PRIMITIVITY OF SKEW POLYNOMIAL
AND SKEW LAURENT POLYNOMIAL RINGS

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ABSTRACT. Let R be a noetherian P.I. ring and S an automorphism of R. Necessary and sufficient conditions for the primitivity of the skew Laurent polynomial ring $R[t; t^{-1}; S]$ and the skew polynomial ring $R[t, S]$ are given.

Introduction

The problem of primitivity and primitive ideals of various kinds of ring extensions has been extensively studied since a few decades (e.g. [D], [GW], [IS], [J1], [J2], [J3], [L], [O], [R1]). One of the problems in this wide framework is to characterize primitivity of Ore extensions. The problem is far from being completely solved but during the last few years essential progress has been made. Goodearl and Warfield [GW] characterized the primitivity of Ore extensions $R[t; d]$ of derivation type for R commutative noetherian. Ouarit [O] extended this result to the case R is a noetherian P.I. ring. Recently, D. Jordan [J3] gave necessary and sufficient conditions for a skew Laurent polynomial ring over a commutative

noetherian ring to be primitive. The aim of the paper is to extend this result of Jordan to the case of a noetherian P.I. coefficient ring. As an application we also obtain a characterization of the primitivity for skew polynomial rings of automorphism type over noetherian P.I. rings. This characterization seems to be new even for commutative noetherian rings. Our approach owes much to that of [J3] and [R1].

Throughout the paper R will denote a unital ring and S will stand for an automorphism of R.

Recall that the skew polynomial ring $R[t; S]$ is a ring of polynomials in t with coefficients in R and subject to the relation $ta = S(a)t$, $a \in R$. The skew Laurent polynomial ring $R[t, t^{-1}; S]$ is a localization of $R[t; S]$ with respect to the set of powers of t.

An ideal I of R is S-stable if $S(I) = I$. Such ideals will also be called S-ideals. We say that R is S-prime if the product of any two non-zero S-ideals is non-zero.

It is well-known that $T = R[t, t^{-1}; S]$ is prime if and only if R is S-prime. Therefore, while investigating primitivity of T, we may assume that R is S-prime.

The following lemma is standard and will be used frequently.

LEMMA 0.1. (cf [G]) Suppose R is right noetherian.

(1) An ideal I of R is S-stable if and only if $S(I) \subseteq I$.
(2) If R is S-prime then R is semiprime and the minimal prime ideals of R form a single finite orbit under the action of S. $\qquad\square$

1 - Sufficient conditions

In this section we will introduce two definitions and analyze them. These notions will give sufficient conditions for $T = R[t, t^{-1}; S]$ to be primitive.

DEFINITIONS 1.1.

(1) A ring R is right S-primitive if there exists a maximal right ideal in R containing no non-zero S-stable ideals.
(2) A ring R is right S-special (resp. S-special central) if there exists $a \in R$ (resp. $a \in Z(R)$, the center of R) such that the following conditions are satisfied
   a) For any $n \geq 1$, $N_n^S(a) := aS(a)\dots S^{n-1}(a) \neq 0$
   b) For any non-zero S-ideal I of R, there exists $n \geq 1$ such that $N_n^S(a) \in I$.

2

When $S = id$, Definition 1.1 (2) gives back the notion of speciality introduced by Rowen [R1] ; when R is commutative we obtain the definition of S-speciality as given by Jordan [J3].

It is clear that if R is either S-special or S-primitive then R is S-prime.

Let us remark that if R is right artinian and S-prime then R is semi-simple (cf. Lemma 0.1). S-primeness of R then implies S-simplicity of R. Therefore for right artinian rings, S-primitivity and S-speciality boil down to S-simplicity.

Let us point out that the notions of S-primitivity and S-speciality pass to matrix rings. Concretely : if R is either S-primitive or S-special then so is $M_n(R)$ for any $n \geq 1$. We leave the easy proof to the reader.

In [J3] D. Jordan presented examples of commutative noetherian domains showing that S-primitivity and S-speciality are logically independent conditions.

In the following lemma we present a useful characterization of S-special rings in the case when R is right noetherian. It is essentially the same as Lemma 2.6 [J3].

LEMMA 1.2. Suppose R is right noetherian and $a \in R$ is such that $N_n^S(a) \neq 0$ for all $n \geq 1$. The following conditions are equivalent :

(1) For every non-zero S-stable ideal I of R, there exists $m \geq 1$ such that $N_m^S(a) \in I$.

