On Induced Modules Over Ore Extensions

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Abstract

Let R be a ring and $S = R[x; \sigma, \delta]$ its Ore extension. For an R-module M_R we investigate the uniform dimension and associated primes of the induced S-module $M \otimes_R S$.

1 Introduction

Let $R \subseteq S$ be a ring extension. Then we have a natural functor $- \otimes_R S$ from the category of right R-modules into the category of right S-modules and it is important to know which properties of the module M_R lift to the induced S-module $M \otimes_R S$. In the literature there are many results in this direction both for special kinds of ring extensions $R \subseteq S$ and for special kinds of modules. Even in the case when $M_R = R_R$ the situation is nontrivial, as it means that we lift module properties from the ring R to that of the ring S.

The aim of the paper is to investigate the passage of properties from a module M_R to its induced S-module $M \otimes_R S$ in the case when $S = R[x; \sigma, \delta]$ is an Ore extension of R. We are mainly concerned with problems related to the uniform dimension and associated prime ideals of the induced module $M \otimes_R S$. In both cases, our approach is based on the use of good polynomials. Such polynomials were first used by R.C. Shock in his classical paper [Sh] for proving that uniform dimensions of a ring R and the polynomial ring $R[x]$ are equal. They were also used in other papers (see for example [A1],[A2],[Ma])

In section 2 we set the notations and present or recall some basic definitions.

Section 3 is devoted to a detailed study of good polynomials. Such polynomials were earlier used either in the context of rings but not modules or under a very strong additional assumptions on the automorphism σ and σ-derivation δ of R. Moreover, the situation drastically changes while passing from the module R_R to an arbitrary R-module M_R. In the first case we still have an action of σ and δ on R_R, in the second case we do not have such action.

In Section 4, under the assumption that the S-module $M \otimes_R S$ is good (see Definition 4.4), we prove that the modules M_R and $M \otimes_R S$ have the same uniform dimension. We show also that when S_S is a good module, then for any R-module M_R, the induced module $M \otimes_R S$ is nonsingular if and only if the module M_R is such. Earlier we provide natural situations when the induced module is good.

It is known that, in general, such equality does not hold. In many previous papers (see for example [BeG], [Gr], [Ma], [Si], [Qu]) similar results were obtained. All of the cited papers except [Ma] concern the case when S is a differential operator ring in one or more variables. In [Ma] such equality was shown when $M_R = R_R$ is a good module. Our results extend most of the above mentioned ones and cover also the classical case of usual polynomial rings ([Sh]). As an application we show in Theorem 4.10 that for a quantum algebra $T(R)$, which can be presented as an iterated Ore extension over R, the induced $T(R)$-module $M \otimes_R T(R)$ is nonsingular and has the same uniform dimension as M_R, assuming that the module M_R is nonsingular.

We close this section with an observation concerning divisibility of the induced module.

In the final Section 5 we investigate, with the help of good polynomials, relations between associated primes of an R-module M_R and that of the induced S-module $M \otimes_R S$. The prime spectrum of an Ore extension $S = R[t, \sigma, \delta]$ and its generalizations are intensively studied (for an exposition and references see [BrG] and [GL]). The structure of the prime spectrum of S is quite complicated and it is natural to investigate certain parts of it. For example, it is known that in the case $S = R[x]$ is a polynomial ring then associated prime ideals of the induced module $M \otimes_R S$ are just extensions of associated primes of the R-module M_R (Cf. [BH], [Fa]). Recently, S. Annin in [A1], [A2] extended this result to certain skew polynomial rings. In Section 5 we show that in many other situations all associated primes of the induced module $M \otimes_R S$ arise from associated primes of the module M_R even in the case S is an iterated Ore extension of R. This, in particular, generalizes the above mentioned result of S. Annin. We also offer examples which show that the assumptions we make are necessary.

2 Preliminaries

In this section we introduce notations and recall some classical definitions.

Throughout the paper R will stand for an associative ring with unity, σ will denote an automorphism of R and δ a σ-derivation of R, i.e. $\delta\colon R \to R$ is an additive map such that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in R$.

S will stand for the Ore extension $R[x; \sigma, \delta]$. Recall that elements of S are polynomials in x with coefficients written on the left. Multiplication in S is given by the multiplication in R and the condition $xa = \sigma(a)x + \delta(a)$ for all $a \in R$.

For any $0 \leq j \leq i$, $f_j^i \in \mathrm{End}(R, +)$ will denote the map which is the sum of all possible words in σ, δ built with j letters σ and $i - j$ letters δ

The following properties of those maps are well-known (Cf. [LLM]).

Lemma 2.1. *For any $a, b \in R$ we have:*

1. $f_j^i(ab) = \sum_{k=j}^{i} f_k^i(a) f_j^k(b)$ *for any* $0 \leq j \leq i$.
2. $x^n a = \sum_{j=0}^{n} f_j^n(a) x^j$ *in the ring* S.

We say that the σ-derivation δ of R is q-quantized if $\delta\sigma = q\sigma\delta$ where q is a central, invertible element of R such that $\sigma(q) = q$ and $\delta(q) = 0$. When δ is q-quantized, then $f_j^i = \binom{i}{j}_q \sigma^j \delta^{i-j}$, where $\binom{i}{j}_q$ means the q-binomial symbol. If $q = 1$ then $\binom{i}{j}_q = \binom{i}{j}$. It is also known (Cf. [BrG] or [LLM]), that if either q is not a root of unity or $q = 1$ and R contains the ring $\mathbb{Z}$ of integers as a subring, then $\binom{i}{j}_q$ is regular in R for any $0 \leq j \leq i$.

Let M_R be a right R-module. Then the right S-module $M \otimes_R S$ will be called the induced module and will be denoted by $\widehat{M}_S$.

Since $R[x; \sigma, \delta]$ is a free left R-module, the elements from $\widehat{M}_S$ can be seen as polynomials in x with coefficients in M_R with natural additive and right S-module structures. For this reason we will sometimes call elements of $\widehat{M}_S$ polynomials and present elements of $\widehat{M}_S$ in the form $\sum_i m_i x^i$, where $m_i \in M_R$ for all i.

In case $M_R = R_R$ this construction gives $\widehat{M}_S = S_S$.

The notion of degree of polynomials from $\widehat{M}_S$ is defined similarly as in the case of polynomials in S.

Let N_R be a submodule of M_R. Then we can consider $\widehat{N}_S$ as a submodule of $\widehat{M}_S$ and it is standard to check that:

Lemma 2.2. *Let N_R be a submodule of an R-module M_R. Then the S-modules $(\widehat{M/N})_S$ and $(\widehat{M}/\widehat{N})_S$ are isomorphic.*

For a subset $\mathcal{A}$ of the module M_R, $\mathrm{ann}_R(\mathcal{A})$ will denote the annihilator of $\mathcal{A}$ in R.

Let N_R be an R-submodule of M_R. Recall that N_R is essential in M_R if for any $0 \neq m \in M_R$ we have $mR \cap N \neq 0$. The singular submodule of M_R, denoted by $Z(M_R)$, is the set of all elements $m \in M_R$ such that $\mathrm{ann}_R(m)$ is an essential right ideal of R. The module M_R is said to be nonsingular if $Z(M_R) = 0$.

The uniform dimension of the module M_R is denoted by $\mathrm{udim} M_R$.

3 Good Polynomials

The notion of good polynomials was first introduced by R.C. Shock (Cf. [Sh]) for proving that uniform dimensions of R and $R[x]$ are equal. They were used also by the second author for proving the same result for Ore extensions $S = R[x; \sigma, \delta]$ (under some mild extra assumptions).

In this section we introduce and carefully investigate good polynomials in the induced module $\widehat{M_S}$.

In the sequel we will use, for an ideal I of R, invariant ideals associated to I under various actions. In particular I_σ will denote the largest σ-invariant ideal of R contained in I, i.e. $I_\sigma := \{a \in I | \sigma^n(a) \in I \text{ for all } n \geq 0\}$. Similarly we define I_δ, $I_{\sigma,\delta}$ and denote

$$I_\Omega := \{a \in I |\ w(a) \in I \text{ for all words } w \text{ in } \sigma \text{ and } \delta\}$$

and

$$I_F := \{a \in I |\ f_j^i(a) \in I \text{ for all } i \in \mathbb{N} \text{ and } 0 \leq j \leq i\}.$$

Lemma 3.1. *With the above notations and definitions we have :*

1. $I_\Omega \subseteq I_F \subseteq I_\sigma \cap I_\delta$ *and all these subsets of* I *are ideals of* R.
2. *If either* $\delta(I_\sigma) \subseteq I_\sigma$ *or* $\sigma(I_\delta) \subseteq I_\delta$ *then* $I_\Omega = I_F = I_\sigma$.
3. $I_\Omega = I_F$ *provided* δ *is a q-quantized* σ*-derivation and* $\binom{i}{j}_q$ *is invertible in* R *for any* $0 \leq j \leq i$, $i \in \mathbb{N}$.
4. *Let* N *be a right* R*-module with* $\mathrm{ann}_R(N) = I$.
   Then:
   (a) $\mathrm{ann}_R(\widehat{N}) = I_F$.
   (b) $\mathrm{ann}_S(\widehat{N})$ *is the largest ideal of* S *contained in* $\mathrm{ann}_S(N)$.
   (c) *If* $\mathrm{ann}_S(\widehat{N}) = JS$ *for some ideal* J *of* R, *then* $J = I_F = \mathrm{ann}_R(\widehat{N})$

Proof. (1) The inclusions are clear. Note also that $\omega(I_\Omega) \subseteq I_\Omega$ for all $\omega \in \Omega$. An easy induction on the length of ω then shows that I_Ω is an ideal of R.

The fact that I_F is an ideal is an easy consequence of Lemma 2.1(1)

(2) Suppose that $\delta(I_\sigma) \subseteq I_\sigma$. Since also $\sigma(I_\sigma) \subseteq I_\sigma$ we obtain $I_\sigma \subseteq I_\Omega$. This together with (1) gives the thesis.

If $\sigma(I_\delta) \subseteq I_\delta$, then a similar argument to the one above gives the thesis.

