3.1.
f) An F-algebraic subset of $\mathcal{F}$ should be called left F-algebraic. A similar definition for right F-algebraic sets can be given. Singleton sets of $\mathcal{F}$ not contained in $U(R)$ are, of course, left and right F-algebraic but there are sets with only 2 elements that are left F-algebraic but not right F-algebraic : This is the case of $\{t, at\} \subset R = k[t; S]$ where k is a field, S is an endomorphism of

k which is not an automorphism and $a \in k \setminus S(k)$. In this paper F-algebraic will always refer to the left notion defined above.

In the following $\mathcal{A}$ will stand for the set of atoms of R.

Proposition 3.4. *Let $\Delta \subseteq \mathcal{F}$ and $d \in \mathcal{F}$ be such that $\Delta \cup \{d\}$ is F-algebraic. Then the following are equivalent:*

(i) *There exist a finite subset Γ of Δ and $p \in \mathcal{A}$ such that $d \in Rp$ and $Rp + \bigcap_{\gamma\in\Gamma} R\gamma \neq R$.*
(ii) *There exists a finite subset Γ of Δ such that $Rd + \bigcap_{\gamma\in\Gamma} R\gamma \neq R$.*
(iii) *There exist a finite subset Γ of Δ and an element $\Gamma_\ell \in \mathcal{F}$ such that $\bigcap_{\gamma\in\Gamma} R\gamma = R\Gamma_\ell$ and $R\Gamma_\ell \bigcap Rd = Rm \neq 0$ with $\ell(m) < \ell(\Gamma_\ell) + \ell(d)$.*

Proof. $(i) \Longrightarrow (ii)$ If d, p and Γ are as in (i) then $Rd + \bigcap_{\gamma\in\Gamma} R\gamma \subseteq Rp + \bigcap_{\gamma\in\Gamma} R\gamma \neq R$.
$(ii) \Longrightarrow (iii)$ Since Γ is a finite subset of $\mathcal{F}$, remark 3.3 b) shows that there exists $\Gamma_\ell \in \mathcal{F}$ such that $\bigcap_{\gamma\in\Gamma} R\gamma = R\Gamma_\ell$ and, since $\{d\} \cup \Gamma \subseteq \{d\} \cup \Delta$ is algebraic, we have $0 \neq Rd \cap R\Gamma_\ell = Rm$, for some m in R. Since R is a *2*-fir, $Rd + R\Gamma_\ell = Ra$ for some a in R. Now, 2.14 c) and d) give us that $a, m \in \mathcal{F}$. From Theorem 2.12 we get $\ell(m) + \ell(a) = \ell(d) + \ell(\Gamma_\ell)$. Since, by (ii), a is not a unit we have $\ell(m) < \ell(d) + \ell(\Gamma_\ell)$ as required.
$(iii) \Longrightarrow (i)$ Since R is a *2*-fir and $R\Gamma_\ell \bigcap Rd = Rm \neq 0$ there exists $a \in R$ such that $R\Gamma_\ell + Rd = Ra$ and we have $\ell(d) + \ell(\Gamma_\ell) = \ell(a) + \ell(m) < \ell(a) + \ell(d) + \ell(\Gamma_\ell)$. Hence $\ell(a) \neq 0$. If $p \in \mathcal{A}$ divides a on the right we have $d \in Rp$ and $Rp + \cap_{\gamma\in\Gamma} R\gamma = Rp + R\Gamma_\ell \subseteq Rp + Ra = Rp \neq R$, as required. □

In view of the above proposition the following definitions appear naturally:

Definition 3.5. Let Δ be an algebraic set.

(a) An element $d \in \mathcal{F}$ is said to be **F-dependent** on Δ if $\Delta \cup \{d\}$ is algebraic and one of the conditions of the above proposition is satisfied.
(b) Δ is **F-independent** if and only if for any $\delta \in \Delta$, δ is not F-dependent over $\Delta \setminus \{\delta\}$.
(c) An F-independent subset $B \subseteq \Delta$ is an **F-basis** if any element of Δ is F-dependent on B.

It is clear that a subset of an F-algebraic set is also F-algebraic and a subset of an F-independent set is also F-independent. The above proposition 3.4(i) shows that it is possible to express F-dependence by means of atoms. We will have more precise information in the next section. For the moment let us notice the following special case.

Proposition 3.6. *Let p be an atom in R. Then p is F-dependent on an F-algebraic set $\Delta \subset \mathcal{F}$ if and only if there exists a finite subset $\Gamma \subseteq \Delta$ such that $\bigcap_{\gamma\in\Gamma} R\gamma \subseteq Rp$.*

The next proposition connects F-independence and length.

Proposition 3.7. *Let R be a 2-fir and $\Delta \subset \mathcal{F}$ be an F-algebraic set. Then Δ is an F-independent set if and only if for any finite subset $\Gamma := \{a_1, a_2, \ldots, a_n\} \subseteq \Delta$ we have $\ell(\Gamma_\ell) = \sum_{i=1}^n \ell(a_i)$ where $\Gamma_\ell \in \mathcal{F}$ is such that $\cap_{i=1}^n Ra_i = R\Gamma_\ell$.*

Proof. If some $a_j \in \{a_1, \ldots, a_n\}$ is F-dependent on $\Gamma_j := \{a_1, \ldots, a_n\} \setminus \{a_j\}$, then $\ell(\Gamma_\ell) < \ell(a_j) + \ell((\Gamma_j)_\ell) \leq \sum_{i=1}^n \ell(a_i)$, where the last equality comes from Theorem 2.12. Conversely, suppose $\{a_1, \ldots, a_n\}$ is F-independent and let $a_j \in \{a_1, \ldots, a_n\}$. By induction, we may assume that $\ell((\Gamma_j)_\ell) = \sum_{i\neq j}^n \ell(a_i)$. Since a_j is not F-dependent on $\Gamma_j = \{a_1, \ldots, a_n\} \setminus \{a_j\}$, $\ell(\Gamma_\ell) = \ell(a_j) + \ell((\Gamma_j)_\ell) = \sum_{i=1}^n \ell(a_i)$. This finishes the proof. □

Corollary 3.8. *Let $\Delta \subseteq \mathcal{F}$ be an F-independent algebraic set in a 2-fir R. Then $|\Delta| < \infty$ if and only if there exists $\Delta_\ell \in \mathcal{F}$ such that $\bigcap_{\delta\in\Delta} R\delta = R\Delta_\ell$. Moreover in this case we have $|\Delta| \leq \ell(\Delta_\ell)$ and the equality occurs if and only if $\Delta \subseteq \mathcal{A}$.*

The above properties are quite nice but we will soon see that the definitions of F-dependence and F-independence have also some drawbacks.
There are some relations between F-bases and maximal F-independent sets. To understand more precisely the relationship, we first prove the following intermediate fact.

Proposition 3.9. *Let $\Delta \cup \{a\} \cup \{b\} \subseteq \mathcal{F}$ be an F-algebraic set in a 2-fir R. Then, if b is F-dependent on $\Delta \cup \{a\}$, but is not F-dependent on Δ, then a is F-dependent on $\Delta \cup \{b\}$.*

Proof. Since b is F-dependent on $\Delta \cup \{a\}$ there exists a finite subset Γ of Δ such that for g , h , $m \in \mathcal{F}$ defined by $\bigcap_{\gamma\in\Gamma} R\gamma = Rg$, $Ra \cap Rg = Rh$ and $Rb \cap Rh = Rm$ we have $\ell(m) < \ell(h) + \ell(b)$. On the other hand the fact that b is not F-dependent on Δ implies that $\ell(c) = \ell(b) + \ell(g)$ where c is such that $Rb \cap Rg = Rc$. We thus have $\ell(a)+\ell(c) = \ell(a)+\ell(b)+\ell(g) \geq \ell(b)+\ell(h) > \ell(m)$. Since we also have that $Rm = Rb \cap Ra \cap Rg = Ra \cap Rc$ we conclude that a is F-dependent on $\Delta \cup \{b\}$, as required. □

Proposition 3.10. *Let Δ be an F-algebraic set in a 2-fir R. Then $B \subset \Delta$ is an F-basis of Δ if and only if B is a maximal F-independent subset of Δ. In particular, any algebraic set has a basis.*

Proof. The only if part is clear. Assume B is a maximal F-independent subset of Δ and let $b \in \Delta \setminus B$. By assumption $B \cup \{b\}$ is not an F-independent set. Hence some element $c \in B \cup \{b\}$ is F-dependent on the others. If $c = b$, b is F-dependent on B as desired. Assume $c \in B$. Then c is not F-dependent on $B \setminus \{c\}$ but F-dependent on $(B \setminus \{c\}) \cup \{b\}$. By the last proposition, b is F-dependent on $(B \setminus \{c\}) \cup \{c\} = B$ as desired. The last statement follows by using Zorn's lemma. $\square$

Examples 3.11. a) Let $R = K[t; S, D]$ be an Ore extension over a division ring K where $S \in End(R)$ and D is an S-derivation. Let $\{a_1, \ldots, a_n\}$ be a subset of K and consider $\Delta := \{t - a_1, \ldots, t - a_n\} \subset R$. Then Δ is algebraic since the $(t - a_i)$'s have a nonzero least common left multiple. In fact, in this case, R is a left principal ideal domain and any finite subset of R is (left) algebraic. These situations have been studied extensively in [8],[9] and [10]. In these papers a basis for an algebraic set Δ was called a P-basis.