(2) R is S-prime and for every non-zero S-prime ideal P of R, there exists $m \geq 1$ such that $N_m^S(a) \in P$.

If moreover a is central in R then :

(3) a is regular and for every non-zero S-ideal J of R there exist $1 \leq n \leq m$, and positive intergers $k_n, \ldots, k_m$ such that $S^n(a^{k_n}) \ldots S^m(a^{k_m}) \in J$.

(4) a is regular and the localization $R_{\mathcal{A}}$ is an S-simple ring, where $\mathcal{A}$ is the S-invariant multiplicatively closed set generated by a.

PROOF. The equivalence $(3) \leftrightarrow (4)$ and the implication $(1) \rightarrow (2)$ are clear.

$(2) \rightarrow (1)$ Assume (1) is not satisfed. Let I be an S-ideal maximal among ideals not satisfying (1). If P, Q are S-ideals strictly containing I, then there exist $m, n \geq 1$ such that $N_n^S(a) \in P$ and $N_m^S(a) \in Q$. Hence $N_{n+m}^S(a) = N_n^S(a)S^n(N_m(a)) \in PQ$ and so $PQ \nsubseteq I$. This means that I is S-prime, and contradicts our hypothesis.

$(1) \rightarrow (3)$ Suppose a is central. Since R is right noetherian and S-prime, R is semiprime and the set of minimal prime ideals of R is of the form $\{Q, S(Q), \ldots, S^{n-1}(Q)\}$ for some minimal prime ideal Q. Let $0 \neq r \in R$. Since

3

$0 = \cap_{i=0}^{n-1} S^i(Q)$, there exists $i \in \{0, 1, \dots, n-1\}$ such that $r \notin S^i(Q)$. If $ar = 0$ then $aRr = 0 \subset S^i(Q)$ and so $a \in S^i(Q)$ and $N_n^S(a) \in Q \cap \dots \cap S^{n-1}(Q) = 0$ (since $S^n(Q) = Q$). This contradiction shows that a is regular. The last assertion of (3) is clear.

(3) $\to$ (2) Let P be a non-zero S-prime ideal of R. Since R is right noetherian, P is semiprime. Now, the fact that a is central gives easily (2). $\qquad\square$

Recall that an automorphism S of an S-prime ring R is of infinite X-inner order if no non-zero power of S becomes inner while extended to the symmetric Martindale quotient ring of R constructed with respect to the filter of non-zero S-ideals (see [MR] 10.6.15 or [P]).

LEMMA 1.3. For an S-prime ring R, the following conditions are equivalent :

(1) Every non-zero ideal of $R[t, t^{-1}; S]$ has a non-zero intersection with R.
(2) S is of infinite X-inner order.

PROOF. This is a direct consequence of Theorem 10.6.17 [MR] and its proof. $\qquad\square$

LEMMA 1.4. Suppose that R is right S-special and S is of infinite X-inner order. If $a \in R$ is the element defining right S-speciality of R then every non-zero ideal of $R[t, t^{-1}; S] = T$ contains a power of at, so T is special.

PROOF. Let I be a non-zero ideal of T. Then, by Lemma 1.3, $I \cap R$ is a non-zero S-stable ideal of R, so $N_n^S(a) \in I$ for some $n \geq 1$. This shows that $0 \neq (at)^n = N_n^S(a)t^n \in I$. $\qquad\square$

THEOREM 1.5. Suppose that S is of infinite X-inner order and R is either right S-primitive or right S-special. Then $T = R[t, t^{-1}; S]$ is right primitive.

PROOF. Suppose R is right S-primitive and let M be a maximal right ideal of R containing no non-zero S-ideals. Let N be a maximal right ideal of T containing MT. Then $N \cap R \supseteq MT \cap R = M$. Since $N \neq T$ and M is a maximal right ideal of R, we get $N \cap R = M$. By Lemma 1.3, every non-zero ideal of T has a non-zero intersection with R. Therefore N does not contain non-zero ideals, otherwise M would contain a non-zero S-stable ideal. This shows that T is right primitive.