(3) Since δ is a q-quantized σ-derivation, we have $f_i^j = \binom{i}{j}_q \sigma^j \delta^{i-j}$. Moreover, by the assumption imposed on q, $\binom{i}{j}_q$'s are central, invertible elements of R. This gives $I_\Omega = I_F$ in this case.

(4) Let N_R be a right R-module with $\mathrm{ann}_R(N) = I$.

(a) Let $a \in I_F$ and $\sum n_i x^i \in \widehat{N}_S$ with $n_i \in N_R$. Then, by Lemma 2.1(2), we have $\sum (n_i x^i) a = \sum_i (\sum_j n_i f_j^i(a) x^j) = 0$ as $f_j^i(a) \in I = \mathrm{ann}_R(N)$ for all i, j. This shows that $I_F \subseteq \mathrm{ann}_R(\widehat{N})$.

For proving the converse inclusion, assume that $a \in \mathrm{ann}_R(\widehat{N})$. In particular, for any $n \in N_R$ and $i \in \mathbb{N}$ we have $0 = nx^i a = \sum_j n f_j^i(a) x^j$. This implies that $\mathrm{ann}_R(\widehat{N}) \subseteq I_F$.

(b) Obviously $\mathrm{ann}_S(\widehat{N})$ is an ideal of S which is contained in $\mathrm{ann}_S(N)$. On the other hand, if J is an ideal of S contained in $\mathrm{ann}_S(N)$ then $\widehat{N} J = 0$ and this yields the conclusion.

(c) Suppose $\mathrm{ann}_S(\widehat{N}) = JS$ for some ideal J of R. Then, by (a), $I_F \subseteq J$. The converse inclusion is a consequence of the fact that $Nx^i J = 0$ for all $i \geq 0$. □

Definition 3.2. We say that a nonzero polynomial $g \in \widehat{M_S}$ is good if for any $r \in R$ $\deg(gr) = \deg(g)$, provided $gr \neq 0$.

In the following example we offer a few natural constructions of good polynomials.

Example 3.3. Let M_R be a right R-module.

1. Any nonzero element from M_R is a good polynomial of degree 0.
2. If $m \in M_R$ is such that $\mathrm{ann}_R(m) = 0$, then any polynomial from $\widehat{M_S}$ with leading coefficient m is good.
3. Let $m \in M_R$ be such that $\mathrm{ann}_R(m) = 0$ and let $0 \neq b \in R$. Then one can easily check that $mx\sigma^{-1}(b) = mbx + m\delta\sigma^{-1}(b)$ is a good polynomial of degree one in $\widehat{M_S}$. More generally, the polynomial $mx^n b \in \widehat{M_S} S = \widehat{M_S}$ is a good polynomial of degree n and leading coefficient $m\sigma^n(b)$.
4. Suppose $\delta = 0$ and $g \in \widehat{M_S}$ is a good polynomial, then the polynomial gx^i is also good for any $i \geq 0$.

In the next lemma we will consider various characterizations of good polynomials. Before stating this lemma, let us introduce the following notation : for a module M_R and an automorphism τ of R we denote by M_τ the τ-twisted R-module defined on the same additive structure $M_\tau = M$ where the action of R is twisted by τ, i.e. $m \cdot r := m\tau(r)$. For $m \in M_R$ it will be convenient to denote the element of M_τ corresponding to m by m_τ.

Lemma 3.4. *Let $g \in \widehat{M_S}$ be a nonzero polynomial of degree l and leading coefficient a. Then the following statements are equivalent :*

(i) g *is good.*

(ii) g *is a polynomial of minimal degree in* gR.

(iii) g *is a polynomial of minimal degree in* gS.

(iv) *For any* $r \in R$, $a\sigma^l(r) = 0$ *if and only if* $gr = 0$.

(v) $\operatorname{ann}_R(g) = \sigma^{-l}(\operatorname{ann}_R(a))$.

(vi) $\operatorname{ann}_S(g) = \sigma^{-l}(\operatorname{ann}_R(a))S$.

(vii) *There exists an* S*-module isomorphism* $\phi\colon gS \longrightarrow \widehat{(aR)_{\sigma^l}}$ *such that* $\phi(g) = a_{\sigma^l}$.

Proof. The easy proofs of the equivalences (i) to (v) are left to the reader.

(v) $\Rightarrow$ (vi): Let $h = b_nx^n + \cdots + b_1x + b_0 \in \operatorname{ann}_S(g)$. Considering the leading coefficient of $gh = 0$ we get $a\sigma^l(b_n) = 0$ and (v) gives us $gb_n = 0$. This yields $g(b_{n-1}x^{n-1} + \cdots + b_0) = 0$. Continuing this process we obtain $\{b_0, \dots, b_n\} \subseteq \sigma^{-l}(\operatorname{ann}_R(a))$ and thus $h \in \sigma^{-l}(\operatorname{ann}_R(a))S$. Hence $\operatorname{ann}_S(g) \subseteq \sigma^{-l}(\operatorname{ann}_R(a))S$. The reverse inclusion is clear.

(vi) $\Rightarrow$ (vii): Using (vi) and Lemma 2.2, we have the following chain of natural isomorphisms of S-modules:

$$\begin{aligned} gS &\cong S/\operatorname{ann}_S(g) = S/\sigma^{-l}(\operatorname{ann}_R(a))S \cong \\ &\cong \Big(R/\sigma^{-l}(\operatorname{ann}_R(a))\Big) \otimes_R R[x;\sigma,\delta] \cong \widehat{(aR)_{\sigma^l}}. \end{aligned}$$

It is easy to check that, by this chain of isomorphisms, the image of g is a_{σ^l}.

(vii) $\Rightarrow$ (iv): Suppose that there is an S-module isomorphism, say ϕ, between gS and $\widehat{(aR)_{\sigma^l}}$ sending g onto a_{σ^l}. Then, for any $r \in R$, $\phi(gr) = a_{\sigma^l} \cdot r = a\sigma^l(r)$ and hence, $gr = 0$ if and only if $a\sigma^l(r) = 0$.

□

Using the characterization of good polynomials given in the above lemma we obtain:

Corollary 3.5.

1. *Let* $0 \neq g \in \widehat{M}_S$. *Then there exists* $r \in R$ *such that* gr *is a good polynomial.*

2. *If* $g \in \widehat{M}_S$ *is a good polynomial, then* $\operatorname{ann}_S(g) = \operatorname{ann}_R(g)S$.

We will need the existence of good polynomials of any degree in a submodule generated by $a \in M_R$. This is the objective of the following lemma which is similar to Lemma 2.1 from [Ma].

Proposition 3.6. *Let* $a \in M_R$ *and* $g \in \widehat{M}_S$ *be a good polynomial of degree* l *and leading coefficient* a. *If the submodule* aR *of* M_R *is nonsingular, then:*

1. aS *contains a good polynomial of degree one.*

2. *For any* $n \geq l$ *there exists a good polynomial in* gS *of degree* n.

Proof. (1) Since the submodule aR is nonsingular, we can pick $0 \neq b \in R$ such that $\operatorname{ann}_R(a) \cap \sigma(b)R = 0$. We claim that the degree one polynomial $axb = a\sigma(b)x + a\delta(b)$ is good. Indeed if $r \in R$ is such that $a\sigma(b)\sigma(r) = 0$ then $\sigma(b)\sigma(r) \in \sigma(b)R \cap \operatorname{ann}_R(a) = 0$. Hence $br = 0$ and $axbr = 0$.

(2) Since g is good, Lemma 3.4 shows that there is an isomorphism of S-modules $\phi : gS \rightarrow \widehat{(aR)_{\sigma^l}}$ such that $\phi(gh) = a_{\sigma^l} \cdot h$. Let us first notice that under this isomorphism good polynomials in gS correspond to good polynomials in $\widehat{(aR)_{\sigma^l}}$. Indeed, since g is good we have $\operatorname{ann}_R(a_{\sigma^l}) = \sigma^{-l}(\operatorname{ann}_R(a)) = \operatorname{ann}_R(g)$. This implies that $\deg(gh) = \deg(a_{\sigma^l} \cdot h) + l$ for any $h \in S$. In particular, thanks to the characterization (ii) in Lemma 3.4, $gh \in \widehat{M}_S$ is good if and only if $a_{\sigma^l} \cdot h \in \widehat{(aR)_{\sigma^l}}$ is good.

By the assumption, aR is nonsingular and obviously so is $(aR)_{\sigma^l}$. Part (1) above gives us a good polynomial, say $a_{\sigma^l} \cdot h$, of degree 1 in $a_{\sigma^l}S$. We can thus conclude that we have a good polynomial gh of degree $l+1$ in gS.

Since the leading coefficient of gh belongs to aR, its leading coefficient still satisfies the hypothesis of the proposition. Thus, by an easy inductive argument, the thesis of (2) follows. □

In the next lemma we will study annihilators of submodules generated by good polynomials.

Lemma 3.7. *Suppose that* $g \in \widehat{M}_S$ *is a good polynomial of degree* l *and* a *is the leading coefficient of* g. *Set* $I := \sigma^{-l}(\operatorname{ann}_R(aR))$. *Then*

1. $\operatorname{ann}_R(gS) \subseteq \operatorname{ann}_S(gS) \subseteq \operatorname{ann}_S(gR)$

2. $\mathrm{ann}_R(gR) = I$ *and* $\mathrm{ann}_S(gR) = IS$
3. $\mathrm{ann}_R(gS) = I_F$
4. $\mathrm{ann}_S(gS)$ *is equal to the largest ideal of* S *contained in the* (R, S)*-bimodule* $\mathrm{ann}_S(gR)$.
5. *If* $\delta(I_\sigma) \subseteq I_\sigma$ *then* $\mathrm{ann}_S(gS) = I_\sigma S$

Proof. (1) All these inclusions are obvious.

(2) Since g is good and $aR\sigma^l(I) = 0$, $gRI = 0$, i.e. $I \subseteq \mathrm{ann}_R(gR)$. The reverse inclusion is clear. This shows the first equality.

The proof of the second equality is similar to the one given for the implication (v) $\Rightarrow$ (vi) in Lemma 3.4.

(3) Notice that we have natural isomorphism $\widehat{gR} = gR \otimes_R S \cong gS$ of R-modules. Now the thesis is a direct consequence of Lemma 3.1(4)(a) and the statement (2).