b) Of course, a basis of an algebraic subset Δ of R might well be infinite. When a basis is finite there exists an element $h \in \mathcal{F}$ such that $\bigcap_{b \in B} Rb = Rh$. But in general there may be no element $g \in R$ such that $\bigcap_{\delta \in \Delta} R\delta = Rg$

c) Let us consider $R = K[t]$ where K is a field. Let a, b be nonzero element of K and $\Delta = \{(t-a)(t-b), (t-b)\}$. So $\Delta_\ell = (t-a)(t-b)$. Moreover $B = \{t - b\}$ and $B' = \{(t-a)(t-b)\}$ are F-bases of Δ. But we have $R\Delta_\ell = RB'_\ell \subsetneq RB_\ell$. This shows that even when an algebraic set Δ is finite the least left common multiple of the element of a basis and the least left common multiple of the elements of Δ may be different. It will be shown later (Cf. Proposition 4.1) that such a situation is impossible in the case when all elements of Δ are atoms.

d) Let p_1, p_2 be different atoms in R such that $Rp_1 \bigcap Rp_2 = Rm \neq 0$. Then the set $\Delta = \{p_1, p_2, m\}$ is an algebraic set. Notice that $\{m\}$ and $\{p_1, p_2\}$ are bases for Δ with different cardinals.

e) Let us consider the *2*-fir $R = \mathbb{Z} + x\mathbb{Q}[[x]]$ (Cf. the example 2.3) and let $\Delta = \{p \in \mathbb{Z} | \text{p is prime and } p > 0\}$. Notice that the elements of Δ are atoms in R. Since $x \in \bigcap_{\delta \in \Delta} R\delta$ we see that Δ is an algebraic subset of R. In fact Δ is a basis of itself. Notice also that $\bigcap_{\delta \in \Delta} R\delta = Rx$. Since $x \notin \mathcal{F}$, this gives the example promised in remark d) of 3.3.

The notions of F-dependence and F-independence are strongly related to the notion of abstract dependence. Let us recall this definition (Cf. [6]).

For a non vacuous set X and a relation Γ from X to the power set $\mathcal{P}(X)$, we write $x \prec S$ if $(x, S) \in \Gamma$. We call Γ a *dependence relation* in X if the following conditions are satisfied :

(i) if $x \in S$, $x \prec S$.

(ii) if $x \prec S$, then $x \prec F$ for some finite subset $F \subset S$.
(iii) if $x \prec S$ and every $y \in S$ satisfies $y \prec T$, then $x \prec T$.
(iv) if $x \prec S$ but $x \not\prec S \setminus \{y\}$ then $y \prec (S \setminus \{y\}) \cup \{x\}$.

In our case $X = \mathcal{F} \setminus U(R)$, S is an F-algebraic set of R and the relation "$\prec$ " is the *F-dependence relation*. Obviously (i) and (ii) are satisfied. The assertion (iv) is given by 3.9. But (iii) is false in general as the following example shows.

Example 3.12. Let a, b, c, d be atoms in a *2*-fir R such that a is not similar to d but $ba = cd$. We then have that a is F-dependent on $\{ba\}$ and $ba = cd$ is F- dependent on $\{d\}$ but a is not F-dependent on $\{d\}$.

The problem of non transitivity disappears if we restrict ourselves to algebraic sets of atoms. Let us recall that $\mathcal{A}$ denotes the set of atoms in R.

Proposition 3.13. *Let Δ, $\Delta' \subseteq \mathcal{A}$ be algebraic sets of atoms. Assume $p \in \mathcal{A}$ is F-dependent on Δ and each element of Δ is F-dependent on Δ'. Then p is F-dependent on Δ'.*

Proof. By hypothesis there exists a finite subset Γ of Δ such that $\bigcap_{\gamma \in \Gamma} R\gamma \subseteq Rp$ (Proposition 3.6). Now each $\gamma \in \Gamma$ is F-dependent on Δ' and since Γ is finite we can find a finite subset Γ' of Δ' such that $\bigcap_{\gamma' \in \Gamma'} R\gamma' \subseteq R\gamma$ for all $\gamma \in \Gamma$. This means that $\bigcap_{\gamma' \in \Gamma'} R\gamma' \subseteq \bigcap_{\gamma \in \Gamma} R\gamma \subseteq Rp$. This shows that p is F-dependent on Δ'. $\square$

So if we restrict to algebraic sets of atoms the notion of F-dependence defines an abstract dependence relation. In this case the general theory shows that a subset B of an algebraic set Δ is a basis if and only if it is minimal such that all elements of Δ are F-dependent on B.
The restriction to subsets of $\mathcal{A}$ is not as bad as it could seem on the first sight. We have already seen that atoms appear naturally while dealing with F-independence (see 3.4 (i)). In Proposition 4.9 we will show more precisely how the notion of F-dependence on elements of $\mathcal{F}$ is controlled by the F-dependence on $\mathcal{A}$.

4. Algebraic set of atoms

In this section we will concentrate on the structure of algebraic subsets of the set $\mathcal{A}$ of atoms. We will introduce the rank of such an algebraic set and also get some connections between F-independence and some usual dimensions of vector spaces over division rings. This will shed some new lights on these notions.

We start this section with some easy facts on algebraic sets of atoms. First let us recall that, in general, even for a finite algebraic set Δ with basis B we might have $RB_\ell \neq R\Delta_\ell$ as we have seen in example 3.11 (c). In case of algebraic sets of atoms we have:

Proposition 4.1. *Let $\Delta \subseteq \mathcal{A}$ be an F-algebraic set with basis B.*

a) $\bigcap_{\delta\in\Delta} R\delta = \bigcap_{b\in B} Rb$.
b) *If $|B| < \infty$ then there exist Δ_ℓ and $B_\ell \in \mathcal{F}$ such that $\bigcap_{\delta\in\Delta} R\delta = R\Delta_\ell = \bigcap_{b\in B} Rb = RB_\ell$ and $\ell(B_\ell) = \ell(\Delta_\ell) = |B|$.*

Proof. a) The inclusion $\bigcap_{\delta\in\Delta} R\delta \subseteq \bigcap_{b\in B} Rb$ is clear. Now if $x \in \bigcap_{b\in B} Rb$ and $\delta \in \Delta$ then, thanks to Proposition 3.6 there exists a finite subset Γ of B such that $\bigcap_{\gamma\in\Gamma} R\gamma \subseteq R\delta$, hence $x \in \bigcap_{b\in B} Rb \subseteq \bigcap_{\gamma\in\Gamma} R\gamma \subseteq R\delta$ for any $\delta \in \Delta$. This shows that $\bigcap_{b\in B} Rb \subseteq \bigcap_{\delta\in\Delta} R\delta$ as desired.
b) This is clear in view of a) above and corollary 3.8. □

In view of the above proposition it is natural to introduce the following notions:

Definitions 4.2. a) *Let Δ be an F-algebraic set of atoms and B be an F-basis for Δ. We define the* **rank** *of Δ, denoted $rk(\Delta)$, by $rk(\Delta) = |B|$.*
b) *For $a \in R \setminus \{0\}$, let*

$$V(a) := \{p \in \mathcal{A} | a \in Rp\}$$

c) *For an algebraic subset Δ of $\mathcal{A}$ we call the* **closure** *of Δ the set of atoms which are F-dependent on Δ and we denote this set by $\overline{\Delta}$.*

Lemma 4.3. *With the above notations and definitions we have :*

a) *$V(a)$ is an F-algebraic set and $rk(V(a)) \leq \ell(a)$.*
b) *If $\Delta \subseteq \mathcal{A}$ is an F-algebraic set with an F-basis B, then $\overline{\Delta}$ is F-algebraic, $\overline{B} = \overline{\Delta}$ and $rk(\Delta) = rk(\overline{\Delta})$. If Δ is of finite rank then $\overline{\Delta} = V(\Delta_\ell) = V(B_\ell) = \overline{B}$.*
c) *Let $a, b \in \mathcal{F} \setminus \{0\}$ be such that $Ra \cap Rb \neq 0$. Then $V(a) \cap V(b) = \emptyset$ if and only if $Ra + Rb = R$ if and only if $\ell([a, b]_\ell) = \ell(a) + \ell(b)$.*
d) *If C is a finite algebraic subset of $\mathcal{A}$ then $rk(C) \leq |C|$ and the equality occurs if and only if C is an F-independent subset of $\mathcal{A}$.*

Proof. a) If a is a unit in R then $V(a) = \emptyset$ and so $V(a)$ is algebraic. If $0 \neq a \in R \setminus U(R)$ we have $0 \neq Ra \subset \bigcap_{p\in V(a)} Rp$. This shows that $V(a)$ is an algebraic set. Proposition 4.1 b) implies that for a finite F-independent set $B \subseteq V(a)$ we have $Ra \subseteq \cap_{b\in B} Rb = RB_\ell$ and so $|B| = \ell(B_\ell) \leq \ell(a))$

b) Using Proposition 3.6 it is easy to remark that $0 \neq (\cap_{\delta\in\Delta} R\delta) = (\cap_{\gamma\in\overline{\Delta}} R\gamma)$ and so $\overline{\Delta}$ is an algebraic set. Obviously $\overline{B} \subseteq \overline{\Delta}$ and the transitivity of F-dependence on sets of atoms gives the reverse inclusion. Hence $\overline{B} = \overline{\Delta}$ and $rk(\Delta) = rk(\overline{\Delta})$. The last statement follows easily.

c) and d) are left to the reader.

□

Theorem 4.4. *Let $\Delta \cup \Gamma \subseteq \mathcal{A}$ be an F-algebraic set of finite rank. Then*

(i) $R(\Delta \cup \Gamma)_\ell = R\Delta_\ell \cap R\Gamma_\ell$ *and* $rk(\Delta \cup \Gamma) \leq rk(\Delta) + rk(\Gamma)$.