Now suppose R is right S-special and let $a \in R$ be as in Definition 1.1 (2). Let us assume that $(1 - at)T = T$. Then there exist integers r and l, $l \leq r$, such

4

that $(1 - at)(a_r t^r + \cdots + a_l t^l) = 1$ for some $a_r, \ldots, a_l \in R$, $a_l \neq 0$. This gives
$-aS(a_r)t^{r+1} + (-aS(a_{r-1}) + a_r)t^r + \cdots + (-aS(a_l) + a_{l+1})t^{l+1} + a_l t^l = 1$ and we
get successively $l = 0$, $a_l = 1$, $a_{l+1} = a, \cdots, a_r = N^S_{r-l}(a)$, $N^S_{r-l+1}(a) = 0$. This
last equality contradicts the choice of a and we conclude that $(1 - at)T \neq T$.
Let N be a maximal right ideal of T containing $(1 - at)T$. If N would contain
a non-zero ideal of T then, by Lemma 1.4 $(at)^n \in N$ for some $n \geq 1$ and
$(at)^{n-1} = (1 - at)(at)^{n-1} + (at)^n \in N$ as well. Hence $1 \in N$. Therefore N does
not contain non-zero ideals and the right T-module T/N is simple faithful. $\quad\square$

In connection with the first part of the proof, let us briefly mention the

EXAMPLE 1.6. It may happen that M is maximal in R but MT is not maximal in $T = R[t, t^{-1}; S]$. Take $R = K(x) \oplus K(y)$ the direct sum of fields of rational functions and define a K-automorphism S of R by $S(p(x), q(y)) = (q(x + 1), p(y + 1))$. $M = K(x)$, then $MT \subsetneq N := MT + (t^2 + (0, y)))T \subsetneq T$ i.e. MT is not maximal. To see that $N \neq T$ notice that for every $f \in T = R[t, t^{-1}; S]$, $(t^2 + (0, y))f - (0, 1) \notin MT$. Notice also that R is S-simple and T is primitive in this example.

2 - Reduction to the prime case

Let R be an S-prime right noetherian ring. Then R is semi-prime and the set of minimal prime ideals of R is the orbit $\{Q, S(Q), \ldots, S^{n-1}(Q)\}$ of some minimal prime ideal Q. Notice that, since $S^n(Q) = Q$, S^n induces an automorphism of R/Q. This automorphism will be denoted also by S^n. In this short section we want to analyze S-properties of R and S^n properties of R/Q with respect to the notions introduced in Definition 1.1. But let us first compare $R[t, t^{-1}; S]$ and $R/Q[t, t^{-1}; S^n]$ with respect to primitivity.

PROPOSITION 2.1. Let R be a noetherian S-prime ring and Q be a minimal prime ideal. The following conditions are equivalent :

  (1) $T = R[t, t^{-1}; S]$ is right primitive
  (2) QU is a right primitive ideal of $U = R[t^n, t^{-n}; S^n] \subseteq T$
  (3) $R/Q[t, t^{-1}; S^n]$ is right primitive

where n denotes the number of minimal prime ideals of R.

PROOF. Q is an S^n-ideal of R. Thus QU is an ideal of U and it is easy to see that the rings $R/Q[t, t^{-1}; S^n]$ and U/QU are isomorphic. This gives the equivalence of (2) and (3).

$(1) \leftrightarrow (2)$ Using the isomorphism described above, it is clear that QU is a prime ideal of U. Let $P \subset QU$ be a minimal prime ideal of U. Since $\cap_{i=0}^{n-1} S^i(Q) = 0$, $0 = \cap_{i=0}^{n-1}(S^i(Q)U) \subseteq P$. Therefore $S^i(Q)U \subseteq P \subset QU$ for some $0 < i \leq n-1$. Then $P = QU$. This shows that QU is a minimal prime of U. Now observing that T is a finite normalizing extensions of U, the equivalence $(1) \leftrightarrow (2)$ is a direct consequence of Corollary 10.4.15 (ii) in [MR]. $\qquad\square$

Notice that if in the above proposition we replace $R[t, t^{-1}; S]$ by $R[t, S]$ and U by $R[t, S^n]$, then exactly the same proof gives us the following

COROLLARY 2.2. Let R and Q be as in Proposition 2.1. The following conditions are equivalent :

(1) $R[t, S]$ is right primitive
(2) $R/Q[t, S^n]$ is right primitive

where n denotes the number of minimal prime ideals of R.

PROPOSITION 2.3. Suppose R is S-prime right noetherian and $\{Q, S(Q), \dots, S^{n-1}(Q)\}$ is the set of minimal prime ideals of R. If R/Q is right S^n-primitive then R is right S-primitive.

PROOF. Let M be a maximal right ideal of R containing Q and such that M/Q does not contain non zero S^n-ideals of R/Q. We claim that M does not contain non-zero S-ideals. Let I be an S-ideal of R contained in M. Then $I + Q$ is an S^n-ideal included in M and so, by the choice of M, $I \subset Q$ and hence $I \subset Q \cap S(Q) \cap \dots \cap S^{n-1}(Q) = 0$. $\qquad\square$

The following lemma is a well-known generalization of Posner's theorem to the case of semiprime rings.