(4) This statement is a direct consequence of Lemma 3.1(4)(b).

(5) Suppose that $\delta(I_\sigma) \subseteq I_\sigma$. Then, by Lemma 3.1(2) $I_\sigma = I_F$. Thus, by making use of (3) and (1) we obtain $I_\sigma S = \mathrm{ann}_R(gS)S \subseteq \mathrm{ann}_S(gS)$.

To see that the reverse inclusion holds, take $h = bx^n + ... \in \mathrm{ann}_S(gS)$. Then $gRx^ih = 0$ for any $i \geq 0$. Comparing the leading coefficients of both sides of the above equation we obtain $\sigma^i(b) \in \sigma^{-l}(\mathrm{ann}_R(aR)) = I$, for all $i \geq 0$, i.e. $b \in I_\sigma$. By Lemma 3.1, $I_\sigma = I_F$. Since g is a good polynomial, we get that $gRx^ib = 0$ for all $i \geq 0$. This means that $gSb = 0$. Then $gS(h - bx^n) = 0$ and $\deg(h - bx^n) < \deg h$. Now, an easy inductive argument yields that $\mathrm{ann}_S(gS) \subseteq I_\sigma S$ and the equality $\mathrm{ann}_S(gS) = I_\sigma S$ follows. □

4 Uniform Dimension and Related Topics

The main objective of this section it to show that in many natural situations we have $\mathrm{udim} M_R = \mathrm{udim} \widehat{M}_S$. In the course of doing this we will examine the lifting of various properties from the module M_R to the induced module $\widehat{M}_S$. In particular, we will be concerned with following properties : uniformity, essentiality and nonsingularity.

If M_R is an R-module then the induced S-module $\widehat{M}_S$ can also be viewed as an R-module. When confusion is possible we will, while referring to these modules structures, write $\widehat{M}_R$ or $\widehat{M}_S$, respectively.

Recall that $Z(M_R)$ denotes the singular submodule of M_R.

Lemma 4.1. *Let* N_R *be a submodule of the module* M_R*. Then:*

1. $\widehat{N}_R$ *is an essential submodule of* $\widehat{M}_R$ *if and only if* $\widehat{N}_S$ *is an essential submodule of* $\widehat{M}_S$.

2. $Z(M_R)=0$ *if and only if* $Z(\widehat{M}_R)=0$

3. If $Z(M_R)=0$*, then* $Z(\widehat{M}_S)=0$.

Proof. (1) Suppose that $\widehat{N}_S$ is an essential submodule of $\widehat{M}_S$ and let $0\neq f\in \widehat{M}\setminus\widehat{N}$. Then $0\neq\overline{f}\in\widehat{M/N}\cong\widehat{M}/\widehat{N}$, where $\overline{f}$ denotes the natural image of f in $(\widehat{M/N})_S$. By Corollary 3.5, there exists $r\in R$ such that $\overline{f}r$ is good in $(\widehat{M/N})_S$. Since $\widehat{N}_S$ is essential in $\widehat{M}_S$, there exists $g=\sum_{i=0}^{i=n}a_ix^i\in S$ such that $0\neq frg\in\widehat{N}_S$. This means that $g\in\mathrm{ann}_S(\overline{f}r)$. The element $\overline{f}r\in(\widehat{M/N})_S$ is good so, by Corollary 3.5(2), we get $\overline{f}ra_i=0$ for every $0\leq i\leq n$. This shows that for any i, $fra_i\in\widehat{N}$. Since $frg\neq 0$ we conclude that there exists a_i such that $0\neq fra_i\in\widehat{N}$. This shows that $\widehat{N}\cap fR\neq 0$, proving that $\widehat{N}_R$ is an essential R-submodule of $\widehat{M}_R$.

The reverse implication is a tautology.

(2) Notice that $Z(M_R)=Z(\widehat{M}_R)\cap M_R$. Thus in order to prove (2), it is enough to show that $Z(\widehat{M}_R)\cap M_R\neq 0$ provided $Z(\widehat{M}_R)\neq 0$. To this end, suppose that $Z(\widehat{M}_R)\neq 0$ and let $0\neq f\in Z(\widehat{M}_R)$ be a polynomial of minimal degree, say $\deg f=n$. Now, Lemma 3.4(ii) shows that f is a good polynomial. This means that the essential right ideal $\mathrm{ann}_R(f)$ of R is equal to $\sigma^{-n}(\mathrm{ann}_R(a))$, where a denotes the leading coefficient of f. This shows that $0\neq a\in Z(\widehat{M}_R)\cap M_R$.

(3) Assume that $Z(M_R)=0$ and $Z(\widehat{M}_S)\neq 0$. Let $0\neq f\in Z(\widehat{M}_S)$ be a polynomial of minimal degree, say $\deg f=n$. Since $Z(\widehat{M}_S)$ is a submodule of $\widehat{M}_S$, we may apply Lemma 3.4(iii) to see that f is a good polynomial. Let a denote the leading coefficient of f. Then, by Corollary 3.5(2), $\mathrm{ann}_S(f)=\sigma^{-n}(I)S$ where $I=\mathrm{ann}_R(a)$. Now, it is easy to conclude that $\sigma^{-n}(I)$ and hence also $I=\mathrm{ann}_R(a)$ are essential right ideals of R, i.e. $0\neq a\in Z(M_R)$. This contradiction yields the result. □

In [A1] and [A2], S.Annin studied properties of the induced S-module $\widehat{M}_S$ under the assumption that for any $a\in M_R$ the annihilator I of a in R is $(\sigma,\sigma^{-1},\delta)$-stable, i.e. $\sigma(I)=I$ and $\delta(I)\subseteq I$. We relax this hypothesis in the following definition.

Definition 4.2. We say that a module M_R satisfies the weak (σ,δ)-compatibility condition if every nonzero submodule N_R of M_R contains an element $a\neq 0$ such that $\mathrm{ann}_R(a)$ is $(\sigma,\sigma^{-1},\delta)$-stable.

In the sequel we will need the following technical observation:

Lemma 4.3. *Let* τ*,* d *denote an automorphism and* τ*-derivation of* R*, respectively. Suppose that* M_R *satisfies the weak* (τ,d)*-compatibility condition. Then:*

1. *For any good polynomial $f \in \widehat{M}_S$ there exists $r \in R$ such that fr is a good polynomial and the annihilator of its leading coefficient is (τ, τ^{-1}, d)-stable.*

2. *Suppose additionally that $\sigma\tau = \tau\sigma$, there exists a central invertible element $q \in R$ such that $\sigma d = qd\sigma$ and τ and d can be extended to $S = R[x; \sigma, \delta]$. Then the S-module $\widehat{M}_S$ satisfies the weak (τ, d)-compatibility condition.*

Proof. (1) Let $f \in \widehat{M}_S$ be a good polynomial of degree l and leading coefficient a. Since M_R satisfies the weak (τ, d)-compatibility condition, there exists $w \in R$ such that $aw \neq 0$ and $I = \mathrm{ann}_R(aw)$ is (τ, τ^{-1}, d)-stable. Then, by Lemma 3.4, the polynomial $f\sigma^{-l}(w)$ is good and I is the annihilator of its leading coefficient.

(2) Let B_S be a nonzero submodule of $\widehat{M}_S$ and $0 \neq f \in B_S$ a polynomial of minimal degree. Then f is good and using (1) above we can replace f by its multiple fr for some suitable $r \in R$ and assume that $I = \mathrm{ann}_R(a)$ is (τ, τ^{-1}, d)-stable, where $a \in M_R$ stands for the leading coefficient of f. Lemma 3.4 implies that $\mathrm{ann}_S(f) = \sigma^{-l}(I)S$, where $l = \deg f$. Now, the additional assumption yields that $\sigma^{-l}(I)S$ is (τ, τ^{-1}, d)-stable. This gives the lemma. □

We have seen in Example 3.3 that quite often the induced S-module $\widehat{M}_S$ contains good polynomials of all degrees $n \geq 0$. In the sequel we will need the following stronger property.

Definition 4.4. An S-submodule B_S of $\widehat{M}_S$ is said to be good if for any good polynomial $g \in B_S$ and any $n \geq \deg(g)$ there exists a good polynomial of degree n in gS.

Let us remark that, a priori, this notion depends on σ and δ. Notice also that if $\widehat{M}_S$ is a good module, then any S-submodule B_S of $\widehat{M}_S$ is also good.

Proposition 4.5. *The induced S-module $\widehat{M}_S$ is good if one of the following conditions is satisfied:*

1. *M_R is nonsingular.*

2. *$M_R = R_R$ and for any nonzero $a \in R$ there exists a good polynomial of degree one in aS_S.*

3. *$\delta = 0$.*

4. *M_R satisfies the weak (σ, δ)-compatibility condition.*

Proof. (1) This statement is a direct consequence of Proposition 3.6.

(2) This statement is a direct consequence of Lemma 2.1. from [Ma].

(3) This was already mentioned in Example 3.3(4)

(4) Let $f \in \widehat{M}_S$ be a good polynomial of degree l. By Lemma 4.3(1), we can pick $w \in R$ such that $g = fw$ is a good polynomial and the annihilator I of its leading coefficient a is $(\sigma, \sigma^{-1}, \delta)$-stable.

We claim that for any $k \geq 0$, gx^k is a good polynomial. To this end, notice that $\deg gx^k = l + k$ and let $r \in R$ be such that $\deg gx^k r < k + l$, i.e. $a\sigma^{l+k}(r) = 0$. This means that $r \in I$, as $\sigma(I) = I$. Since also $\delta(I) \subseteq I$, $f_i^k(r) \in I$ for all $0 \leq i \leq k$. Therefore $gx^k r = g \sum_{i=0}^{k} f_i^k(r)x^i \in gIS = 0$. This proves the claim and gives the thesis. □

In the next theorem we come to the question of preservation of essentiality and uniformity while passing from M_R to $\widehat{M}_S$. As a corollary we will compare the uniform dimension of M_R and that of $\widehat{M}_S$. The idea of the proof is similar to the one of Theorem 2.3 from the second's author paper [Ma] and, in fact, it goes back to the paper [Sh] of R.C. Shock.