(ii) $V(\Delta_\ell) \cap V(\Gamma_\ell) = \emptyset$ *if and only if equality holds in* (i).

Proof. (i) We have $R(\Delta \cup \Gamma)_\ell = \cap_{\epsilon \in \Delta \cup \Gamma} R\epsilon = (\cap_{\delta \in \Delta} R\delta) \bigcap (\cap_{\gamma \in \Gamma} R\gamma) = R\Delta_\ell \cap R\Gamma_\ell$. The statement about rank follows from Theorem 2.12 and Proposition 4.1(b). To prove (ii) assume $V(\Delta_\ell) \cap V(\Gamma_\ell) = \emptyset$. Then the previous lemma and (i) above imply that $\ell((\Delta \cup \Gamma)_\ell) = \ell(\Delta_\ell) + \ell(\Gamma_\ell)$. Hence $rk(\Delta \cup \Gamma) = rk(\Delta) + rk(\Gamma)$.

□

Theorem 4.5. *Let $\Delta \cup \Gamma$ be an F-algebraic set in $\mathcal{A}$. Denote by B, B' respectively F-bases for Δ and Γ. Then we have $\overline{\Delta} \cap \overline{\Gamma} = \emptyset$ if and only if $B \cup B'$ is an F-basis for $\Delta \cup \Gamma$.*

Proof. Using Lemma 4.3 we get $\overline{\Delta} \bigcap \overline{\Gamma} = \emptyset$ if and only if $\overline{B} \bigcap \overline{B}' = \emptyset$ if and only if $\overline{C} \bigcap \overline{C}' = \emptyset$ for any finite subsets C, C' of B and B' respectively. This is equivalent to $V(C_\ell) \bigcap V(C'_\ell) = \emptyset$ i.e., using Theorem 4.4, $rk(C \cup C') = rk(C) + rk(C') = |C| + |C'|$ for any finite subsets C, C' of B and B' respectively. From the above we conclude that $\overline{\Delta} \bigcap \overline{\Gamma} = \emptyset$ if and only if for any finite subsets C, C' of B and B' respectively we have that $rk(C \cup C') = |C \cup C'|$ i.e. if and only if $C \cup C'$ is F-independent. Hence $\overline{\Delta} \bigcap \overline{\Gamma} = \emptyset$ if and only if $B \cup B'$ is F-independent. Now, since B and B' are F-bases for Δ and Γ respectively it is easy to finish the proof. □

Let us recall, from section 2, that for $a, b \in R$ such that $Ra \cap Rb \neq 0$ we wrote $Ra \cap Rb = Ra^b b = Rb^a a$.

Proposition 4.6. *Let $\Gamma \subseteq \mathcal{F}$ be an F-algebraic set of atoms in a 2-fir R such that $\bigcap_{\gamma \in \Gamma} R\gamma = Rh$ for some element $h \in \mathcal{F}$. Then*

(i) *h is a product of atoms similar to atoms in Γ.*

(ii) *any right atomic factor of h is similar to some atom in Γ.*

In particular, this applies to any finite subset of an F- algebraic set $\Delta \subseteq \mathcal{A}$.

Proof. Let B be a basis for Γ. From Corollary 3.8 and Proposition 4.1 we have that $\ell(h) = |B| < \infty$. Let us put $B = \{b_1, \dots, b_s\}$, we will show by induction on s that h is a product of atoms similar to the b_i's. From 4.1 we know that $\bigcap_{i=1}^{s} Rb_i = \bigcap_{\gamma\in\Gamma} R\gamma = Rh$.

If $s = 1$ we have $Rb_1 = Rh$ and h must be an atom associated to b_1.

If $s > 1$ we have $Rh \subsetneq Rb_1$ and we can write $h = h_1 b_1$. We then have $Rh_1 b_1 = Rh = Rb_1 \bigcap (\cap_{i=2}^{s} Rb_i) = \bigcap_{i=2}^{s}(Rb_1 \cap Rb_i) = \bigcap_{i=2}^{s} Rb_i^{b_1} b_1$. This gives that $Rh_1 = \bigcap_{i=2}^{s} Rb_i^{b_1}$. Now $\{b_2^{b_1}, \dots, b_s^{b_1}\}$ is an algebraic set and the induction hypothesis implies that h_1 is a product of atoms which are similar to the $b_i^{b_1}$'s and hence similar to the b_i's for $i \in \{2, \dots, s\}$. Since $h = h_1 b_1$ we can conclude.

(ii) Let us use the same notations as in (i) above and assume that $h = ga$ where $g \in \mathcal{F}$ and $a \in \mathcal{A}$. We want to show that a is similar to one of the b_i's. We proceed by induction on s. We have $h = h_1 b_1 = ga \in Ra$ with $\bigcap_{i=2}^{s} Rb_i^{b_1} = Rh_1$. Hence by 2.10 either $b_1 \in Ra$ or $h_1 \in Ra^{b_1}$. In the first case we conclude that a is associated to b_1 and hence a and b are similar. In the second case the induction hypothesis shows that a^{b_1} is similar to one of the $b_i^{b_1}$'s. The transitivity of similarity yields the conclusion. $\square$

The following definition will be useful for us:

Definition 4.7. An F-algebraic subset Γ of a set Δ is **full** in Δ if any element of Δ which is F-dependent on Γ is already in Γ.

Lemma 4.8. *Let Δ be an F-algebraic set of atoms and Γ be a full subset of Δ. If $Rh = \cap_{\gamma\in\Gamma} R\gamma$ and $Rf = \cap_{\delta\in\Delta} R\delta$ then $Rf = Rgh$ where $Rg = \cap_{d\in\Delta\setminus\Gamma} Rd^h$.*

Proof. Since $\Gamma \subseteq \Delta$ we know that there exists $g \in R$ such that $f = gh$, and we must show that $Rg = \cap_{d\in\Delta\setminus\Gamma} Rd^h$. Now, for any $d \in \Delta \setminus \Gamma$ we know that $f = gh \in Rd$, but since Γ is full in Δ we have that $h \notin Rd$ hence by Lemma 2.10 $g \in Rd^h$. This shows that $Rg \subseteq \cap_{d\in\Delta\setminus\Gamma} Rd^h$. On the other hand, if $p \in \cap_{d\in\Delta\setminus\Gamma} Rd^h$ then $ph \in \cap_{d\in\Delta\setminus\Gamma} Rd^h h = \cap_{d\in\Delta\setminus\Gamma}(Rd \cap Rh) = \cap_{d\in\Delta} Rd$ and hence, $ph \in Rf = Rgh$. This implies that $p \in Rg$, as required.

$\square$

We will study the influence of the decomposition into similarity classes on the notions of F-independence and rank. Let us first start with the promised expression of F-independence of an element in terms of the F-independence of the atoms appearing in its factorization.

Let us first introduce the following notation : for $\Delta \subseteq R$ and $u \in R \setminus \{0\}$ we denote $\Delta^u = \{g \in R | \exists \delta \in \Delta : Rgu = R\delta \cap Ru \neq 0\}$ (to justify this notation let us notice that in 2.9 we wrote $R\delta \cap Ru = R\delta^u u$).

Proposition 4.9. *Let R be 2-fir and $a = p_1 p_2 \cdots p_n$ be a factorization of an element $a \in \mathcal{F}$ into atoms. If $\Delta \subseteq \mathcal{F}$ is F-algebraic then a is F-dependent on Δ*

if and only if either p_n is F-dependent on Δ or there exists $s \in \{1, 2, \ldots, n-1\}$ such that p_s is F- dependent on $\Delta^{p_{s+1}\cdots p_n}$.

Proof. Assume a is F-dependent on Δ. We have $0 \neq (\bigcap_{\delta\in\Delta} R\delta) \bigcap Ra \subseteq (\bigcap_{\delta\in\Delta} R\delta) \bigcap Rp_n$, so that $\Delta \cup \{p_n\}$ is algebraic. If $n = 1$ the result is clear. So let us assume that $n > 1$ and that p_n is not F-dependent on Δ. We leave it to the reader to check that Δ^{p_n} is algebraic. Now there exists a finite subset Γ of Δ and a non unit $d \in R$ such that $R\Gamma_\ell + Ra = Rd$. We claim that $p_1p_2...p_{n-1}$ is F-dependent on Δ^{p_n}. First let us remark that $\Delta^{p_n} \cup \{p_1p_2...p_{n-1}\}$ is algebraic since $((\bigcap_{\delta\in\Delta} R\delta^{p_n}) \bigcap Rp_1p_2...p_{n-1})p_n = \bigcap_{\delta\in\Delta} R\delta^{p_n}p_n \bigcap Ra = (\bigcap_{\delta\in\Delta} R\delta \bigcap Rp_n) \bigcap Ra = \bigcap_{\delta\in\Delta} R\delta \bigcap Ra \neq 0$. Now assume that $R\Gamma_\ell^{p_n} + Rp_1p_2...p_{n-1} = R$, then $Rp_n = (R\Gamma_\ell^{p_n} + Rp_1p_2...p_{n-1})p_n = R\Gamma_\ell^{p_n}p_n + Ra = (R\Gamma_\ell \bigcap Rp_n) + Ra \subseteq Rd$. Since p_n is an atom and d is not a unit this leads to $Rd = Rp_n$, but then $Ra + R\Gamma_\ell = Rp_n$ and hence $Rp_n + R\Gamma_\ell = Rp_n$. This contradicts the fact that p_n is not F-dependent over Δ and proves the claim. Now the induction hypothesis and the formula $(\Delta^{p_n})^q = \Delta^{qp_n}$ for any q such that $Rq \cap \bigcap_{\delta\in\Delta} R\delta \neq 0$ allow us to conclude easily.