LEMMA 2.4. (see 1.7.22 [R3]) Let R be a semiprime, right noetherian, P.I. ring. Then the localization of R with respect to the set of all central regular elements is a semi-simple ring. $\qquad\square$

PROPOSITION 2.5. Suppose R is an S-prime, right noetherian, P.I. ring and $\{Q, S(Q), \dots, S^{n-1}(Q)\}$ is the set of minimal prime ideals of R. If R/Q is S^n-special central then R is S-special central.

PROOF. By hypothesis there exits $c \in R$ such that :

(1) $\bar{c} = c + Q \in R/Q$ is central,
(2) for any $l \in \mathbb{N}$, $N_l^{S^n}(c) \notin Q$,
(3) every S^n-ideal I of R such that $Q \subsetneq I$ contains $N_l^{S^n}(c)$ for some $l \geq 1$.

Let us first show that we can assume that c is regular and belongs to $Z(R)$, the center of R. Since $\bar{c} \in R/Q$ is central, $cR + Q$ is a two-sided ideal of R. Now, if $cR + Q \subseteq S^i(Q)$ for some $i \in \{0, 1, \dots, n-1\}$ then $Q \subseteq S^i(Q)$. Thus $i = 0$ and $c \in Q$, a contradiction. Hence for any $i \in \{0, 1, \dots, n-1\}$, $cR + Q \not\subseteq S^i(Q)$. So if A is the right annihilator of $cR + Q$, then $(cR + Q)A = 0 \subset S^i(Q)$ and therefore $A \subset \cap_{i=0}^{n-1} S^i(Q) = 0$. This shows in particular that the two-sided ideal $cR + Q$ of the semi-prime ring R is essential. Since R is right noetherian and P.I., Lemma 2.4 implies that there exists a central regular element $d \in cR + Q$. Clearly $N_l^{S^n}(d) \notin Q$ for any $l \geq 1$, as this element is central and regular. Moreover, since $\bar{c}$ is central in R/Q, for any $l \geq 1$ there is $\alpha_l \in R$ such that $N_l^{S^n}(d) = dS^n(d)\dots S^{(l-1)n}(d) \in N_l^{S^n}(c)\alpha_l + Q$. The above shows that eventually replacing c by d, we may additionally assume that c is a central regular element of R.

Now let I be a non-zero S-ideal of R. Then $I \not\subset Q$ and $I + Q$ is an S^n-ideal of R. By hypothesis there exists $l > 0$ such that $N_l^{S^n}(S^i(c)) = S^i(N_l^{S^n}(c)) \in S^i(I + Q) = I + S^i(Q)$, for all i. This yields $\prod_{i=0}^{n-1} N_l^{S^n}(S^i(c)) \in I$ and shows that I intersects the S-invariant multiplicatively closed set generated by c. So R is S-special central by Lemma 1.2. $\qquad\square$

3 - Necessary and sufficient conditions for the primitivity of $R[t, t^{-1}; S]$ and $R[t; S]$

We will assume, up to the end of the paper, that R is an S-prime, right noetherian P.I. ring having n minimal prime ideals : $Q, S(Q), \dots, S^{n-1}(Q)$ (cf. Lemma 0.1). $Z(R)$ will stand for the center of R and RZ^{-1} denotes the localization of R with respect to the regular elements of $Z(R)$. These notations will remain fixed. In Section 1, we observed that if S is of infinite X-inner order and R is either right S-primitive or right S-special then $T = R[t, t^{-1}; S]$ is primitive. We will now show that these properties are in fact necessary.

LEMMA 3.1. Suppose R is an S-prime right noetherian P.I. ring. Then the following conditions are equivalent :

(i) the restriction of S to $Z(R)$ is of finite order,
(ii) there exist $k \geq 1$ and an invertible element $d \in RZ^{-1}$ such that $S(d) = d$ and $S^k(r) = drd^{-1}$ for all $r \in RZ^{-1}$.

PROOF. Clearly (ii) implies (i).