Theorem 4.6. *Let N_R be an R-submodule of M_R such that $\widehat{N}_S$ is good. Then:*

1. *N_R is essential in M_R if and only if $\widehat{N}_S$ is essential in $\widehat{M}_S$.*
2. *N_R is uniform if and only if $\widehat{N}_S$ is uniform.*

Proof. (1) If T_R is a submodule of M_R such that $N_R \cap T_R = 0$ then clearly $\widehat{N}_S \cap \widehat{T}_S = 0$. This gives one implication.

Suppose that N_R is an essential submodule of M_R. Thanks to Lemma 4.1(1) it is enough to show that $\widehat{N}_R$ is essential in $\widehat{M}_R$, i.e. for any $0 \neq p \in \widehat{M}$ we must show that $\widehat{N} \cap pR \neq 0$. For doing so, we proceed by induction on $n = \deg p$. First notice that, by Corollary 3.5, we may replace p by pr for some suitable $r \in R$ and assume that p is good. If $n = 0$, then $p \in M$ and $N \cap pR \neq 0$, by the assumption.

Suppose $n > 0$. Since N_R is essential in M_R, we can pick $r \in R$ such that the leading coefficient, say a, of pr belongs to $N \setminus \{0\}$. The element $a \in N \subseteq \widehat{N}_S$ is a good polynomial and because $\widehat{N}_S$ is a good module, we can find a good polynomial $g \in aS \subseteq \widehat{N}_S$ such that $\deg(g) = \deg(pr)$. The leading coefficient of g belongs to aR, hence there exists $w \in R$ such that prw and g have the same leading coefficient. If $g = prw$, then $g \in \widehat{N} \cap pR$ and we are done. Suppose $g \neq prw$. Then $prw - g \neq 0$ and $\deg(prw - g) < \deg(p)$. By the inductive hypothesis, we can find $s \in R$ such that $0 \neq (prw - g)s =: h \in \widehat{N}$. Since prw and g are good polynomials of the same degree and with the same leading coefficients they have the same annihilator in R. This yields that $prws \neq 0$, as otherwise also $gs = 0$ and h would be equal to 0. Therefore we get $0 \neq prws = gs + h \in \widehat{N} \cap pR$.

(2) If $\widehat{N}_S$ is a uniform module than it is easy to see that N_R has to be uniform as well.

Suppose now, that N_R is a uniform R-module and assume that $\widehat{N}_S$ is not uniform. Thus there are nonzero polynomials $p, g \in \widehat{N}$ such that $pS \cap gS = 0$. Among such polynomials choose p and g such that $\deg p + \deg g$ is as small as possible. By Corollary 3.5, we may assume that both p and g are good polynomials. Suppose also that $n = \deg p \geq \deg g = m$.

Because $\widehat{N}_S$ is a good submodule of $\widehat{M}_S$, there is a nice polynomial $z \in gS$ with $\deg z = n$. Let a and b denote the leading coefficients of polynomials p and z, respectively. Since $a, b \in N$ and N_R is uniform, there exist $c, d \in R$ such that $ac = bd \neq 0$. Then the polynomial $h = p\sigma^{-n}(c) - z\sigma^{-n}(d) \in \widehat{N}_S$ is of degree smaller than n. Notice also that $h \neq 0$, as otherwise $0 \neq p\sigma^{-n}(c) = z\sigma^{-n}(d) \in pS \cap gS = 0$. Now, the choice of p and g gives $hS \cap gS \neq 0$. Therefore, there are polynomials $v_1, v_2 \in S$ such that

$$0 \neq gv_2 = hv_1 = p\sigma^{-n}(c)v_1 - z\sigma^{-n}(d)v_1 \tag{4.I}$$

Notice that, by the above equation, $p\sigma^{-n}(c)v_1 \in pS \cap gS = 0$. Since $p\sigma^{-n}(c)$ and $z\sigma^{-n}(d)$ are good polynomials of the same degree and the same leading coefficients they have the same annihilators in S. Therefore $z\sigma^{-n}(d)v_1 = 0$ as well. This means that the right hand side of the equation (4.I) is 0. The obtained contradiction shows that $\widehat{N}_S$ is a uniform submodule of $\widehat{M}_S$. □

As a direct consequence of the above theorem and Lemma 4.1 we obtain the following:

Theorem 4.7. *Suppose that the module $\widehat{R}_S = S_S$ is good. Then for any R-module M_R, the induced module $\widehat{M}_S$ is nonsingular if and only if M_R is a nonsingular module.*

Proof. By Theorem 4.6(1) applied to the R-module R_R, we know that if I is an essential right ideal of R, then IS is an essential right ideal of S. This implies that $Z(M_R) \subseteq Z(\widehat{M}_S)$. Now, the thesis is an easy consequence of Lemma 4.1(3). □

We have seen in Lemma 4.1(3), that for any nonsingular module M_R, the induced module $\widehat{M}_S$ is also nonsingular. However the fact that $\widehat{M}_S$ is nonsingular does not imply that M_R is nonsingular, as the following example shows.

Example 4.8. Let $R = K[t]/(t^p)$, where $K[t]$ denotes the polynomial ring over a field K of nonzero characteristic p. Set $S = R[x; \delta]$, where δ stands for the derivation of R induced by the standard derivation $\delta/\delta t$ of $K[t]$. Then it is known (see, e.g., [Go] Proposition 7.5), that S is isomorphic to a full $p \times p$ matrix ring over the ring $K[x]$. Therefore $\widehat{R}_S = S_S$ is a nonsingular module. One can easily check that $Z(R_R) = (\bar{t})$, where $\bar{t}$ denotes the natural image of t in R.

Let us remark that in the above example we also have $\mathrm{udim}\widehat{R}_S = p$, while $\mathrm{udim}R_R = 1$.

Theorem 4.9. *Suppose that $\widehat{M}_S$ is a good module. Then* $\mathrm{udim}\widehat{M}_S = \mathrm{udim}M_R$.

Proof. Notice that direct sums of submodules of M_R lift to direct sums of submodules of $\widehat{M}_S$, i.e. if $\sum_i K_i \subseteq M_R$ is direct, then $\sum_i K_iS \subseteq \widehat{M}_S$ is direct. Now the thesis is a consequence of Theorem 4.6. □

Let us recall that Proposition 4.5 describes situations when $\widehat{M}_S$ is a good module, so the above theorem can be applied.

The classical result of R.C. Shock is a special case of the above theorem, when $M_R = R_R$ and S is the usual polynomial ring $R[x]$.

The main result of [Ma] states that the equality $\mathrm{udim}\widehat{M}_S = \mathrm{udim}M_R$ holds when $M_R = R_R$ satisfies (2) of Proposition 4.5. On the other hand, it is well-known (see for example [Si], [Ma]) that in general the equality $\mathrm{udim}\widehat{M}_S = \mathrm{udim}M_R$ does not hold even in the case $M_R = R_R$ and $S = R[x;\delta]$ is a differential polynomial ring.

The following is an application of Theorem 4.9 and Lemma 4.1:

Theorem 4.10. *Let $T = R[x_1;\sigma_1,\delta_1]\ldots[x_n;\sigma_n.\delta_n]$ be an iterated Ore extension. Suppose that M_R is a nonsingular R-module. Then:*

1. *$M \otimes_R T$ is a nonsingular right T-module.*
2. $\mathrm{udim}(M \otimes_R T)_T = \mathrm{udim}M_R$.

Proof. Set $T_0 = R$, $M_0 = M_R$ and define $T_k = R[x_1;\sigma_1,\delta_1]\ldots[x_k;\sigma_k,\delta_k]$ for $1 \leq k \leq n$. Let M_k denotes the right T_k-module $M_{k-1} \otimes_{T_{k-1}} T_k$, i.e. $M_k = (\widehat{M_{k-1}})_{T_k}$.

An easy inductive argument shows that the T_k-modules M_k and $M \otimes_R T_k$ are isomorphic for all $0 \leq k \leq n$. In particular, $T_n = T$ and the T-modules M_n and $M \otimes_R T$ are isomorphic.

(1) Applying Lemma 4.1(4) k-times, we see that the T_k-module M_k is nonsingular, for any $1 \leq k \leq n$. In particular $(M_n)_{T_n}$ is a nonsingular module over $T_n = T$. Since T-modules $(M \otimes_R T)_T$ and $(M_n)_T$ are isomorphic, the thesis follows.

(2) By the above, M_k is a nonsingular T_k-module for $0 \leq k \leq n$. Therefore, Theorem 4.9 together with Proposition 4.5(1) imply that $\mathrm{udim}M_R = \mathrm{udim}(M_k)_{T_k}$ for $k \in \{0,\ldots,n\}$. This yields the thesis. □

In the case the iterated Ore extension T from the above theorem is an iterated differential operator ring, i.e. all automorphisms σ_i are identities, the second part of the above theorem was proved by A.D. Bell and K.R. Goodearl in [BeG].

Notice that quantum algebras very often can be presented as iterated Ore extensions so Theorem 4.10 can be applied in such cases.

Lemma 4.3(2) enables us to use an inductive argument for showing that udimM_R = udim$(M \otimes_R T)_T$ for certain iterated skew polynomial extensions T of R provided M_R satisfies some weak compatibility conditions. Instead of presenting a general theorem, we present a concrete application. Recall that many rings related to quantum algebras can be presented as ambiskew polynomial rings. Such extensions were introduced by D.A. Jordan (see [BJ] for references). The construction is as follows: let σ be an automorphism of R, v a central element and p a central invertible element of R. Then the ambiskew polynomial ring $A = A(R, \sigma, v, p)$ is the iterated skew polynomial ring $R[x; \sigma][y; \sigma^{-1}, \delta]$, where the automorphism σ of R is extended to $R[x; \sigma]$ by setting $\sigma(x) = p^{-1}x$ and δ is the σ-derivation of $R[x; \sigma]$ given by $\delta(R) = 0$ and $\delta(x) = v$. Keeping this notation we have:

Proposition 4.11. *Suppose that the R-module M_R satisfies the weak $(\sigma, 0)$-compatibility condition. Then* udimM_R = udim$(M \otimes_R A)_A$*, where $A = A(R, \sigma, v, p)$ is the ambiskew polynomial extension of R.*

Proof. Notice that the σ^{-1}-derivation δ of $R[x; \sigma]$ is the extension of the σ^{-1}-derivation 0 of R. Thus, by Lemma 4.3(2), the $R[x, \sigma]$-module $M \otimes_R R[x, \sigma]$ satisfies the weak (σ^{-1}, δ)-compatibility condition. Proposition 4.5 and Theorem 4.9 applied twice, give udimM_R = udim$(M \otimes_R R[x, \sigma])_{R[x,\sigma]}$ = udim$(M \otimes_R R[x, \sigma]) \otimes_{R[x,\sigma]} A)_A$ = udim$(M \otimes_R A)_A$. □

We close this section with an observation concerning divisibility of the induced module. This observation is independent from the main stream of the paper. Let us begin with the following general easy result which is probably part of folklore.