Conversely, assume first that p_n is F-dependent on Δ and consider Γ a finite subset of Δ such that $\cap_{\gamma\in\Gamma} R\gamma + Rp_n \neq R$. Then, $\cap_{\gamma\in\Gamma} R\gamma + Ra \subset \cap_{\gamma\in\Gamma} R\gamma + Rp_n \neq R$.

Now, assume that p_n is F-dependent on Δ, but there exists $s \in 1, 2, \ldots, n-1$ such that p_s is F-dependent on $\Delta^{p_{s+1}\cdots p_n}$. This means that there exists a finite subset $\Gamma_0 \subseteq \Delta$ such that $\cap_{\gamma\in\Gamma_0} R\gamma^{p_{s+1}\cdots p_n} \subseteq Rp_s$. We want to show that a is F-dependent on Δ. Assume that this is not the case. Then, for all finite subset $\Gamma \subseteq \Delta, \cap_{\gamma\in\Gamma} R\gamma + Ra = R$. In particular, $\cap_{\gamma\in\Gamma_0} R\gamma + Ra = R$. Hence we have $(\cap_{\gamma\in\Gamma_0} R\gamma + Ra) \cap Rp_{s+1}\cdots p_n = Rp_{s+1}\cdots p_n$. Since $Ra \subseteq Rp_{s+1}\cdots p_n$, this gives $Rp_{s+1}\cdots p_n = ((\cap_{\gamma\in\Gamma_0} R\gamma) \cap Rp_{s+1}\cdots p_n) + Ra = (\cap_{\gamma\in\Gamma_0}(R\gamma \cap Rp_{s+1}\cdots p_n)) + Ra = \cap_{\gamma\in\Gamma_0} R\gamma^{p_{s+1}\cdots p_n}p_{s+1}\cdots p_n + Ra$. Since $\cap_{\gamma\in\Gamma_0} R\gamma^{p_{s+1}\cdots p_n} \subseteq Rp_s$, we finally get $Rp_{s+1}\cdots p_n \subseteq Rp_sp_{s+1}\cdots p_n$. This contradiction yields the result. $\square$

For an element a in R we denote $\Delta(a)$ the set of elements which are similar to a.

Theorem 4.10. *Let Δ be an algebraic set of atoms in a 2-fir R. If an atom a is F-dependent on Δ then a is F-dependent on $\Delta \cap \Delta(a)$.*

Proof. Since a is F-dependent on a finite subset of Δ we may assume that Δ is finite. Put $\Gamma := \Delta \cap \Delta(a)$, $h := \Gamma_\ell$ and denote by $f := \Delta_\ell$. We must show that $h \in Ra$. Let us notice that for any element $d \in \Delta \setminus \Delta(a) = \Delta \setminus \Gamma$ we have $h \notin Rd$ (since by Proposition 4.6 the factors of h are similar to a) hence Γ is full in Δ. Now we can write, as in the lemma 4.8, $f = gh$ where g is such that $Rg = \cap_{d\in\Delta\setminus\Gamma} Rd^h$. For any $d \in \Delta \setminus \Gamma$, we must have $g \in Rd^h$. On the other hand, since a is F-dependent on Δ, we have that $gh \in Ra$. Assume now that $h \notin Ra$. Then, thanks to Lemma 2.10, $g \in Ra^h$. But this would mean that an element of $\Delta(a)$ is a factor of g. This contradicts the definition of g and shows that h must be in Ra, as desired.

□

This theorem has an immediate useful corollary which will essentially reduce the study of an algebraic set of atoms to the case of an algebraic set contained in a similarity class.

Corollary 4.11. *Let $\Delta \subset \mathcal{A}$ be an algebraic set of finite rank. Then Δ intersects a finite number of similarity classes. More precisely : if $r = rk(\Delta)$, there exist $n \leq r$ non similar atoms $p_1, \dots, p_n \in \mathcal{A}$ such that $\Delta = \bigcup_{i=1}^n \Delta_i$ where $\Delta_i = \Delta \cap \Delta(p_i)$ for $i \in \{1, \dots, n\}$. Moreover if B_i is an F-basis for Δ_i then $B := \bigcup B_i$ is an F-basis for Δ and*

$$rk(\Delta) = \sum_{i=1}^{n} rk(\Delta_i) \qquad ; \qquad \overline{\Delta} = \bigcup_{i=1}^{n} \overline{\Delta_i}$$

In particular, if $f \in \mathcal{F}$ then $V(f)$ intersects at most $l(f)$ similarity classes.

Proof. Assume at the contrary that Δ intersects more than $r = rk(\Delta)$ similarity classes and let $a_1, \dots, a_{r+1}$ be elements of Δ belonging to distinct similarity classes. Then the above theorem shows that $\{a_1, \dots, a_{r+1}\}$ are F-independent, hence $rk(\Delta) \geq r + 1$, a contradiction. Now if $x \in \Delta$, then the above theorem shows that x is F-dependent on Δ_i for some $i \in \{1, \dots, n\}$; i.e. x is F-dependent on some B_i. On the other hand if $y \in \bigcup_{i=1}^n B_i$ is F-dependent on $\bigcup_{i=1}^n B_i \setminus \{y\}$ then $y \in B_i$ for some i and is F-dependent on $B_i \setminus \{y\}$. This contradiction allows us to conclude that B is an F-basis for Δ.
It remains to prove that $\overline{\Delta} \subseteq \bigcup_{i=1}^n \overline{\Delta_i}$ (the other inclusion being obvious). Let $p \in \mathcal{A}$ be an element which is F-dependent on $\Delta = \bigcup_{i=1}^n \Delta_i$. By the above theorem 4.10 we know that p is F-dependent on $\Delta(p) \cap (\bigcup_{i=1}^n \Delta_i) = \bigcup_{i=1}^n (\Delta_i \cap \Delta(p))$. Since the p_i's are non similar all but one of these intersections are empty and so there exists $j \in \{1, \dots, n\}$ such that p is F-dependent on Δ_j.

□

The notion of F-independence will be particularly explicit inside the similarity classes $\Delta(p_i)$. Let us recall that for an atom $p \in \mathcal{A}$, the ring $End_R(R/Rp)$, denoted $C(p)$, is in fact a division ring (Cf. Corollary 1.5). It turns out that in the similarity class $\Delta(p)$ of an atom p the notion of F-independence can be translated in terms of usual linear dependence over this division ring $C(p)$. Let us also recall that R/Rp has a natural structure of right $C(p)$-vector space. In the following definition we introduce a very useful map.

Definition 4.12. Let p be an atom and $f \in R$. We define

$$\lambda_{f,p} : R/Rp \longrightarrow R/Rp : x + Rp \mapsto fx + Rp\,.$$

Theorem 4.13. (a) *Let f be an element in $\mathcal{F}$ the map $\lambda_{f,p}$ is a right $C(p)$-linear map with $Ker(\lambda_{f,p}) = \{x + Rp \mid fx \in Rp\}$ and we have*

$$dim_{C(p)} Ker(\lambda_{f,p}) \leq \ell(f).$$

(b) *Let Δ be an algebraic subset contained in the similarity class $\Delta(p)$ of an atom p. Let $\{p_1, p_2, ..., p_n\}$ be a subset of Δ and for $i \in \{1, 2, ...n\}$ let $\phi_i : R/Rp_i \longrightarrow R/Rp$ be isomorphisms of left R-modules. Then the set $\{p_1, p_2, ..., p_n\}$ is F-dependent if and only if the set $\{\phi_1(1+Rp_1), \phi_2(1+Rp_2), ..., \phi_n(1 + Rp_n)\}$ is right $C(p)$-dependent.*

(c) *For $f \in \mathcal{F}$ and $p \in \mathcal{A}$, we have*

$$dim_{C(p)} Ker(\lambda_{f,p}) = rk(V(f) \cap \Delta(p)).$$

Proof. (a) We leave it to the reader to check that for the natural structure of right $C(p)$-vector space on R/Rp, the map $\lambda_{f,p}$ is a right homomorphism. The given description of $ker\lambda_{f,p}$ is straightforward and we only need prove that $dim_{C(p)} ker\lambda_{f,p} \leq \ell(f)$. We proceed by induction on $\ell(f)$. The claim is obvious if $\ell(f) = 0$. If f is an atom and $x+Rp$, $y+Rp$ are nonzero elements in $ker\lambda_{f,p}$ then $fx \in Rp$ and $fy \in Rp$. Using the notations of Lemma 2.10 we have $f \in Rp^x \cap Rp^y$. Since $x \notin Rp$ and $y \notin Rp$, p^x and p^y are not units in R and, f being an atom we conclude that $f = \alpha p^x = \beta p^y$ for units α and β in R. Define the isomorphisms $\phi_x : \frac{R}{Rp^x} \longrightarrow \frac{R}{Rp} : 1+Rp^x \mapsto x+Rp$ and similarly for ϕ_y. Now, since $Rp^x = Rf = Rp^y$ we have that $\frac{R}{Rp^x} = \frac{R}{Rp^y}$ and the map $\gamma := (\phi_x)^{-1} \circ \phi_y \in End_R(R/Rp)$ is such that $\gamma(x + Rp) = y + Rp$. Hence $x + Rp$ and $y + Rp$ are right $C(p)$-dependent. This shows that $dim_{C(p)} ker\lambda_{f,p} \leq 1$ as desired. For the general case we remark that if $f = f_1 f_2 ... f_r$ is an atomic decomposition of f then we have $\lambda_{f,p} = \lambda_{f_1,p} \circ \lambda_{f_2,p} \cdots \circ \lambda_{f_r,p}$. Hence $dim_{C(p)} ker\lambda_{f,p} \leq \sum_{i=1}^{r} dim_{C(p)} ker\lambda_{f_i,p} \leq r = \ell(f)$, as desired.