Suppose that (i) holds. The extension of S to RZ^{-1} is also of finite order on $Z(RZ^{-1})$, say of order l. By Lemma 2.4 RZ^{-1} is a semisimple ring. Let

$e_1, \ldots, e_m$ be central primitive idempotents of RZ^{-1} such that $\sum_{i=1}^{m} e_i = 1$. Then, for any $i \in \{1, \ldots, m\}$, $S_i = S^l|_{e_i R} Z^{-1}$ is an automorphism of a central simple algebra $e_i RZ^{-1}$. Moreover S_i is identity on the center of $e_i RZ^{-1}$. Therefore, by Skolem-Noether Theorem, S_i is an inner automorphism of $e_i RZ^{-1}$. Let $q_i \in e_i R$ denote the element determining S_i. Then $q = \sum_{i=1}^{m} q_i$ is invertible in RZ^{-1} and $S^l = I_q$ is the inner automorphism determined by q. Set $d := N_l^S(q)$ and observe that $S(d) = S(qS(q) \ldots S^{l-1}(q)) = (q^{-1}q)S(q) \ldots S^{l-1}(q)q = N_l^S(S^{-l}(q)) = N_l^S(q) = d$. Moreover, since $S^i S^l S^{-i} = S^l$ for any $i \geq 1$ we have $I_{S^i(q)} = I_q$ and also $S^{l^2} = I_q I_{S(q)} \ldots I_{S^{l-1}(q)} = I_{N_l^S(q)} = I_d$. Setting $k = l^2$, we obtain $S^k = I_d$ for some $d \in RZ^{-1}$ satisfying $S(d) = d$. $\qquad \square$

PROPOSITION 3.2. Suppose R is a right noetherian P.I. ring. If $T = R[t, t^{-1}; S]$ is primitive, then S is of infinite order in $Z(R)$.

PROOF. Since T is primitive, R is S-prime. Assume that S is of finite order on $Z(R)$. Then, by Lemma 3.1, there exist $k \geq 1$, and an invertible element $d \in RZ^{-1}$ such that $S(d) = d$ and S^k is the inner automorphism determined by d. This means that the elements $d^{-1}t^k$ and dt^{-k} are central in $RZ^{-1}[t, t^{-1}; S]$ and shows that $RZ^{-1}[d^{-1}t^k, dt^{-k}]$ is a P.I. ring. Since $RZ^{-1}[t, t^{-1}; S]$ is a module of finite type over $RZ^{-1}[d^{-1}t^k, dt^{-k}]$ we conclude that $RZ^{-1}[t, t^{-1}; S]$ is P.I. Therefore $T = R[t, t^{-1}; S]$ is a primitive P.I. ring and Kaplansky's Theorem implies that T is a central simple algebra finite dimensional over its center. Since T is not right artinian we get a contradiction and this shows that S cannot be of finite order on $Z(R)$. $\qquad \square$

Before proving that the sufficient conditions given in Section 1 are also necessary, let us introduce two technical lemmas.

LEMMA 3.3. Let R be an S-prime right noetherian ring and let M be a maximal right ideal of $T = R[t, t^{-1}; S]$. The set of leading (resp. minimal) coefficients of elements from M is an essential right ideal of R.

PROOF. Define $U := \{a \in R|$ there are $n \in \mathbb{Z}$ and elements $a_1, a_2, \ldots \in R$ almost all equal to zero, such that $at^n + a_1 t^{n-1} + \ldots \in M\}$ and also $D := \{a \in R|$ there are $n \in \mathbb{Z}$ and elements $a_1, a_2, \ldots \in R$ almost all equal to zero such that $at^n + a_1 t^{n+1} + \ldots \in M\}$. Obviously U and D are right ideals of R. We will now show that U is an essential right ideal. The proof for D is the same. Assume U is not essential and let I be a non-zero right ideal of R such that $U \cap I = 0$. We deduce easily that $IT \oplus M = T$. Therefore $0 \neq IT$ is contained in the right socle $soc(T)$ of T. The assumption on R implies that T is a prime right noetherian

8

ring. We have seen that $soc(T) \neq 0$ and Theorem 1.24 [CH] shows that T is simple artinian. This contradiction yields that U is an essential right ideal of R. $\qquad\square$

LEMMA 3.4. Let T be the skew Laurent polynomial ring $T = R[t, t^{-1}; S]$ and X_T be a right T-module. Suppose $0 \neq x \in X$ is such that $(\operatorname{ann}_R(x))T \subsetneqq \operatorname{ann}_T(x)$. Then there exists $n \geq 1$ such that for any $c \in Z(R) \cap \operatorname{ann}_R(x)$ we have $\operatorname{ann}_R(x) \subsetneqq \operatorname{ann}_R(xS^{-n}(c))$.