Lemma 4.12. *If $M_R = \bigcup_{i \geq 0} M_i$ is a filtered R-module such that its associated graded R-module $Gr(M) = \bigoplus_{i \geq 0} M_i/M_{i-1}$ (with $M_{-1} = 0$) is divisible then M_R is also divisible.*

Proof. Suppose that the associated graded R-module $Gr(M) = \bigoplus_{i \geq 0} N_i$, where $N_i = M_i/M_{i-1}$, is divisible. In particular, all modules N_i are divisible.

Let $r \in R$ and $f \in M$ be such that $\mathrm{ann}_R(r) \subseteq \mathrm{ann}_R(f)$. We must prove that $f \in Mr$. We will proceed by induction on n such that $f \in M_n \setminus M_{n-1}$. The case when $n = 0$, i.e. $f \in M_0 = N_0$ is trivial. Suppose that $n \geq 1$ and $f \in M_n \setminus M_{n-1}$. Let $\overline{f}$ denotes the natural image of f in N_n. Then $\mathrm{ann}_R(r) \subseteq \mathrm{ann}_R(f) \subseteq \mathrm{ann}_R(\overline{f})$. Because N_n is divisible, $\overline{f} \in N_n r$. Thus there exists $g \in M_n$ such that $\overline{f} = \overline{g}r$, i.e. $f - gr \in M_{n-1}$. Now, for any $s \in \mathrm{ann}_R(r)$ we have $fs = 0$ and so $(f - gr)s = 0$. This means that $\mathrm{ann}_R(r) \subseteq \mathrm{ann}_R(f - gr)$

and the induction hypothesis yields $f - gr \in Mr$ and shows that $f \in Mr$, as required. □

Corollary 4.13. *Suppose that M_R is a divisible module. Then the induced module $\widehat{M}_R$ is also divisible.*

Proof. $\widehat{M}$, treated as the right R-module, has a natural filtration given by the degree in x. It is easy to check that the associated graded module is isomorphic to $\bigoplus_{i\geq 0} M_{\sigma^i}$. Since M_R is divisible, the σ^i-twisted R-modules M_{σ^i} are also divisible and the above lemma gives the thesis. □

5 Associated Primes

In this section we will investigate the relationships between the associated prime ideals of the R-module M_R and that of the induced S-module $\widehat{M}_S$.

Recall that an R-module N_R is prime if $\mathrm{ann}_R(N) = \mathrm{ann}_R(N')$ for any nonzero submodule N'_R of N_R. If N_R is prime, $\mathrm{ann}_R(N)$ is necessarily a prime ideal of R. For a module M_R, a prime ideal of R which is the annihilator of some prime submodule of M is called an associated prime of the module M. The set of all associated primes of M_R is denoted by $\mathrm{Ass}(M_R)$.

Clearly, if we do not impose some extra conditions either on the module M_R or on the ring R then it may happen that $\mathrm{Ass}(M_R)$ is empty. We will see in Example 5.8 that in such situation we can also have $\mathrm{Ass}(\widehat{M}_S) \neq \emptyset$ even in the case $S = R[x;\sigma]$ is a skew polynomial ring of automorphism type. This situation differs from the classical case when $S = R[x]$ is the usual polynomial ring. It is known (Cf. [A1], [Fa]) that always $\mathrm{Ass}(\widehat{M}_{R[x]}) = \{PR[x] \mid P \in \mathrm{Ass}(M_R)\}$.

In general, it may also happen that $\mathrm{Ass}(M_R)$ is not empty but $\mathrm{Ass}(N_R) = \emptyset$ for some nonzero submodule N_R of M_R. Indeed, if the module N_R is such that $\mathrm{Ass}(N_R) = \emptyset$ and B_R is a prime module with $\mathrm{Ass}(B_R) = \{P\}$, then $\mathrm{Ass}(N_R \oplus B_R) = \{P\}$.

Due to the above, we will work with R-modules M_R such that the set $\mathrm{Ass}(N_R)$ is not empty for all nonzero submodules N_R of M_R. This statement is equivalent to the following:

Definition 5.1. We will say that the module M_R has enough prime submodules if any nonzero submodule of M_R contains a prime submodule.

When the ring R satisfies ACC on annihilators of submodules of M_R, then clearly M_R has enough prime submodules. This holds, in particular, for any R-module M_R if the ring R satisfies ACC on two-sided ideals.

The class of R-modules having enough prime submodules is obviously closed with respect to taking submodules. However, as we will see in Example 5.9, it is not homomorphically closed.

In the following proposition we show that the class of R-modules having enough prime submodules is closed with respect to direct sums.

Proposition 5.2. *Let $\{M_i\}_{i\in\mathcal{S}}$ be the set of R-modules having enough prime submodules. Then $\bigoplus_{i\in\mathcal{S}} M_i$ has enough prime submodules.*

Proof. First notice that it is enough to prove the proposition only in the case $\mathcal{S}=\{1,2\}$. Indeed, let $0\neq a\in\bigoplus_{i\in\mathcal{S}} M_i$. The element a has only finite number of nonzero entries, thus for showing that aR contains a prime submodule, we may assume that the set $\mathcal{S}$ is finite. Now an easy inductive argument reduces the situation to the case $\mathcal{S}$ consists of two elements.

Suppose $\mathcal{S}=\{1,2\}$ and let $(a_1,a_2)\in M_1\oplus M_2$. Since M_i, for $i=1,2$, contains enough prime submodules, replacing (a_1,a_2) by $(a_1,a_2)r$ for some suitable $r\in R$ we may reduce the situation further to the case the M_i's are prime R-modules. Let $\mathrm{Ass}(M_i)=\{P_i\}$ for $i=1,2$.

Case 1. Suppose $P_1=P_2=P$. Then it is easy to see that the annihilator of any nonzero submodule of $M_1\oplus M_2$ is equal to P, so $M_1\oplus M_2$ is a prime module.

Case 2. Suppose $P_1\neq P_2$, say there is $r\in P_1\setminus P_2$. Then there exists $s\in R$ such that $a_2sr\neq 0$ and $(a_1,a_2)srR=(0,a_2sr)R$ is a prime submodule of $(a_1,a_2)R$. □

Notice also that if the R-module M_R is uniform then M_R has enough prime submodules if and only if $\mathrm{Ass}(M_R)\neq\emptyset$. One can also easily check that if $N_R\subseteq M_R$ is an essential extension of modules then N_R has enough prime submodules if and only if M_R has enough prime submodules. These remarks together with the above proposition give us the following:

Corollary 5.3. *Suppose that every submodule of the R-module M_R contains a uniform submodule (this holds, in particular, when* $\mathrm{udim} M_R$ *is finite), then M_R has enough prime submodules if and only if* $\mathrm{Ass}(N_R)$ *is not empty for every uniform submodule N_R of M_R*

Now let us state two useful lemmas relating prime submodules and good polynomials.

Lemma 5.4. *Suppose that M_R has enough prime submodules. Then any nonzero submodule B_S of $\widehat{M_S}$ contains a good polynomial g with leading coefficient a such that aR is a prime submodule of M_R.*

Proof. Let B_S be a nonzero submodule of $\widehat{M_S}$ and let $g'=bx^n+\ldots\in B_S$ be a nonzero polynomial of minimal degree. Lemma 3.4 shows that g' is good and since M_R contains enough prime submodules, bR contains a nonzero prime submodule aR where $a=br\neq 0$ for some $r\in R$. It then follows that the polynomial $g=g'\sigma^{-n}(r)\in B_S$ is good with leading coefficient a such that aR is a prime submodule of M_R as required. □

The above lemma shows that if M_R has enough prime submodules, then any associated prime ideal of $\widehat{M}_S$ is the annihilator of a submodule of the form gS where g is a good polynomial such that its leading coefficient generates a prime submodule of M_R. The next lemma offers a characterization when gS is a prime submodule of $\widehat{M}_S$. Recall that for an ideal I of R, I_F denotes the largest ideal contained in I which is invariant under the action of all maps f^i_j.

Lemma 5.5. *Let $g \in \widehat{M}_S$ be a good polynomial of degree l and leading coefficient a. Then:*

1. *gS is a prime submodule of $\widehat{M}_S$ if and only $a_{\sigma^l} \cdot S$ is a prime submodule of $(\widehat{M_{\sigma^l}})_S$.*

2. *gR is a prime submodule of $\widehat{M}_R$ if and only if $a_{\sigma^l} \cdot R$ is a prime submodule of $(M_{\sigma^l})_R$.*

Suppose additionally that for all cyclic submodules A and B of $(M_{\sigma^l})_R$ the inclusion $(\mathrm{ann}_R(A))_F \subseteq (\mathrm{ann}_R(B))_F$ implies $\mathrm{ann}_R(A) \subseteq \mathrm{ann}_R(B)$. Then:

(3) *If gS is a prime submodule of $\widehat{M}_S$ then gR is a prime submodule of $\widehat{M}_R$.*

Proof. The statement (1) is a direct consequence of Lemma 3.4(vii).

(2) Since $g \in \widehat{M}_S$ is good, $\mathrm{ann}_R(g) = \mathrm{ann}_R(a_{\sigma^l})$. Using this equality, similarly as in the proof of Lemma 3.4(vii), one can check that gR and $a_{\sigma^l} \cdot R$ are isomorphic as right R-modules. This gives (2).