(b) Let us put $x_i := \phi_i(1 + Rp_i)$. We then have $p_i x_i = 0 \in R/Rp$. First let us assume that the x_i's are right $C(p)$-dependent and let $\sum_{i=1}^{n} x_i \gamma_i = 0$ be a dependence relation. Without loss of generality we may assume that $\gamma_n \neq 0$ and thus write $x_n = \sum_{i=1}^{n-1} x_i \psi_i$ for some $\psi_i \in C(p)$. Since Δ is algebraic there exists f in R such that $\bigcap_{i=1}^{i=n-1} Rp_i = Rf$ and we will show that $Rf \subseteq Rp_n$. We know there exist $f_1, f_2, \ldots, f_{n-1}$ such that $f = f_i p_i$ for $i = 1, 2, \ldots, n-1$ and since $p_i x_i = 0$ in R/Rp, we get $f x_i = f_i p_i x_i = 0$ for $i = 1, 2, \ldots, n-1$. This leads to $\phi_n(f + Rp_n) = f x_n = \sum_{i=1}^{n-1} f x_i \psi_i = 0$ and so $f \in Rp_n$, as desired.

Conversely let us suppose that $p_1, p_2, \ldots, p_n$ are F-dependent. Since these elements are contained in an algebraic set, we have $\cap_{i=1}^{n} Rp_i = Rf$ for some $f \in \mathcal{F}$ and since they are F-dependent we know by proposition 4.1(b) that $\ell(f) \leq n-1$. For $i = 1, 2, \ldots, n$ let us write $f = f_i p_i$. Now, since for

$i = 1, 2, \ldots, n$ we have $p_i x_i = 0 \in R/Rp$, we have $\lambda_{f,p}(x_i) = fx_i = f_i p_i x_i = 0$ i.e. $\{x_1, \ldots, x_n\} \subseteq ker\lambda_{f,p}$. By part (a) above we have that $dim_{C(p)} ker\lambda_{f,p} \leq \ell(f) \leq n-1$ and we conclude that $\{x_1, x_2, \ldots, x_n\}$ are right $C(p)$-dependent.

(c) Let $\{p_1, \ldots, p_n\}$ be an F-basis for $V(f) \cap \Delta(p)$ and put $R/Rp_i \overset{\phi_i}{\cong} R/Rp : y + Rp_i \mapsto yx_i + Rp$ for some $x_i \in R$. Let $y_i \in R$ be such that $p_i x_i = y_i p$; since $f \in Rp_i$, we have $fx_i \in Ry_i p$ and so $\phi_i(1 + Rp_i) = x_i + Rp \in ker(\lambda_{f,p})$ and part (b) above shows that these elements are $C(p)$-independent. We thus conclude that $rk(V(f) \cap \Delta(p)) \leq dim_{C(p)} ker(\lambda_{f,p})$.
Conversely if $x_1 + Rp, \ldots, x_n + Rp$ are $C(p)$-independent in $ker(\lambda_{f,p})$ then $fx_i \in Rp$ and since $x_i \notin Rp$ we get $f \in Rp^{x_i}$ and from part b) again we easily conclude that $p^{x_1}, \ldots, p^{x_n}$ are F-independent elements in $V(f) \cap \Delta(p)$. $\square$

Part a) in the above theorem was obtained by P.M.Cohn [3, Theorem 5.8, P.233] and part b) was inspired by similar results obtained for Ore extensions [9].
With the help of the previous theorem we are ready to present, as a corollary, the full computation of the rank of an algebraic subset $\Delta \subseteq \mathcal{A}$ as well as the description of the closure $\overline{\Delta}$. Recall that for an algebraic set of finite rank corollary 4.11 shows that Δ intersects a finite number of similarity classes $\Delta(p_1), \ldots, \Delta(p_n)$ and we can write $\Delta = \bigcup_{i=1}^n \Delta_i$ where $\Delta_i = \Delta \cap \Delta(p_i)$ for $i \in \{1, \ldots, n\}$. Now, for any $\gamma \in \Delta_i$ let ϕ_γ be an isomorphism $R/R\gamma \cong R/Rp_i$ and denote by Y_i the right $C(p_i)-$subspace of R/Rp_i defined by $Y_i := \sum_{\gamma \in \Delta_i} \phi_\gamma(1 + R\gamma)C(p_i)$. With these notations we can state:

Corollary 4.14. *Let Δ be an algebraic subset of $\mathcal{A}$. Then $rk(\Delta) = \infty$ if and only if one of the following holds :*
a)Δ contains infinitely many non similar atoms.
b)There exist $p \in \mathcal{A}$ and infinitely many atoms p_i in $\Delta \cap \Delta(p)$ such that their images into R/Rp by the isomorphisms $\phi_i : R/Rp_i \cong R/Rp$ generate an infinite dimensional vector space over $C(p)$.
If none of these conditions is satisfied then Δ is of finite rank and, using the above notations, we have :

$$\Delta = \bigcup_{i=1}^n \Delta_i \qquad rk(\Delta) = \sum_{i=1}^n dim_{C(p_i)} Y_i \qquad \overline{\Delta} = \bigcup_{i=1}^n \overline{\Delta_i}.$$

In particular if $f \in \mathcal{F}$ and $V(f) = \cup_{i=1}^r (V(f) \cap \Delta(p_i))$ is the decomposition of $V(f)$ into similarity classes one has

$$rkV(f) = \sum_{i=1}^r dim_{C(p_i)} ker(\lambda_{f,p_i}).$$

Proof. The proof uses 4.11 and theorem 4.13(b),(c). $\square$

The next result, although a bit technical, will be helpful.

Proposition 4.15. *Let h be a nonzero element in R and $\{a_1, \dots, a_n\}$ be an F-basis for $V(h)$. If $\{b_1, \dots, b_m\} \subset R \setminus V(h)$, then $\{a_1, \dots, a_n, b_1, \dots, b_m\}$ is F-independent if and only if $\{b_1^h, \dots, b_m^h\}$ is F-independent.*

Proof. If h is a unit $V(h)$ is empty and $\{b_1, b_2, \dots, b_n\}$ is F-independent if and only if $\{b_1^h, b_2^h, \dots, b_n^h\}$ is F-independent. We may thus assume that h is not a unit and we begin with the " only if" part. By Theorem 4.10, we know that elements in different similarity classes are F-independent. Lemma 2.9 shows that $b_j^h \sim b_j$, hence we may assume that $\{a_1, \dots, a_n, b_1, \dots, b_m\}$ is contained in a single conjugacy class, say $\Delta(p)$. Let, for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$,

- $\phi_i : R/Ra_i \longrightarrow R/Rp : 1 + Ra_i \mapsto \alpha_i + Rp$,
- $\psi_j : R/Rb_j \longrightarrow R/Rp : 1 + Rb_j \mapsto \beta_j + Rp$
- $\sigma_j : R/Rb_j^h \longrightarrow R/Rb_j : 1 + Rb_j^h \mapsto h + Rb_j$

be isomorphisms of left R-modules. Then $\psi_j \circ \sigma_j$ is an R-isomorphism of left modules between R/Rb_j^h and R/Rp such that $\psi_j \circ \sigma_j(1 + Rb_j^h) = h\beta_j + Rp$. Now, assume that $\{b_1^h, \dots, b_m^h\}$ is F-dependent. Then Theorem 4.13 (b) shows that $\{h\beta_1 + Rp, \dots, h\beta_m + Rp\}$ is right $C(p)$-dependent. So, there exist $\eta_1, \dots, \eta_m \in C(p)$ not all zero such that

$$Rp = \sum_{j=1}^{m}((h\beta_j + Rp)\eta_j) = h(\sum_{j=1}^{m}(\beta_j + Rp)\eta_j).$$

Let us write $(\beta_j + Rp)\eta_j = \beta_j' + Rp$. So we get

$$h(\sum_{j=1}^{m}\beta_j') \in Rp.$$

Let us remark that by Theorem 4.13 (b), we know that $\{\beta_1 + Rp, \dots, \beta_m + Rp\}$ is right $C(p)$-independent. This implies that $\sum_{j=1}^{m}\beta_j' + Rp \neq Rp$, and so, $\sum_{j=1}^{m}\beta_j' \notin Rp$. Using Lemma 2.10, we get $h \in Rp^{\sum \beta_j'}$. This shows that $p^{\sum \beta_j'} \in V(h)$ and so, $\{p^{\sum \beta_j'}, a_1, \dots, a_n\}$ is F-dependent. Considering the ϕ_i's and the isomorphism of left R-modules $R/Rp^{\sum \beta_j'} \longrightarrow R/Rp : 1 + Rp^{\sum \beta_j'} \mapsto \sum_{j=1}^{m}\beta_j' + Rp$, Theorem 4.13 (b) again shows that $\{\sum_{j=1}^{m}\beta_j' + Rp, \alpha_1 + Rp, \dots, \alpha_n + Rp\}$ is right $C(p)$-dependent. In other words $\sum_{j=1}^{m}\beta_j' + Rp = \sum_{j=1}^{m}(\beta_j + Rp)\eta_j$ is right $C(p)-$dependent on $\{\alpha_1 + Rp, \dots, \alpha_n + Rp\}$ and so $\{\alpha_1 + Rp, \dots, \alpha_n + Rp, \beta_1 + Rp, \dots, \beta_m + Rp\}$ is also right $C(p)$-dependent. This gives a contradiction, by Theorem 4.13 (b), since $\{a_1, \dots, a_n, b_1, \dots, b_m\}$ is F-independent.