PROOF. Let $f \in \operatorname{ann}_T(x) \smallsetminus (\operatorname{ann}_R(x))T$ be of minimal length (if $f(t) = \sum_{i=i_0}^{i_1} a_i t^i$, length $(f) = i_1 - i_0$). We may assume that $f = at^n + \cdots + b$ where $n > 0$ and $a, b \notin \operatorname{ann}_R(x)$. Let $c \in \operatorname{ann}_R(x) \cap Z(R)$. Then $cf - fS^{-n}(c) \in \operatorname{ann}_T(x)$ and is of smaller length than f, so $cf - fS^{-n}(c) \in (\operatorname{ann}_R(x))T$. Hence, in particular, $x(c - S^n(c))b = 0$ and thus $xS^{-n}(c)b = 0$, i.e. $b \in \operatorname{ann}_R(xS^{-n}(c)) \smallsetminus \operatorname{ann}_R(x)$. $\qquad\square$

DEFINITION 3.5. A right $T = R[t, t^{-1}; S]$-module X is induced if X is isomorphic to T/IT for some right ideal I of R.

It is easy to check that if a right induced T-module T/IT is simple and faithful then I is a maximal right ideal of R containing no non-zero S-stable ideal and hence R is S-primitive.

PROPOSITION 3.6. Let R be a prime right noetherian ring and X is a simple faithful right $T = R[t, t^{-1}; S]$-module. If X is not induced then X is $Z(R)$-torsion free.

PROOF. Suppose that the simple faithful T-module X is not induced. Since R is right noetherian, we can choose $x \in X$ such that $\operatorname{ann}_R(x)$ is maximal among annihilators of non-zero elements from X. Put $M = \operatorname{ann}_T(x)$. The simplicity of X implies that $X \cong T/M$ and since X is not induced, $(\operatorname{ann}_R(x))T \subsetneqq M = \operatorname{ann}_T(x)$. Lemma 3.5 shows that there exists $n \geq 1$ such that $\operatorname{ann}_R(x) \subsetneqq \operatorname{ann}_R(xS^{-n}(c))$ for any $c \in \operatorname{ann}_{Z(R)}(x)$. Since $\operatorname{ann}_R(x)$ is maximal, we must have $xS^{-n}(c) = 0$ for any $c \in \operatorname{ann}_{Z(R)}(x)$ i.e. $S^{-n}(\operatorname{ann}_{Z(R)}(x)) \subseteq \operatorname{ann}_{Z(R)}(x)$. Now R is right noetherian and hence the two sided ideal $I = \cap_{i=0}^{n-1} S^{-i}(\operatorname{ann}_{Z(R)}(x)R)$ which satisfies $S^{-1}(I) \subset I$ is in fact S-stable. Hence IT is an ideal of T and obviously $IT \subseteq (\operatorname{ann}_{Z(R)}(x))T \subseteq \operatorname{ann}_T(x) = M$. Since $X = T/M$ is faithful, we obtain $I = 0$. Because $Z(R)$ is a domain, the above shows that $\operatorname{ann}_{Z(R)}(x) = 0$. So we have proved that if $x \in X$ is such that $\operatorname{ann}_R(x)$ is maximal then $\operatorname{ann}_{Z(R)}(x) = 0$. Now, it is easy to see that for any $y \in X$, $\operatorname{ann}_{Z(R)}(y) = 0$ i.e. X is $Z(R)$-torsion free.

9

For any central subset C of a prime ring R, R_C will denote the localization of R at the multiplicatively closed set generated by C. For any right R-module X let $X_C = X \otimes_R R_C$ be the corresponding localization of X.

LEMMA 3.7. Let R be a prime P.I. noetherian ring and X a right $T = R[t, t^{-1}; S]$-module.

  (a) If X is simple then there exists $a \in Z(R)$ such that $X_{\{a\}}$ is a finitely generated $Z(R)_{\{a\}}$-module
  (b) If X is simple faithful and R is not S-special central then X is induced.