(3) Suppose that the additional assumption holds and gS is a prime submodule of $\widehat{M}_S$. Then, by (1), $a_{\sigma^l} \cdot S$ is a prime submodule of $(\widehat{M_{\sigma^l}})_S$. Thus, for any $b \in R$ such that $a_{\sigma^l} \cdot b \neq 0$, we have $\mathrm{ann}_S(a_{\sigma^l} \cdot bS) = \mathrm{ann}_S(a_{\sigma^l} \cdot S)$. Hence, using Lemma 3.7(3), we obtain $(\mathrm{ann}_R(a_{\sigma^l} \cdot bR))_F = (\mathrm{ann}_R(a_{\sigma^l} \cdot R))_F$ and our additional hypothesis gives $\mathrm{ann}_R(a_{\sigma^l} \cdot bR) = \mathrm{ann}_R(a_{\sigma^l} \cdot R)$. This, in turn, can be translated to $a_{\sigma^l} \cdot R$ is a prime submodule of $(\widehat{M_{\sigma^l}})_R$ □

Lemma 5.6. *Suppose that $f \in \widehat{M}_S$ is a good polynomial of degree n and leading coefficient a. Set $P = \mathrm{ann}_R(aR)$. Then:*

1. *If $\sigma(P) = P$ and $\delta(P) \subseteq P$, then $\mathrm{ann}_S(fS) = PS$*

2. *If δ is a q-quantized σ-derivation and $\sigma(P) = P$, then $\mathrm{ann}_S(fS) = P_\delta S$.*

3. *If δ is a q-quantized σ-derivation, aR is a prime submodule of M_R and fS contains good polynomials of any degree greater than n, then $\mathrm{ann}_S(fS) = \sigma^{-n}(P_{\sigma,\delta})S$.*

Proof. (1) Since $\sigma(P) = P$, Lemma 3.7 (5) gives the desired conclusion.

(2) Suppose that δ is q-quantized and $\sigma(P) = P$. Then, for any $x \in P_\delta$ and $i \geq 0$, we have $\sigma(\delta^i(x)) \in \sigma(P) = P$ so that $\delta^i(\sigma(x)) = q^i\sigma(\delta^i(x)) \in P$. This shows that $\sigma(P_\delta) \subseteq P_\delta$. Hence $P_\delta S$ is a two-sided ideal of S and $P_\delta S \subseteq \mathrm{ann}_S(fS)$ easily follows.

Now we claim that the reverse inclusion holds. Since δ is q-quantized, $f^i_j = \binom{i}{j}_q \sigma^j\delta^{i-j}$. Therefore, as $\sigma(P) \subseteq P$, for any $b \in R$ and any $k \geq 1$ the following property holds:

(5.I) If $\{b, \delta(b) \cdots, \delta^{k-1}(b)\} \subseteq P$ then $\{f^k_j(b) \mid 1 \leq j \leq k\} \subseteq P$.

Let $g = \sum_{s=0}^{l} b_s x^s \in \mathrm{ann}_S(fS)$. We show, by induction on k, that for any $k \geq 0$ and $0 \leq s \leq l$, $\delta^k(b_s) \in P$. Since f is good and $\sigma(P) = P$, Lemma 3.4 yields that $\mathrm{ann}_S(fR) = \sigma^{-n}(P)S = PS$, where $n = \deg f$. This gives the above statement for $k = 0$.

Assume that $k \geq 1$ and $\{b_s, \delta(b_s), \cdots, \delta^{k-1}(b_s)\} \subseteq P$ for any $0 \leq s \leq l$. Then, by (5.I), $\{f^k_j(b_s) \mid 1 \leq j \leq k\} \subseteq P$ for all $0 \leq s \leq l$. Now, since $fRx^kg = 0$ and $\mathrm{ann}_S(fR) = PS$ we get:

$$0 = frx^k(\sum_{s=0}^{l} b_s x^s) = \sum_{s=0}^{l}\sum_{j=0}^{k} frf^k_j(b_s)x^{s+j} = \sum_{s=0}^{l} frf^k_0(b_s)x^s$$

for any $r \in R$. This means that $fR\sum_{s=0}^{l}\delta^k(b_s)x^s = 0$ and shows that $\delta^k(b_s) \in P$ for any $k \geq 0$ and $0 \leq s \leq l$. This implies that $g \in P_\delta S$, as desired.

(3) Suppose that all assumptions of (3) are satisfied. In particular, for any $i \geq n = \deg f$ there exists a good polynomial $f_i \in fS$ of degree i. Let α_i denote the leading coefficient of f_i. Obviously $\alpha_i \in aR$.

By assumption, aR is a prime submodule of M_R, so $\mathrm{ann}_R(\alpha_i R) = P$ for all $i \geq n$. The elements f_i are good so, by Lemma 3.4, $\mathrm{ann}_S(f_iR) = \sigma^{-i}(P)S$ for all $i \geq n$. Let E denote the R-submodule of $\widehat{M}_R$ defined by $E = \sum_{i\geq n} f_iR$. Then $E \subseteq fS$ and

(5.II) $$\mathrm{ann}_S(fS) \subseteq \mathrm{ann}_S(E) = \bigcap_{i\geq n}\sigma^{-i}(P)S = \sigma^{-n}(P_\sigma)S.$$

Let $p = \sum_{s=0}^{l} p_s x^s \in \mathrm{ann}_S(fS)$. We will prove, by induction on k, that for all $k \geq 0$ and $0 \leq s \leq l$ we have $\delta^k(p_s) \in \sigma^{-n}(P_\sigma)$. The case when $k = 0$ is given by (5.II).

Suppose that $k \geq 1$ and $\delta^i(p_s) \in \sigma^{-n}(P_\sigma)$ for $i < k$ and $0 \leq s \leq l$. Since $\sigma(\sigma^{-n}(P_\sigma)) \subseteq \sigma^{-n}(P_\sigma)$, we can apply (5.I) to $\sigma^{-n}(P_\sigma)$ obtaining $\{f^k_j(p_s) \mid 1 \leq j \leq k\} \subseteq \sigma^{-n}(P_\sigma)$ for $0 \leq s \leq l$. Therefore, as $Ex^k \subseteq fS$, we get:

$$0 = Ex^kp = \sum_{s=0}^{l}\sum_{j=0}^{k} Ef^k_j(p_s)x^{j+s} = E\sum_{s=0}^{l} f^k_0(p_s)x^s.$$

By (5.II), $\mathrm{ann}_S(E) = \sigma^{-n}(P_\sigma)S$ and the above equation implies that $\delta^k(p_s) \in \sigma^{-n}(P_\sigma)$ for all $0 \le s \le l$ and $k \ge 0$, as desired. This means that $\mathrm{ann}_S(fS) \subseteq (\sigma^{-n}(P_\sigma))_\delta S$. It is easy to check, using the equality $\delta\sigma = q\sigma\delta$, that $(\sigma^{-n}(P_\sigma))_\delta = \sigma^{-n}(P_{\sigma,\delta})$. This shows that $\mathrm{ann}_S(fS) \subseteq \sigma^{-n}(P_{\sigma,\delta})S$.

In order to prove the reverse inclusion let $b \in \sigma^{-n}(P_{\sigma,\delta})$. Then $\sigma^n(\sigma^j\delta^k(b)) \in P$ for any $j, k \ge 0$. Hence, as f is a good polynomial of degree n and leading coefficient a, $fR\sigma^j\delta^k(b) = 0$. Now, it is easy to check that for any $i \ge 0$, $fRx^ib = 0$. From this we conclude easily that $\sigma^{-n}(P_{\sigma,\delta})S \subseteq \mathrm{ann}_S(fS)$, as desired. □

With the help of the above lemma we easily get the following:

Theorem 5.7. *Suppose that M_R contains enough prime submodules and let $Q \in Ass\widehat{M}_S$. Then:*

1. *If for every $P \in \mathrm{Ass}(M_R)$ $\sigma(P) = P$ and $\delta(P) \subseteq P$, then $Q = PS$ for some $P \in \mathrm{Ass}(M_R)$.*
2. *If δ is q-quantized and $\sigma(P) = P$ for all $P \in \mathrm{Ass}(M_R)$, then $Q = P_\delta S$ for some $P \in \mathrm{Ass}(M_R)$.*
3. *If δ is q-quantized and $\widehat{M}_S$ is a good module, then $Q = P_{\sigma,\delta}S$ for some $P \in \mathrm{Ass}(M_R)$ and $\sigma(P_{\sigma,\delta}) = P_{\sigma,\delta}$*

Proof. Let B_S be a prime submodule of $\widehat{M}_S$ such that $\mathrm{ann}_S(B) = Q$. Then, by Lemma 5.4, we can pick a good polynomial $f \in B_S$ such that aR is prime submodule of M_R, where a denotes the leading coefficient of f. Let $P = \mathrm{ann}_R(aR) \in \mathrm{Ass}(M_R)$.

Since B_S is a prime module, its submodule fS is also prime and $\mathrm{ann}_S(fS) = \mathrm{ann}_S(B) = Q$. Therefore, replacing B by fS, we may assume that $B_S = fS$.

Now, the statements (1) and (2) of the theorem are direct consequences of Lemma 5.6 (1) and (2), respectively.

(3) Suppose that the assumptions of (3) are satisfied. Let $n = \deg f$. Since the module $\widehat{M}_S$ is good, $B = fS$ contains a good polynomial g of degree $n+1$. Notice that $\mathrm{ann}_S(gS) = Q$, because fS is prime module. On the other hand, by Lemma 5.6(3), we have $\mathrm{ann}_S(gS) = \sigma^{-n-1}(P_{\sigma,\delta})S$ and $\mathrm{ann}_S(fS) = \sigma^{-n}(P_{\sigma,\delta})S$. This implies that $\sigma(P_{\sigma,\delta}) = P_{\sigma,\delta}$ and, consequently, $Q = \mathrm{ann}_S(fS) = P_{\sigma,\delta}S$, as required. □

In the following example we present a ring R with an automorphism σ and an R-module M_R such that $\mathrm{Ass}(M_R) = \emptyset$ but $\mathrm{Ass}(\widehat{M})_S$ is not empty, where $S = R[t;\sigma]$ is a skew polynomial ring of automorphism type. Therefore the assumption that M_R contains enough prime submodules is necessary in the above theorem.