For the "if" part assume $\{b_1^h, \dots, b_m^h\}$ is F-independent but $\Delta = \{a_1, \dots, a_n, b_1, \dots, b_m\}$ is F-dependent. Let us suppose that a_i is F-dependent on $\Delta \setminus \{a_i\}$. Let Δ_i be a minimal subset of $\Delta \setminus \{a_i\}$ such that a_i is F-dependent on Δ_i. As $\{a_1, \dots, a_n\}$ is F-independent, some b_j belongs to Δ_i. Now Proposition 3.9 shows that b_j is F-dependent on $(\Delta_i \cup \{a_i\}) \setminus \{b_j\}$. So we may assume that some b_j is F-dependent on $\{a_1, \dots, a_n, b_1, \dots, b_m\} \setminus \{b_j\}$, say b_m. Let us define $Rf := R[b_1^h, \dots, b_{m-1}^h]_\ell h$. Thanks to Lemma 2.10, we know that f is a least left common multiple of $\{a_1, \dots, a_n, b_1, \dots, b_{m-1}\}$. As b_m is F-dependent on $\{a_1, \dots, a_n, b_1, \dots, b_{m-1}\}$, f is also a left multiple of b_m. But, since $b_m \notin V(h)$, Lemma 2.10 shows that $R[b_1^h, \dots, b_{m-1}^h]_\ell \subset Rb_m^h$ i.e. b_m^h is F-dependent on $\{b_1^h, \dots, b_{m-1}^h\}$. This gives a contradiction. □

5. Fully reducible elements

Definition 5.1. An element $f \in \mathcal{F}$ is fully reducible if there exist atoms $p_1, \dots, p_n \in R$ such that $Rf = \bigcap_{i=1}^n Rp_i$

This notion was introduced by Ore for skew polynomials [12] and for *2*-firs by P.M.Cohn [3]. It was also used for product of linear polynomials in Ore extensions (under the name of separate zeros) by J.Treur [14] and G.Cauchon [2] and (under the name of Wedderburn polynomials) by T.Y.Lam and A.Leroy [9] and [10].

The set of fully reducible elements will be denoted by $\mathcal{R}$.

Lemma 5.2. *Let f, g be nonzero elements of a 2-fir R and suppose that $g \in \mathcal{R}$. Then*

a) *If $\phi : R/Rf \longrightarrow R/Rg$ is an injective R-morphism then $f \in \mathcal{R}$.*
b) *If $\psi : R/gR \longrightarrow R/fR$ is a surjective R-morphism then $f \in \mathcal{R}$.*

In particular, if $f \sim g$ then $f \in \mathcal{R}$ and in this case if $Rg = \cap_{i=1}^n Rp_i$, then $Rf = \cap_{i=1}^n Rp_i'$ where, for $1 \leq i \leq n$, $p_i' \sim p_i$.

Proof. a) Let $x \in R$ be such that $\phi(1+Rf) = x+Rg$ and let $y \in R$ be such that $fx = yg$. Lemma 1.3 shows that ϕ is injective if and only if $Rx \cap Rg = Rfx$. Since, by hypothesis, $g \in \mathcal{R}$ there exist atoms p_i's such that $Rg = \cap_{i=1}^n Rp_i$. We thus have $Rfx = Rx \cap (\cap_{i=1}^n Rp_i) = \cap_i (Rx \cap Rp_i) = \cap_i Rp_i^x x$. Hence we get $Rf = \cap_i Rp_i^x$. This yields that f is fully reducible, as requested.
b) This follows from Lemma 1.3.
The particular case is due to the fact that in the above proof $p_i^x \sim p_i$. □

Before stating the next theorem let us mention a nice consequence of the above lemma based on the results of section 1.

Corollary 5.3. *Let f, g be nonzero elements of a 2-fir R and suppose that $g \in \mathcal{R}$. If $Rf \cap Rg = Rg'f$ then $g' \in \mathcal{R}$.*
In particular, with our standard notation, we have $g^f \in \mathcal{R}$.

Proof. This is an easy consequence of Lemmas 5.2 and 1.3.
The particular case is merely a translation of the statement using our previous notation. □

Let us now come to the promised theorem showing that the notion of reducibility is symmetric. A constructive proof was given in [8, Theorem 3.6]. We include here a short one based on Lemma 5.2.

Theorem 5.4. *Suppose $0 \neq Rf = Rp_1 \cap \cdots \cap Rp_n$ is an irredundant intersection, where the p_i's are atoms in R. If we write $\bigcap_{j\neq i} Rp_j = Rg_i$ and $f = p'_i g_i$ $(1 \leq i \leq n)$, then*

1. *for each i, p'_i is an atom similar to p_i;*
2. *$fR = \bigcap_{i=1}^n p'_i R$;*
3. *the intersection representation for fR in (2) is irredundant.*

Proof. We will proceed by induction on n. If $n = 1$, $f = up_1 = p'_1$, $u \in U(R)$, and $fR = p'_1 R$. Now, if $n > 1$, Lemma 1.1 shows that $Rf = Rp_1 \cap Rp_2 \cap \cdots \cap Rp_n = Rp_1 \cap Rg_1 = Rp'_1 g_1 = Rg'_1 p_1$, $p_1 \sim p'_1$, $g_1 \sim g'_1$ and $fR = g'_1 R \cap p'_1 R$. Since $Rg_1 = \cap_{j\geq 2} Rp_j$, we know that g_1 is fully reducible and the above lemma 5.2 shows that g'_1 is also fully reducible i.e. $Rg'_1 = \cap_{j\geq 2} Rq_j$ where the q_j's are similar to the p_j's. The induction hypothesis then gives $g'_1 R = \cap_{j\geq 2} p'_j R$ where p'_j are atoms and $p'_j \sim q_j \sim p_j$. We then get $fR = g'_1 R \cap p'_1 R = \cap_{i=1}^n p'_i R$, with $p_i \sim p'_i$ for $1 \leq i \leq n$, as desired.

□

Corollary 5.5. *Let f, g be nonzero elements of a 2-fir R and suppose that $g \in \mathcal{R}$. Then*

a) *If $\phi : R/fR \longrightarrow R/gR$ is a an injective R-morphism then $f \in \mathcal{R}$.*
b) *If $\psi : R/Rg \longrightarrow R/Rf$ is a surjective R-morphism then $f \in \mathcal{R}$.*

Corollary 5.6. *Let f, g be nonzero elements of a 2-fir R and suppose that $g \in \mathcal{R}$. If $fR \cap gR = fg'R$ then $g' \in \mathcal{R}$.*

The following result is easy but useful :

Lemma 5.7. *Let $\{p_1, \ldots, p_n\} \subseteq \mathcal{A}$ be a finite set of atoms and f an element of $\mathcal{F}$. The following are equivalent :*

i) *$Rf = \cap_{i=1}^n Rp_i$ where the intersection is irredundant.*
ii) *$n = l(f)$ and $\{p_1, \ldots, p_n\}$ is an F-basis for $V(f)$.*

In particular, $f \in R$ is fully reducible if and only if $rkV(f) = \ell(f)$.

Proof. i) $\Rightarrow$ ii) Of course, $p_i \in V(f)$ and if $a \in V(f)$ then $\cap_{i=1}^n Rp_i = Rf \subseteq Ra$ and Proposition 3.6 shows that a is F-dependent on $B := \{p_1, \dots, p_n\}$. This means that $V(f)$ is F-dependent on B. The fact that the intersection is irredundant implies that B is an F-independent subset of R, and the conclusion follows.
ii) $\Rightarrow$ i) Obviously we have $Rf \subseteq \cap_{i=1}^n Rp_i$ and this last intersection is irrredundant since the set $\{p_1, \dots, p_n\}$ is F-independent. There exists $g \in R$ such that $\cap_{i=1}^n Rp_i = Rg$. The implication proved above shows that $l(g) = n = l(f)$ and we conclude that $Rg = Rf$.
The final statement is now obvious. $\square$

In the next theorem we will give a few more characterizations of fully reducible elements and further analyze the structure of the set $\mathcal{R}$ of these elements. In this theorem we will use the following notations: $\Delta(p)$ will stand for the similarity class determined by an element p. For an element $f \in R$ we will write as in 4.11 and 4.14 $V(f) = \cup_{i=1}^r \Delta_i$ where for $i \in \{1, 2, \dots, r\}$, $\Delta_i = V(f) \cap \Delta(q_i)$ is the intersection of $V(f)$ with the similarity class $\Delta(q_i)$ of some atoms $q_i \in \mathcal{A}$. By the term a "factor" of $f \in R$ we mean an element $g \in R \setminus U(R)$ such that there exist $p, q \in R$ with $f = pgq$. We say that g and h are neighbouring factors of an element f if there exist $p, q \in R$ such that $f = pghq$. Let us recall from Corollary 1.5 that for any $i \in \{1, 2, ..., r\}$, $C(q_i) := End(R/Rq_i)$ is a division ring and remark that R/Rq_i is a right $C(q_i)$-vector space.

