Noncommutative Symmetric functions and W-polynomials

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7th June 2006

Abstract

Let K, S, D be a division ring an endomorphism and a S-derivation of K, respectively. In this setting we introduce generalized noncommutative symmetric functions and obtain Viète formula and decompositions of differential operators. W-polynomials show up naturally, their connections with P-independency, Vandermonde and Wronskian matrices are briefly studied. The different linear factorizations of W-polynomials are analysed. Connections between the existence of LLCM of monic linear polynomials with coefficients in a ring and the left duo property are established at the end of the paper.

1 Introduction

Let K be a division ring with center k. Wedderburn (Cf.[24]) proved, among other results, that if $f(t) \in k[t]$ is irreducible but has a root d_1 in K, then $f(t)$ splits linearly in $K[t]$:$f(t) = (t - d_n) \cdots (t - d_1)$. Moreover, Wedderburn showed that if $f_i(t)$ stands for $(t - d_i) \cdots (t - d_1)$ and d is a conjugate of d_1 such that $f_i(d) \neq 0$, then one can choose $d_{i+1} := f_i(d) d f_i(d)^{-1}$;. In particular, the elements $d_1, \ldots, d_n$ are all conjugate to d_1. This result was extended by Jacobson using module theory (Cf. [10]) and extended to polynomials in an Ore extension over a division ring by Lam and the second author. Rowen and Haile (Cf. [9]) studied in details factorizations of central polynomials which are minimal polynomials of some elements in a division rings. They introduced good and very good factorizations and gave applications to the structure of division rings. Haile and Knus used Wedderburn mathod to study algebras of degree 3 with involutions 5[8]. Rowen and Segev ([23]) also applied the Wedderburn method for the study of the multiplicative group of a finitedimensional division ring.

About 12 years ago, Gelfand and Retakh introduced noncommutative symmetric functions via linear factorizations of quasideterminant of Vandermonde matrices over division rings. The form of these symmetric functions already appears in the coefficients of the above factorization of a central polynomial $f(t) = (t - d_n) \cdots (t - d_1)$ if the elements $d_1, \ldots, d_n$ are expressed in terms of the roots of $f(t)$. Gelfand and Retakh proved the required symmetry property and obtained Viète formula, Bezout and Miura decompositions through computations with quasideterminants (Cf [3], [4], [5]). Later Wilson gave a remarkable generalization of the fundamental theorem on symmetric functions (Cf. [25], see also example 2.6).

One of our aim in this paper is to show the link between the Gelfand, Retakh, Wilson symmetric functions and the Wedderburn polynomials (abbreviated W-polynomials) studied by Lam, Ozturk and the second author. The generalized Vandermonde matrices and Wronskian matrices naturally show up in this context and enable us to simplify presentation and generalize some of the results mentioned above. This is the content of section 2 and 3.

In section 4, we show how to quickly introduce quasideterminants in the different factorizations obtaining in this way exactly the same expressions as the one given by Gelfand and Retakh. We also compute the LU decompositions of generalized Vandermonde and Wronskian matrices.

The notion of P-independence was introduced in earlier works by Lam and the second author (Cf.[13] [14] or [15]). This notion is recalled and studied briefly in section 5 in order to present the relations between Vandermonde and Wronskian matrices and W-polynomials. This leads to an algorithmic test for P-independence (Cf. Proposition 5.7). In fact, the P-independence appears also in a somewhat hidden form in [5]. The results of section 5 give a new insight on the hypotheses used in this latter reference (see remark 5.5 b)).

In section 6, we study all the factorizations of a W-polynomial. These are shown to be in bijection with complete flags in some spaces (Cf. Theorem 6.4 and Theorem 6.7)

The last section is a first step towards considering more general base ring than division rings. It shows that a natural choice could be to investigate the Wedderburn polynomials and the symmetric functions with coefficients in a left duo or more generally in an (S, D) left duo ring.

2 Noncommutative Symmetric functions

Let K, S, D be a division ring, an endomorphism of K and a S-derivation of K respectively. Throughout the paper we denote by $R = K[t; S, D]$ the skew polynomial ring whose elements are polynomials of the form $\sum_{i=0}^{n} a_i t^i$, where $a_0, \dots, a_n \in K$. The additive structure of R is the usual one and multiplication is bilinear and based on the commutation rule :

$$ta = S(a)t + D(a) \quad \text{for } a \in K.$$

If $\{x_1, \dots, x_n\}$ are elements of K we can compute the least left common multiple $p_n(t)$ of the polynomials $t - x_i$ in $R := K[t; S, D]$. There are several ways of conducting the computations and they will lead to different factorizations of the polynomial $p_n(t)$. We will denote $[f, g]_l$ (or simply $[f, g]$) the monic polynomial which is a least left common multiple of f and g i.e. $[f, g]_l$ is a monic polynomial such that $Rf \cap Rg = R[f, g]_l$.