PROOF. a) Let M be a maximal right ideal of T. Due to Lemma 3.3 we know that $U = \{b \in R | b$ is the leading coefficient of some element from $M\}$ and $D = \{c \in R | c$ is the minimal coefficient of some element from $M\}$ are essential right ideals of R. Posner's theorem implies that there is a non-zero $b \in U \cap D \cap Z(R)$. Now, R has $P.I.$ degree n and if g_n denotes the Formanek central polynomial (cf [R2] Chapter 6.1) then $0 \neq g_n(R) \subset Z(R)$. Choose any $0 \neq s \in g_n(R)$ and remark that $0 \neq a = bs \in U \cap D \cap g_n(R)$. Using Corollary 6.1.36 [R2] we conclude that $R_{\{a\}}$ is a finitely generated $Z(R)_{\{a\}}$-module. Since $a \in U \cap D$, there are $n \geq 1$ and $f, g \in M$ such that $f = at^n +$ terms of smaller degree ; $g = at^{-n} +$ terms of higher degree. Therefore $X_{\{a\}} = T/M \otimes_R R_{\{a\}}$ is a finitely generated right $R_{\{a\}}$-module, being generated by images of t^k, $|k| < n$. The above shows that $X_{\{a\}}$ is also finitely generated as a module over $Z(R)_{\{a\}}$.

b) Assume, at the contrary, that $X = T/M$ is not induced. Part a) above shows that there exists $0 \neq a \in Z(R)$ such that $X_{\{a\}}$ is a finite generated $Z(R)_{\{a\}}$-module. Let $\mathcal{A}$ denote the S-stable multiplicatively closed set generated by a. Then $X_\mathcal{A}$ is a central localization of $X_{\{a\}}$ and so $X_\mathcal{A}$ is also finitely generated as a module over $Z(R)_\mathcal{A}$. We will show that $R_\mathcal{A}$ is S-simple. Clearly it is enough to show that any non-zero S-ideal of R intersects $\mathcal{A}$ non trivially. Let I be a non-zero S-stable ideal of R. Since R is prime P.I., $J = I \cap Z(R)$ is a non-zero ideal of $Z(R)$, moreover J is S-stable as I is such. From this we conclude that XJ is a T-submodule of a simple faithful T-module X. Therefore $XJ = X$. Localizing this equality at $\mathcal{A}$, we get $X_\mathcal{A} J_\mathcal{A} = X_\mathcal{A}$. $X_\mathcal{A}$ is a finitely generated module over $Z(R)_\mathcal{A}$ thus, by Nakayama's Lemma [MA, p. 8], there exists $z \in Z(R)_\mathcal{A}$ such that $X_\mathcal{A} z = 0$ and $z - 1 \in J_\mathcal{A}$. Writing $z = bd^{-1}$ for some $b \in Z(R)$ and $d \in \mathcal{A}$, we have $X_\mathcal{A} b = 0$. Since X is not induced, Proposition 3.6 yields that X is $Z(R)$-torsion free. Hence $X \subset X_\mathcal{A}$ and $Xb = 0$. This implies $b = 0$ and consequently, $z = 0$. Therefore $1 \in J_\mathcal{A}$. This means that $\emptyset \neq J \cap \mathcal{A} \subseteq I \cap \mathcal{A}$ i.e. $R_\mathcal{A}$ is S-simple. In view of Lemma 1.2 this shows that

10

R is S-special central, a contradiction. $\qquad\square$

PROPOSITION 3.8. Let R be a prime $P.I.$ noetherian ring. If $T = R[t, t^{-1}; S]$ is right primitive then R is either right S-primitive or S-special central.

PROOF. Assume R is not right S-primitive. Let X be a simple faithful T-module, then X is not induced and part b) of the above lemma yields that R is S-special central. $\qquad\square$

Combining the above proposition and results of Section 2, we obtain the following theorem

THEOREM 3.9. Suppose that R is a right noetherian P.I. ring. For the skew Laurent polynomial ring $T = R[t, t^{-1}; S]$, the following conditions are equivalent :

(1) T is right primitive,

(2) T is left primitive,

(3) R is either right S-special or right S-primitive and S is of infinite order on the center of R,

(4) R is either S-special central or right S-primitive and S is of infinite order on the center of R.

PROOF. $(1) \to (4)$ Suppose that T is right primitive. Proposition 3.2 shows that S is of infinite order on $Z(R)$. Let Q be a minimal prime ideal of R. Then, by Proposition 2.1, $R/Q[t, t^{-1}; S^n]$ is right primitive, where n is the number of minimal primes in R. Therefore Proposition 3.8 implies that R/Q is either right S^n-primitive or S^n-special central. Now, Proposition 2.3 and 2.5 yield that R is either right S-primitive or S-special central.

The implication $(4) \to (3)$ is a tautology, while $(3) \to (1)$ is a consequence of Theorem 1.5. This shows that conditions (1), (3) and (4) are equivalent.