Theorem 5.8. *Let R be a 2-fir and let $f \in \mathcal{F}$. Then the following are equivalent:*

(i) *f is fully reducible.*
(ii) *$rkV(f) = \ell(f)$.*
(iii) *Let $V(f) = \cup_{i=1}^r (V(f) \cap \Delta(q_i)))$ be the decomposition of $V(f)$ into similarity classes then $\ell(f) = \sum_{i=1}^r dim_{C(q_i)} ker(\lambda_{f,q_i})$, where $C(q_i) = End_R(R/Rq_i)$ is a division ring.*
(iv) *There exist atoms $p_1, p_2, \dots, p_n$ such that $R/Rf \cong \bigoplus_{i=1}^n R/Rp_i$.*
(v) *All factors of f are fully reducible.*
(vi) *Every product of two neighbouring factors of f is fully reducible.*
(vii) *Every product of two neighbouring atomic factors of f is fully reducible.*
(viii) *For any $g \in R$ if $V(f) \subseteq V(g)$ then $g \in Rf$.*

Proof. $(i) \Leftrightarrow (ii)$ This comes from Lemma 5.7.
$(ii) \Leftrightarrow (iii)$ This is an immediate consequence of Corollary 4.14.
$(i) \implies (iv)$ Assume $Rf = \cap_{i=1}^n Rp_i$ where $p_i \in \mathcal{A}$ and the intersection is irredundant. We shall show, by induction on n, that $R/Rf \cong \oplus_{i=1}^n R/Rp_i$. If $n = 1$, the result is clear. Let us write $Rf_n = \cap_{i=1}^{n-1} Rp_i$. We then have $Rf = Rp_n \cap Rf_n$ and $Rp_n + Rf_n = R$ so that $R/Rf \cong R/Rp_n \oplus R/Rf_n$. The induction hypothesis gives $R/Rf_n \cong \bigoplus_{i=1}^{n-1} R/Rp_i$ and enables us to conclude.

$(iv) \Longrightarrow (i)$ We assume $R/Rf \xrightarrow{\phi} \bigoplus_{i=1}^{n} R/Rp_i$ is an isomorphism. Let $x_1, \dots, x_n \in R$ be such that $\phi(1 + Rf) = (x_1 + Rp_1, \dots, x_n + Rp_n)$. Since ϕ is well defined and onto we have, for all $i \in \{1, \dots, n\}$, $fx_i \in Rp_i$ and $x_i \notin Rp_i$. This leads to the fact that $f \in \cap_{i=1}^{n} Rp_i^{x_i}$. Hence there exists a $g \in R$ such that $Rf \subseteq \cap_{i=1}^{n} Rp_i^{x_i} = Rg$. In particular we have $gx_i \in Rp_i$ for all $i \in \{1, \dots, n\}$ and $\phi(g + Rf) = 0$. Since ϕ is injective we conclude that $g \in Rf$ and $Rg \subseteq Rf$. This shows that $Rf = Rg = \cap_{i=1}^{n} Rp_i^{x_i}$ and, since the $p_i^{x_i}$'s are atoms, we have that $f \in \mathcal{R}$, as desired.
$(i) \Rightarrow (v)$ Assume $f = gh$. We then have an injective map of left R-modules $: R/Rg \xrightarrow{.h} R/Rf$ and Lemma 5.2 shows that $g \in \mathcal{R}$. Similarly the injective map of right R-modules $R/hR \xrightarrow{g.} R/fR$ implies that $h \in \mathcal{R}$. The case of a middle factor is then clear.
$(v) \Rightarrow (vi)$ and $(vi) \Rightarrow (vii)$ These are clear.
$(vii) \Rightarrow (ii)$ We proceed by induction on $n = \ell(f)$. If $n = 1$, f is an atom hence belongs to $\mathcal{R}$. If $n > 1$ we can write $f = ga$ for some $a \in \mathcal{A}$ and $g \in R$ such that $\ell(g) = n - 1$. Clearly g also satisfies the condition in (vii) and the induction hypothesis implies that $g \in \mathcal{R}$. Let us write $Rg = \cap_{i=1}^{n-1} Rp_i$ where the p_i's are in $\mathcal{A}$ and form an F-basis for $V(g)$ (cf. lemma 5.7). Then $Rga = \cap_{i=1}^{n-1} Rp_i a$ and the hypothesis shows that $p_i a \in \mathcal{R}$ so that there exist $c_1, \dots, c_{n-1} \in R$ with $Rp_i a = Rc_i \cap Ra = Rc_i^a a$ and we get $Rp_i = Rc_i^a$ which shows that the c_i^a's form an F-basis for $V(g)$. Proposition 4.15 then implies that $\{c_1, \dots, c_{n-1}, a\}$ are F-independent. Remarking also that $\{c_1, \dots, c_{n-1}, a\} \subseteq V(ga) = V(f)$, we thus have $rk(V(f)) \geq n = \ell(f)$. Since the inequality $rk(V(f)) \leq \ell(f)$ is always true we get that $rk(V(f)) = \ell(f)$, as desired.
$(i) \Rightarrow (viii)$ Assume $f \in \mathcal{R}$ and let us write $Rf = \cap_{i=1}^{n} Rp_i$. Hence $p_i \in V(f) \subseteq V(g)$ and $g \in \cap_{i=1}^{n} Rp_i = Rf$.
$(viii) \Rightarrow (ii)$ Let us put $\ell(f) = n$. By $(viii)$ any element which is a left common multiple of an F-basis of $V(f)$ has length $\geq n$ thus $rkV(f) \geq n$. Since the converse inequality always holds we get (ii).

□

Remark 5.9. It is worth to mention the relations between the p_i's and the q_i's appearing in the above theorem. First let us notice that it is clear from the proof that, if f is fully reducible and $Rf = \cap_{i=1}^{n} Rp_i$ is an irredundant representation where the p_i's are atoms, then these atoms are exactly those appearing in statement (iv) of the theorem. Let us also recall that we know from 5.7 that these atoms form an F-basis for $V(f)$. It is then clear that every similarity class intersecting non trivially $V(f)$ contains at least one of the p_i's. Since the q_i's must represent these similarity classes we can just choose the q_i's amongst the p_i's.

The following corollary gives more precise information on the equivalence $(i) \Leftrightarrow (iv)$ of the above theorem.

Corollary 5.10. *For an element f in a 2-fir R, we have $f \in \mathcal{R}$ and $\ell(f) = n$ if and only if there exist $p_1, \dots, p_n \in \mathcal{A}$ such that $R/Rf \cong \oplus_{i=1}^n R/Rp_i$.*

Proof. If $f \in \mathcal{R}$ and $\ell(f) = n$ then Lemma 5.7 implies that there exists an irredundant representation $Rf = \cap_{i=1}^n Rp_i$ with $p_i \in \mathcal{A}$ and the proof of the implication $(i) \Rightarrow (iv)$ of the above theorem shows that $R/Rf \cong \oplus_{i=1}^n R/Rp_i$. Conversely if $R/Rf \xrightarrow{\phi} \bigoplus_{i=1}^n R/Rp_i$ is an isomorphism then using the same notations as in the proof of the above theorem we have $Rf = \cap_{i=1}^n Rp_i^{x_i}$ and $f \in \mathcal{R}$. We must only show that $n = \ell(f)$. From Lemma 5.7 this is equivalent to showing that this representation is irredundant. Assume at the contrary that this is not the case, without loss of generality we may assume that $\cap_{i=1}^{n-1} Rp_i^{x_i} \subseteq Rp_n^{x_n}$. Now, since ϕ is an isomorphism there exists $h \in R$ such that $\phi(h+Rf) = (0, \dots, 0, 1+Rp_n)$ i.e. $hx_i \in Rp_i$ for $i = 1, \dots, n-1$ and $hx_n = 1 + Rp_n$. Since $x_i \notin Rp_i$ we must have $h \in \cap_{i=1}^{n-1} Rp_i^{x_i} \subseteq Rp_n^{x_n}$. This implies $hx_n \in Rp_n$ a contradiction.

□

In the following theorem we present different characterizations for a product to be fully reducible. Let us first introduce two relevant definitions :

Definitions 5.11. *For $a \in R$,*

a) $V'(a) := \{p \in \mathcal{A} | \, a \in pR\}$.
b) $\mathbb{I}_R(Ra) = \{f \in R | \, af \in Ra\}$.

Theorem 5.12. *For $a, b \in R \setminus U(R)$ the following are equivalent :*

(i) *ab is fully reducible.*
(ii) *a, b are fully reducible and $R/Rab \cong R/Ra \oplus R/Rb$.*
(iii) *a, b are fully reducible and $1 \in Ra + bR$.*
(iv) *a, b are fully reducible and for all $p \in V(a)$, pb is fully reducible.*
(v) *a, b are fully reducible and for any F-basis $\{p_1, \dots, p_\ell\}$ of $V(a)$, $p_i b$ is fully reducible for $i = 1, \dots, \ell$.*
(vi) *a, b are fully reducible and for any $p \in V(a)$ and any $q \in V'(b)$, pq is fully reducible.*
(vii) *a, b are fully reducible and $\mathbb{I}_R(Ra) \subset Ra + bR$.*

Proof. $(i) \implies (ii)$ Since ab is fully reducible theorem 5.8 (v) shows that a and b are also fully reducible. Since $V(b) \subseteq V(ab)$, we can present an F-basis for $V(ab)$ in the form $p_1, \dots, p_r, q_1, \dots, q_s$ where the p_i's form an F-basis for $V(b)$. Using Theorem 5.8 we obtain the isomorphism $R/Rab \cong$

$(\oplus_{j=1}^r R/Rp_j)\bigoplus(\oplus_{i=1}^s R/Rq_i)$. Notice that we have $r+s = rkV(ab) = \ell(ab) = \ell(a)+\ell(b)$ and $r = rkV(b) = \ell(b)$ so that $s = \ell(a) = rkV(a)$. Now, by Lemma 5.7, we get $Rb = \cap_{j=1}^r Rp_j$ and $Rab = (\cap_{i=1}^s Rq_i)\bigcap(\cap_{j=1}^r Rp_j) = (\cap_{i=1}^s Rq_i)\cap Rb = \cap(Rq_i\cap Rb) = \cap Rq_i^b b$. This gives $Ra = \cap_{i=1}^s Rq_i^b$. Now, since $\ell(a) = s$ we have (Cf. Remark 5.9), that $R/Ra \cong \oplus R/Rq_i^b \cong \oplus R/Rq_i$ and $R/Rab \cong (\oplus R/Rq_i)\bigoplus(\oplus R/Rp_j) \cong R/Ra\oplus R/Rb$.