Example 5.8. Let K be a field and M a K-linear space with basis $\{v_i\}_{i\in\mathbb{Z}}$. For $k \in \mathbb{Z}$ define $\varphi_k \in \mathrm{End}_K(V)$ by setting

$$\varphi_k(v_i) = \begin{cases} v_{i+1} & \text{if } i \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Then M has an $R = K\langle X\rangle$ module structure, where $K\langle X\rangle$ denotes the free algebra over K on the set $X = \{x_i \mid i \in \mathbb{Z}\}$, given by $v_i x_k = \varphi_k(v_i)$.

Let $0 \neq m \in M_R$. Then m can be written in the form $\sum_{i\geq n} \lambda_i v_i$, where $n \in \mathbb{Z}$, $\lambda_i \in K$ are equal to zero for all but finite number of indexes and $\lambda_n \neq 0$. Notice that for such m, $mx_n = \lambda_n v_{n+1} \neq 0$. Thus the submodule mR contains a decreasing sequence of submodules $M_l = \mathrm{span}_K\{v_i \mid i \geq l\}$, $l = n+1, n+2, \ldots$, with strictly increasing sequence of annihilators. This shows that $\mathrm{Ass}(M_R) = \emptyset$.

Let σ denote the right shifting K-automorphism of R, i.e. $\sigma(x_k) = x_{k+1}$ for any $k \in \mathbb{Z}$ and define $S = R[t, \sigma]$. Then $\widehat{M_S}$ is a prime S-module with $\mathrm{Ass}(\widehat{M_S}) = \{0\}$. Indeed, for any nonzero elements $m \in M$ and $w \in R$ one can choose a big enough $k \in \mathbb{N}$ such that $mt^k w \neq 0$. This easily yields that every S-submodule of $\widehat{M_S}$ is faithful.

In the following example we present a module N_S with a submodule A such that every submodule of N_S is faithful but the factor module $(N/A)_S$ has no enough prime submodules, i.e. the class of modules having enough prime submodules is not homomorphically closed.

Example 5.9. Let $R = K\langle X\rangle$, $S = R[t;\sigma]$, M_R and $N_S = \widehat{M_S}$ be as in Example 5.8. We have seen that every nonzero submodule of N_S is faithful, so M_S has enough prime submodules.

We claim, that $(N/Nt^2S)_S$ has no enough prime submodules. To this end, remark that N/Nt^2S as an S-module is isomorphic to $M + Mt$ where the action of t on $M + Mt$ is given by $(m_0 + m_1)t = m_0 t$. Mt is an S-submodule of $(N/Nt^2S)_S$ which, as an R-module is isomorphic to the σ-twisted module M_σ of the R-module M_R. As we have seen in Example 5.8, M_R has no prime submodules, so Mt_R also has no prime submodules. However every R-submodule B of Mt is also an S-submodule and $\mathrm{ann}_S(Mt) = \mathrm{ann}_R(Mt) + St$. This implies that the S-submodule Mt of $(N/Nt^2S)_S$ has no prime submodules.

We will now consider building associated primes of $\widehat{M_S}$ starting with the ones of M_R.

Theorem 5.10. *Let M_R be a prime module with $P = \mathrm{ann}_R(M)$. Then:*

1. *Suppose that $\sigma(P) = P$ and $\delta(P) \subseteq P$. Then the induced module $\widehat{M_S}$ is prime with the associated prime ideal equal to $PS = Q$.*

2. *Suppose that δ is a q-quantized σ-derivation and $\sigma(P)=P$. Then $\widehat{M}_S$ is prime with the associated prime ideal equal to $P_\delta S=Q$.*

3. *Suppose that δ is a q-quantized σ-derivation and the module $\widehat{M}_S$ is good. Then $\widehat{M}_S$ is a prime module if and only if $\sigma(P_{\sigma,\delta})=P_{\sigma,\delta}$. Moreover, if $\widehat{M}_S$ is prime, then its associated prime ideal is equal to $P_{\sigma,\delta}S=Q$.*

Proof. One can easily check that in all cases $\widehat{M}_S Q=0$, i.e. $Q\subseteq \mathrm{ann}_S(\widehat{M})$. Let B_S be a nonzero submodule of $\widehat{M}_S$ and $0\neq f\in B_S$ be a polynomial of minimal degree among elements from B_S. Then, by Lemma 3.4, f is a good polynomial. Let a denote the leading coefficient of f. Since M_R is prime, aR is a prime submodule of M_R with $\mathrm{ann}_R(aR)=P$.

Assume now, that while considering case (3) of the theorem, $\sigma(P_{\sigma,\delta})=P_{\sigma,\delta}$. Then in all cases, using Lemma 5.6, we get $\mathrm{ann}_S(fS)=Q$. Hence we have:

$$Q\subseteq \mathrm{ann}_S(\widehat{M})\subseteq \mathrm{ann}_S(B)\subseteq \mathrm{ann}_S(fS)=Q.$$

This shows that $\mathrm{ann}_S(B)=Q$ for any nonzero submodule B_S of $\widehat{M}_S$, i.e. $\widehat{M}_S$ is a prime module with the associated prime ideal equal to Q.

The above gives the proof of (1), (2) and one implication of (3). Thus, to finish the proof of the theorem, it is enough to remark that if $\widehat{M}_S$ is a prime module, then $\sigma(P_{\sigma,\delta})=P_{\sigma,\delta}$. However, when $\widehat{M}_S$ is prime, then fS is also a prime submodule with the same annihilator and Theorem 5.7(3) yields $\sigma(P_{\sigma,\delta})=P_{\sigma,\delta}$. □

The assumptions of Theorems 5.7 and 5.10 are very technical, so let us record a few direct consequences of those theorems. Recall that the global assumption that the module M_R has enough prime submodules holds always when the ring R satisfies the ACC on ideals. Parts (1) and (2) of the above mentioned theorems give the following:

Corollary 5.11. *Let M_R be a module having enough prime submodules. Then:*

1. *Suppose that $\sigma(P)=P$ and $\delta(P)\subseteq P$ for any $P\in\mathrm{Ass}(M_R)$. Then there is one-to-one correspondence between $\mathrm{Ass}(M_R)$ and $\mathrm{Ass}(\widehat{M}_S)$ given by extension and contraction to R.*

2. *Suppose that δ is a q-quantized σ-derivation and $\sigma(P)=P$ for all $P\in\mathrm{Ass}(M_R)$. Then the map $\Psi\colon \mathrm{Ass}(M_R)\to\mathrm{Ass}(\widehat{M}_S)$ given by $\Psi(P)=P_\delta S$ is onto.*

The analogue of (1) in the above corollary was proved by S. Annin in [A2] in case σ is a monomorphism under a stronger assumption that for any element $m\in M_R$, $\sigma(\mathrm{ann}_R(m))=\mathrm{ann}_R(m)$ and $\delta(\mathrm{ann}_R(m))\subseteq\mathrm{ann}_R(m)$.

A usual derivation d of the ring R is an id_R-derivation. Therefore the statement (2) of the above corollary give us:

Corollary 5.12. *Let M_R be a module having enough prime submodules and $S = R[t;d]$ be a differential polynomial ring. Then the map* $\Psi\colon \mathrm{Ass}(M_R) \to \mathrm{Ass}(\widehat{M}_S)$ *given by* $\Psi(P) = P_d S$ *is onto.*

This corollary can also be seen as a counterpart of Theorem 1.3 from [Ši]. It is shown there that for an ideal I of the finite dimensional Lie algebra L over a field of characteristic 0 and an associated prime P of a finitely generated module V over the enveloping algebra $U(I)$, $PU(L)$ is an associated prime of the induced module $V \otimes_{U(I)} U(L)$.

Recall that in Proposition 4.5 we described situations when the induced module is good.

Corollary 5.13. *Suppose that δ is a q-quantized σ-derivation. Let M_R be a nonsingular R-module over a noetherian ring R. Then the map* $\Psi\colon \mathrm{Ass}(M_R) \to \mathrm{Ass}(\widehat{M}_S)$ *given by* $\Psi(P) = P_{\sigma,\delta} S$ *is onto.*

Proof. If $\sigma(I) \subseteq I$ for some ideal I of a noetherian ring, then $\sigma(I) = I$. Now the thesis is a direct consequence of Proposition 4.5 and Theorem 5.7. □

Suppose that R and M_R are as in the above corollary. Then $S = R[t;\sigma,\delta]$ is also noetherian and, by Lemma 4.1(4), the induced module $\widehat{M}_S$ is nonsingular. This means that when $T = R[t_1;\sigma_1,\delta_1]\ldots[t_n,\sigma_n,\delta_n]$ is an iterated Ore extension such that all δ_i's are quantized skew derivations, then Corollary 5.13 induces an appropriate map $\Phi\colon \mathrm{Ass}(M_R) \to \mathrm{Ass}((M \otimes_R T))_T$ which is onto.

Let us record one more application.

Theorem 5.14. *Let $A = A(R,\sigma,v,p)$ be an ambiskew polynomial extension of R. Suppose that the module M_R has enough prime submodules and satisfies the weak $(\sigma,0)$-compatibility condition. Then* $\mathrm{Ass}((M \otimes_R A)_A) = \{PA \mid P \in \mathrm{Ass}(M_R)\}$.

Proof. We begin the proof with two general observations. Let τ and d denote an automorphism and an τ-derivation of R, respectively.

First remark that if M_R satisfies the weak (τ,d)-compatibility condition, then $\tau(P) = P$ and $d(P) \subseteq P$ for any $P \in \mathrm{Ass}(M_R)$. Indeed, let N_R be a prime submodule of M_R with annihilator P. Then there exists $0 \neq a \in N_R$ such that $\mathrm{ann}_R(a)$ is (τ,τ^{-1},d)-stable. Then it is easy to show that $\tau(I) = I$ and $d(I) \subseteq I$, where $I = \mathrm{ann}_R(aR)$. Since N_R is a prime submodule, $I = P$.

Secondly, observe that if the module M_R has enough prime submodules then the induced S-module $\widehat{M}_S$, where $S = R[x;\tau,d]$, also has enough prime submodules. Indeed, if B_S is a nonzero submodule of $\widehat{M}_S$ then, by Lemma 5.4, there exists a good polynomial $g \in B_S$ with leading coefficient $a \in M_R$ such that aR is a prime submodule of M_R. Let P denote the annihilator of aR in R. By the first observation $\tau(P) = P$ and $d(P) \subseteq P$. Now consider the submodule $a_{\tau^n} \cdot R$ of the twisted R-module M_{τ^n}. Then $a_{\tau^n} \cdot R$ is also a prime module with the associated prime ideal P. Therefore, by Theorem 5.10(1), $(\widehat{a_{\tau^n} \cdot R})_S$ is a prime module and PS is its associated prime ideal. However, by Lemma 3.4, $(\widehat{a_{\tau^n} \cdot R})_S$ is isomorphic to the S-module gS. This shows that gS is prime and finishes the proof of the second observation.