$(1) \leftrightarrow (2)$ If T is (right or left) primitive then R is S-prime and the assumptions imposed on R yields that R is semiprime right noetherian P.I. ring. Thus R is also left noetherian (see [R2] vol II, pp. 174, ex. 24). Now, we can apply the equivalence $(1) \leftrightarrow (4)$ to the opposite ring $T^{op} = R^{op}[t, t^{-1}; S^{-1}]$ obtaining T is left primitive if and only if T^{op} is right primitive if and only if R^{op} is either right S-primitive or S-special central and S is of infinite order on $Z(R^{op}) = Z(R)$. Therefore for proving the equivalence $(1) \leftrightarrow (2)$, it is enough to show that R is right S-primitive if and only if R is left S-primitive.

Suppose R is right S-primitive. Let M be a maximal right ideal of R containing no non zero S-stable ideals. Then R/I is a right primitive P.I. ring where

$I = \mathrm{ann}_R(R/M)$. By Kaplansky's Theorem R/I is left primitive so there is a maximal left ideal N of R such that $I = \mathrm{ann}_R(R/N)$. Clearly N does not contain non-zero S-ideals. The argument is left right symmetric. $\square$.

The above theorem enables us to prove the following.

THEOREM 3.10. Suppose that R is a right noetherian P.I. ring. For the skew polynomial ring $W = R[t; S]$ the following conditions are equivalent :

(1) W is right primitive
(2) W is left primitive
(3) R is right S-special and S is of infinite order on the center of R,
(4) R is S-special central and S is of infinite order on the center of R.

PROOF. $(1) \rightarrow (4)$ Suppose $W = R[t; S]$ is right primitive. Let M be a maximal right ideal of W containing no non-zero ideals. Notice that for any $n \geq 1$, $t^n \notin M$ as otherwise M would contain the non-zero two sided ideal $t^n W$. Therefore MT is a proper right ideal of $T = R[t, t^{-1}; S]$. Also for every proper right ideal K containg MT we have $K \cap W = M$ and $(K \cap W)T = K$, and we conclude that MT is maximal. Moreover MT does not contain non-zero ideals as M has such property in W. The above shows that T is right primitive. Thus, by Theorem 3.9, S is of infinite order on the center of R and R is either S-special central or right S-primitive. We distinguish two cases.

Case 1 : R is prime. Assume R is not S-special central. Then, by Lemma 3.7, every simple faithful T-module is induced from R. This means that $MT = NT$ for some maximal right ideal N of R. Thus $M = W \cap MT = W \cap NT = NW \subsetneq NW + tW \neq W$. This contradicts maximality of M and yields $(1) \rightarrow (4)$ in the case when R is prime.

Case 2 : R is semiprime. Let Q be a minimal prime ideal of R and n the number of minimal primes. By Corollary 2.2, $R/Q[t; S^n]$ is primitive. Thus, by Case 1, R/Q is S^n-special central. Now, Proposition 2.5 shows that R is S-special central and yields $(1) \rightarrow (4)$ in this case as well.

The implication $(4) \rightarrow (3)$ is a tautology.

$(3) \rightarrow (1)$ Suppose (3) holds. Lemma 1.4 implies that T is special. In fact every non-zero ideal of T contains some power of at where $a \in R$ is the element defining S-speciality of R. Now, let I be a non-zero ideal of W. Then IT is a two sided ideal of T as W is right noetherian and T is a localization of W with respect to powers of t. Therefore $(at)^n \in IT \cap W$ for some $n \geq 1$. It means that $(at)^n t^k \in I$ for some $k \geq 0$ and $(at)^{n+k} \in I$ follows. This shows that every non-zero ideal of W contains some power of at, so W is special. Now, as in the

12

proof of Theorem 1.5, one can show that W/M is a simple faithful W-module where M is a maximal right ideal of W containing $(1 - at)W$.

The above shows that conditions (1), (3) and (4) are equivalent. The equivalence of (1) and (2) can be obtained in the same way as equivalence between (1) and (2) in Theorem 3.9. □

EXAMPLE 3.11.. Let $R = \mathbb{C}[x, x^{-1}, y, y^{-1}]$ be the Laurent polynomial ring in two commuting variables over the field of complex numbers. Define the $\mathbb{C}$-automorphism S of R by $S(y) = x$ and $S(x) = yx^{-1}$. It is shown in [J3, Prop. 7.13] that R is S-primitive but not S-special. Theorems 3.9 and 3.10 show that $R[t, t^{-1}; S]$ is primitive but $R[t; S]$ is not.

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