$(ii) \Longrightarrow (i)$ This is an immediate consequence of 5.8 (iv).

$(i) \Longrightarrow (iii)$ Using the same notations as in the proof of $(i) \Longrightarrow (ii)$ above, let us fix an F-basis $\{p_1,\dots,p_r,q_1,\dots,q_s\}$ for $V(ab)$ such that the p_i's form an F-basis for $V(b)$. We thus have $Rb = \cap_{j=1}^r Rp_j$ and we define b' via $Rb' = \cap_{i=1}^s Rq_i$. Theorem 4.4 then shows that $V(b)\cap V(b') = \emptyset$ and from lemma 4.3 c) we get that there exist $u, v\in R$ such that $ub'+vb = 1$. Left multiplying by b, we get $bub'+bvb = b$, in particular $(bv-1)b\in Rb'$. Therefore $(bv-1)b\in Rb'\cap Rb = Rab$, and $(bv-1)\in Ra$. This shows that $1\in Ra+bR$ as desired.

$(iii) \Longrightarrow (iv)$ Let p be an atom in $V(a)$. Since $1\in Ra+bR$, $1\in Rp+bR$. Then there exist $u, v\in R$ such that $up+bv = 1$. Notice that this shows that $bv\notin Rp$ and hence $b\notin Rp^v$. Left multiplying $up+bv = 1$ by p, we get $pup+pbv = p$. So $pbv\in Rp\cap Rbv$ and hence $pbv\in Rp^{bv}bv$ and $p\in Rp^{bv}$. Since p is an atom we conclude that we must have $Rp = Rp^{bv} = R(p^v)^b$ where the last equality comes from 2.9 (b). We finally get $Rpb = R(p^v)^b b = Rp^v\cap Rb$ which shows that pb is fully reducible since b is fully reducible.

$(iv) \Longrightarrow (v)$ is obvious.

$(v) \Longrightarrow (i)$ let A be an F-basis for $V(a)$, say $A = \{a_1,\dots,a_n\}$. By hypothesis, $\exists b_1,\dots,b_n\in\mathcal{A}$ such that $Rb_i\cap Rb = Ra_ib$; let $B = \{b_1,\dots,b_n\}$. We have $Rab = (\cap Ra_i)b = \cap Ra_ib = \cap(Rb_i\cap Rb) = \cap Rb_i\cap Rb$. Since b is fully reducible, this shows that $ab\in\mathcal{R}$.

$(iv) \Longrightarrow (vi)$ is obvious and $(vi) \Longrightarrow (iv)$ follows from 5.8 (vii).

$(iii) \Longrightarrow (vii)$ Assume $1\in Ra+bR$ and let $c\in\mathbb{I}_R(Ra)$. We have $ac\in Ra$, and so

$$c = 1c\in(Ra+bR)c\subseteq Rac+bR\subseteq Ra+bR$$

as desired.

$(vii) \Longrightarrow (iii)$ is trivial since $1\in\mathbb{I}_R(Ra)$. $\square$

6. Rank Theorems

In this final short section we will give some formulas for computing the rank of algebraic sets of atoms. Let us first recall from 4.7 that a subset $\Delta\subseteq\mathcal{A}$ is full (in $\mathcal{A}$) if any atom which is F-dependent on Δ is already in Δ.

Proposition 6.1. *Let $\{\Delta_j : j \in J\}$ be full algebraic sets of atoms. Then*

a) $\cap_{j\in J}\Delta_j$ *is full algebraic.*

b) *If there exists $j_0 \in J$ such that Δ_{j_0} is of finite rank then $\cap_{j\in J}\Delta_j$ is of finite rank and*

$$R(\cap_{j\in J}\Delta_j)_\ell = \sum_{j\in J} R(\Delta_j \cap \Delta_{j_0})_\ell.$$

Proof. a) Since $\emptyset = \overline{\emptyset}$ is a full algebraic set, we may assume that $\Delta := \cap_{j\in J}\Delta_j \neq \emptyset$. If $p \in \mathcal{A}$ is F-dependent on Δ then p is F-dependent on each Δ_j and hence $p \in \Delta_j$. So $p \in \bigcap_{j\in J}\Delta_j = \Delta$. This shows that Δ is a full algebraic set.
b) Obviously, for any $j \in J$, $\Delta_j \cap \Delta_{j_0}$ is algebraic of finite rank and since $\cap_j \Delta_j = \cap_j(\Delta_j \cap \Delta_{j_0})$, we may assume that in fact all the Δ_j's are algebraic of finite rank and full (by a) above). Let us put $\Delta = \cap_{j\in J}\Delta_j$. Let $f \in R$ and for $j \in J$, let $f_j \in R$ be such that $Rf = \cap_{\delta\in\Delta}R\delta$ and $Rf_j = \cap_{\delta\in\Delta_j}R\delta$. We must show that $R\Delta_\ell = \sum_{j\in J} R(\Delta_j)_\ell$, i.e. $Rf = \sum_{j\in J} Rf_j$. Since, for $j \in J$, $\Delta \subseteq \Delta_j$, we have $Rf_j \subseteq Rf$. On the other hand if $h \in R$ is such that $\sum_{j\in J} Rf_j = Rh$ we have

$$V(h) \subset \bigcap_{j\in J} V(f_j) = \bigcap_{j\in J}\Delta_j = \Delta = V(f)$$

since Δ_j's and Δ are full algebraic sets. Therefore, Theorem 5.8 again implies that h is a right divisor of f. This shows that $Rf \subseteq Rh$ and we conclude $\sum_{j\in J} Rf_j = Rf$, as desired. □

The next theorem gives more precise information than Theorem 4.4.

Theorem 6.2. *For any algebraic set of atoms Δ and Γ, we have*

$$rk(\Delta) + rk(\Gamma) = rk(\Delta \cup \Gamma) + rk(\overline{\Delta} \cap \overline{\Gamma}).$$

Proof. If Δ or Γ is of infinite rank the formula is clear. We may thus assume that both Δ and Γ are of finite rank. Let us write $Rp = R\Gamma_\ell + R\Delta_\ell$ and $Rq = R\Gamma_\ell \cap R\Delta_\ell$. Then $Rq = R(\Delta \cup \Gamma)_\ell$ by 4.4(i) and $Rp = R(\overline{\Delta} \cap \overline{\Gamma})_\ell$ by the last proposition. Theorem 2.12 gives then

$$\ell(\Gamma_\ell) + \ell(\Delta_\ell) = \ell(q) + \ell(p).$$

In other words,

$$rk(\Delta) + rk(\Gamma) = rk(\Delta \cup \Gamma) + rk(\overline{\Delta} \cap \overline{\Gamma}).$$

□

In order to express the rank of $V(ab)$, let us introduce the following set : for $a \in R$ we define $I_a := \{q \in \mathcal{A} \mid \exists p \in \mathcal{A}\ ; 0 \neq Rp \cap Ra = Rqa\}$. Let us also recall our notations : $Rp \cap Ra = Rp^a a$.

Theorem 6.3. *Let $a, b \in R$. Then*

$$rkV(ba) = rkV(a) + rk(I_a \cap V(b)).$$

In particular $rkV(ba) \leq rkV(b) + rkV(a)$.

Proof. If $V(a) = V(ba)$ we claim that $I_a \cap V(b) = \emptyset$. Indeed assume $q \in I_a \cap V(b)$, then there exists $p \in \mathcal{A}$ and $a' \in R$ such that $0 \neq Ra \cap Rp = Rqa = Ra'p$. In particular there exists $u \in U(R)$ such that $qa = ua'p$. Since $q \in V(b)$ we can write $b = b'q$ for some $b' \in R$. Multiplying by a on the right gives $ba = b'qa = b'ua'p$. This shows that $p \in V(ba) = V(a)$ and hence $Ra \subseteq Rp$. We thus get $0 \neq Rqa = Ra \cap Rp = Ra$ and finally $q \in U(R)$, this is the required contradiction. We may thus assume that the inclusion $V(a) \subset V(ba)$ is proper. Let $\{a_1, \ldots, a_n\}$ be an F-basis for $V(a)$ and extend it into an F-basis for $V(ba)$, say $\{a_1, \ldots, a_n, b_1, \ldots, b_m\}$. For $i \in \{1, \ldots, m\}$ we have that $a \notin Rb_i$ and $ba \in Rb_i$. Then by Lemma 2.10, $b \in Rb_i^a$ and by Proposition 4.15, $\{b_1^a, \ldots, b_m^a\}$ is F-independent. This shows that $rkV(ba) \leq rkV(a) + rk(I_a \cap V(b))$. For the other inequality let $\{b_1^a, \ldots, b_m^a\}$ be an F-basis for $I_a \cap V(b)$. Then by Proposition 4.15 $\{a_1, \ldots, a_n, b_1, \ldots, b_m\} \subset V(ba)$ is F-independent. This shows that $rkV(ba) \geq rkV(a) + rk(I_a \cap V(b))$. □

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