Now we are ready to finish the proof. Recall that $A = R[x;\sigma][y;\sigma^{-1},\delta]$. By assumption, M_R satisfies $(\sigma, 0)$-compatibility condition. Thus, the first observation together with Corollary 5.11(1) yield that $\mathrm{Ass}((M \otimes_R R[x;\sigma])_{R[x;\sigma]}) = \{PR[x;\sigma] \mid P \in \mathrm{Ass}(M_R)\}$. Recall also that the σ^{-1}-derivation δ of $R[x;\sigma]$ is the extension of the σ^{-1}-derivation 0 of R. Thus, Lemma 4.3(2) implies that the $R[x,\sigma]$-module $M \otimes_R R[x,\sigma]$ satisfies the weak (σ^{-1}, δ)-compatibility condition. By the second observation we know also that this $R[x;\sigma]$-module has enough prime submodules. Therefore, using the first observation together with Corollary 5.11(1) again, we obtain $\mathrm{Ass}((M \otimes_R R[x,\sigma] \otimes_{R[x,\sigma]} A)_A) = \{QA \mid Q \in \mathrm{Ass}((M \otimes_R R[x;\sigma])_{R[x;\sigma]})\}$. Hence $\mathrm{Ass}((M \otimes_R A)_A) = \{PA \mid P \in \mathrm{Ass}(M_R)\}$ follows. □

The following example will show that the hypotheses in Theorem 5.10 are not superfluous. Namely, we will construct a ring R with an automorphism σ and a prime R-module M_R such that $\mathrm{Ass}(M_R) = \{P\}$ and $\sigma(P) \subset P$ but nevertheless the induced module $\widehat{M}_S$ has no associated primes, where $S = R[t;\sigma]$ is a skew polynomial ring of automorphism type.

Example 5.15. Let $X = \{x_i \mid i \in \mathbb{Z}\}$ be the set of commuting indeterminates and $R = K[X]$ denote the polynomial ring on X over a field K. Let $P = (x_i \mid i \geq 0)$ denote the ideal of R generated by the set $\{x_i \mid i \geq 0\}$ and set $M_R = R/P$. Then M_R is a prime R-module and P is the associated prime of M_R. For any $k \geq 1$, let y_k denote the canonical image of x_{-k} in M_R. Then

$$y_k x_l = \begin{cases} 0 & \text{if } l \geq 0, \\ y_k y_{-l} & \text{otherwise.} \end{cases}$$

Now consider the K-automorphism σ of R defined by $\sigma(x_i) = x_{i+1}$ for all $i \in \mathbb{Z}$. Then clearly $\sigma(P) \subseteq P$.

Let $S = R[t;\sigma]$ and $\widehat{M}_S = M \otimes_R S$ be the induced module. The additive structure of $\widehat{M}_S$ is naturally isomorphic to that of the polynomial ring $K[y_k \mid k \geq 1][t]$ and we will write nonzero elements of $\widehat{M}_S$ in the form $m = \sum_{i=l}^n \alpha_i t^i$ where all α_i's are from $K[y_k \mid k \geq 1]$ and $\alpha_l \neq 0$. We claim that $\mathrm{Ass}(\widehat{M}_S) = \emptyset$. Indeed, assume that the submodule mS is prime for some $m \in \widehat{M}_S$. We may assume that $m = \sum_{i=l}^n \alpha_i t^i$ with $\alpha_l \neq 0$. Then we have $mx_{-l-1} = \alpha_l y_1 t^l \neq 0$ and so $\alpha_l y_1 t^l S$ is also a prime submodule of $\widehat{M}_S$. But this is impossible since $x_{-l-1} \in \mathrm{ann}_R(\alpha_l y_1 t^{l+1} S) \setminus \mathrm{ann}_R(\alpha_l y_1 t^l S)$, i.e. $\mathrm{ann}_R(\alpha_l y_1 t^l S) \subset \mathrm{ann}_R(\alpha_l y_1 t^{l+1} S)$.

There is another notion of associated prime ideals a module (Cf. [MR]). Namely, when N_R is a uniform R-module, then the u-associated prime of N_R is equal to the sum of all annihilators of proper submodules of N_R. When M_R is an arbitrary R-module, then an ideal P of R is an u-associated prime of M_R if P is an u-associated prime of some uniform submodule N_R of M_R. By $\mathrm{Ass}_u(M_R)$ we will denote the set of all u-associated primes of the module M_R.

It is easy to see that u-associated prime ideals are really prime and when the ring R is noetherian, then the above new notion coincides with the classical one, i.e. $\mathrm{Ass}_u(M_R) = \mathrm{Ass}(M_R)$ in this case.

In general those two notions are different. In the following proposition we will show that there is not too much hope to relate $\mathrm{Ass}_u(M_R)$ and $\mathrm{Ass}_u(\widehat{M_S})$ even in the case when $S = R[x;\sigma]$ is a skew polynomial ring of automorphism type.

Proposition 5.16. *There exist a ring R with an automorphism τ and a uniform R-module M_R such that:*

1. $\mathrm{Ass}_u(M_R) = \{P\}$*, where P is nonzero and $\tau(P) = P$.*
2. $\mathrm{Ass}_u((M \otimes_R R[y;\tau])_{R[y;\tau]}) = \{0\}$
3. $\mathrm{Ass}_u((M \otimes_R R[y;\tau^{-1}])_{R[y;\tau^{-1}]}) = \{PR[y,\tau^{-1}]\}$

Proof. Let R be the same ring as the ring S in Example 5.15, and M_R denote the module $\widehat{M_S}$ from this example, i.e. $R = K[X][t;\sigma]$, where $\sigma(x_i) = x_{i+1}$ for all $x_i \in X$. $M = K[y_k \mid k \geq 1][t]$ and the action of R on M is given by

$$y_k t^s x_l = y_k x_{l+s} = \begin{cases} 0 & \text{if } l+s \geq 0, \\ y_k y_{-l-s} & \text{otherwise.} \end{cases}$$

We have seen in Example 5.15 that for any $0 \neq m = \sum_{i=l}^n \alpha_i v t^i \in M_R$ where all α_i's are from $K[y_k \mid k \geq 1]$ and $\alpha_l \neq 0$, we have $mx_{-l-1} = \alpha_l y_1 t^l$. Using this, it is easy to check that $m_1R \cap m_2R \neq 0$ for any nonzero elements $m_1, m_2 \in M_R$. Thus M_R is uniform.

Let τ denote the automorphism of R defined as the extension of the automorphism σ^{-1} of $K[X]$ by setting $\tau(t) = t$.

(1) Notice that $\mathrm{ann}_R(y_1 t^k R) = \sum_{i \geq -k} x_i R$. This yields that $\mathrm{Ass}_u(M_R) = \{P\}$, where $P = (X)R$ and (X) denotes the augmentation ideal of the polynomial ring $K[X]$. Clearly $\tau(P) = P$.

(2) Let $S = R[y;\tau]$ and $\widehat{M_S} = M \otimes_R S$. Then any element from $\widehat{M_S}$ can be written it the form $m = \sum_{i,j=0}^z \alpha_{ij} t^i y^j$, for some $z \in \mathbb{N}$, where $\alpha_{ij} \in K[y_k \mid k \geq 1]$. For such $0 \neq m \in \widehat{M_S}$ and $s \geq z + |r| + 1$ we have

$$my^s x_r = \sum_{i,j=0}^z \alpha_{ij} x_{r-s-j+i} t^i y^{j+s} = \sum_{i,j=0}^z \alpha_{ij} y_{r-s-j+i} t^i y^{j+s} \neq 0,$$

since all indexes appearing at x are negative. Using this, it is easy to see that for any $0 \neq \omega \in S$ and $0 \neq m \in \widehat{M}_S$, there exists $n \geq 0$ such that $my^n\omega \neq 0$.This shows that $\mathrm{ann}_S(mS) = 0$, for any nonzero $m \in \widehat{M}_S$ and $\mathrm{Ass}_u(\widehat{M}_S) = \{0\}$.

(3) Let $T = R[y;\tau^{-1}]$. Notice that for any $x_k \in X$ we have $yx_k = \tau^{-1}(x_k)y = \sigma(x_k)y$ and $tx_k = \sigma(x_k)t$. Therefore, using similar arguments as in (1), one can show that $\mathrm{Ass}_u((M \otimes_R T)_T) = \{(X)T\} = \{PT\}$. □

As we have seen in Example 5.15, for a prime module M_R the induced module $\widehat{M}_S$ does not have to be prime even if the σ-derivation is q-quantized and the module $\widehat{M}_S$ is good. While considering u-associated primes, uniform modules play the role of prime modules. By Theorem 4.6, for a good module M_R, the module $\widehat{M}_S$ is uniform if and only if M_R is uniform. In this case we have:

Proposition 5.17. *Suppose that δ is q-quantized, M_R is a uniform, prime module and the induced module $\widehat{M}_S$ is good. Let $\mathrm{Ass}_u(M_R) = \{P\}$. Then $\mathrm{Ass}_u(\widehat{M}_S) = \{Q\}$, where $Q = (\bigcup_{n\geq 0} \sigma^{-n}(P_{\sigma,\delta}))S$.*

Proof. Since every nonzero submodule of $\widehat{M}_S$ contains a good polynomial, $Q = \bigcup_f \mathrm{ann}_S(fS)$, where the sum ranges over all good polynomials from $\widehat{M}_S$. The assumptions imposed on $\widehat{M}_S$ and Lemma 5.6(3) imply that $Ann_S(fS) = \sigma^{-n}(P_{\sigma,\delta})$ for any good polynomial $f \in \widehat{M}_S$ of degree n. Now the thesis follows as, by the assumption, $\widehat{M}_S$ contains good polynomials of arbitrary degree. □

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