For $f \in R$ and $a \in K$, we denote by $f(a) \in K$ the remainder of f right divided by $t - a$, i.e. $f(a)$ is the unique element in K such that $f - f(a) \in R(t - a)$. This notion was introduced in [18]. For the sake of completeness let us recall the important product formula. In the sequel, for $a \in K$ and $c \in K \setminus \{0\}$, we write a^c for the (S, D) conjugation of a by c : $a^c := S(c)ac^{-1} + D(c)c^{-1}$.

Lemma 2.1. (Product formula) Let f, $g \in R := K[t; S, D]$ and $a \in K$:

1.
$$fg(a) = \begin{cases} 0 & if \quad g(a) = 0 \\ f(a^{g(a)})g(a) & if \quad g(a) \neq 0 \end{cases}$$

2.
$$[f, t - a]_l = \begin{cases} f & if \quad f(a) = 0 \\ (t - a^{f(a)})f & if \quad f(a) \neq 0 \end{cases}$$

Proof. 1. Right dividing $g(t)$ by $t - a$ we have $g(t) = p(t)(t - a) + g(a)$ for some polynomial $p(t) \in R$. Assuming $g(a) \neq 0$, we have $f(t) = q(t)(t - a^{g(a)}) + f(a^{g(a)})$ for some $q(t) \in R$. This gives $f(t)g(t) = (q(t)(t - a^{g(a)}) + f(a^{g(a)}))p(t)(t - a) + q(t)(t - a^{g(a)})g(a) + f(a^{g(a)})g(a))$. The fact that $(t - a^{g(a)})g(a) = S(g(a))(t - a)$ then leads quickly to the conclusion.
2. Is is easy to check that, if $f(a) \neq 0$, $(t - a^{f(a)})f \in R(t - a)$. This gives the result. □

Let us notice that $deg([f, t - a]_l) \leq deg(f) + 1$ and that equality occurs if and only if $f(a) \neq 0$. This will be used freely in the paper.
For $\{x_1, \dots, x_n\} \subseteq K$, we can now construct the *llcm* of $t - x_1, \dots, t - x_n \in R = K[t; S, D]$.

Example 2.2. Suppose $x_1 \neq x_2$ are elements in K. We have :

$$[t - x_1, t - x_2]_l = (t - x_1^{x_1 - x_2})(t - x_2) = (t - x_2^{x_2 - x_1})(t - x_1)$$

Comparing coefficients of degree 0 and 1, this immediately leads to the following generalized symmetric functions :

$$\Lambda_1(x_1,x_2) = x_1^{x_1-x_2} + S(x_2) = x_2^{x_2-x_1} + S(x_1)$$

$$\Lambda_2(x_1,x_2) = x_1^{x_1-x_2}.x_2 - D(x_2) = x_2^{x_2-x_1}.x_1 - D(x_1)$$

In order to exhibit the symmetric functions in $\{x_1,\dots,x_n\} \subseteq K$ with $n \geq 2$, we introduce some notations and a definition:
Let us put $p_j = [t - x_i \,|\, i \leq j]_l$ for $j = 1,\dots,n$. It is useful to also define $p_0 := 1$. Using the Lemma 2.1 2., we then get

$$p_n(t) = \begin{cases} p_{n-1}(t) \quad if \quad p_{n-1}(x_n) = 0 \\ (t - x_n^{p_{n-1}(x_n)})p_{n-1}(t) \quad if \quad p_{n-1}(x_n) \neq 0 \end{cases}$$

This enables us to compute p_n by induction on n starting with $p_0(t) = 1$.

Notice that $x_1,\dots,x_n$ are right roots of $p_n(t)$ and the above formula means that $p_i(t)$ can be computed from $p_{i-1}(t)$ by requiring that x_i is a root of $p_i(t)$. It is obvious that $p_n(t)$ can be computed in different ways depending on the order in which the roots are added. In example 2.2 we have seen the 2 different expressions for $p_2(t) = [t - x_1, t - x_2]$ when $x_1 \neq x_2$.

Definition 2.3. We say that the set $\{x_1,\dots,x_n\} \subseteq K$ is P-independent if $deg([t - x_i|1 \leq i \leq n]) = n$.

We leave the easy proof of the next lemma to the reader.

Lemma 2.4. *$\{x_1,\dots,x_n\} \subseteq K$ is P-independent if and only if for any $i \in \{1,\dots,n-1\}$, $p_i(x_{i+1}) \neq 0$.*

An equivalent definition was given in [13]. This notion will be studied more deeply in section 5 (Cf. Theorem 5.2). Of course, if $\{x_1,\dots,x_n\}$ is a P-independent set then for every $i \leq n$ the set $\{x_1,\dots,x_i\}$ is P-independent as well. In this case we put

$$y_i = x_i^{p_{i-1}(x_i)} \text{ for } i \in \{1,\dots,n\}$$

We then have:

$$[t - x_j \,|\, j = 1,\dots,i]_l = p_i(t) = (t - y_i)\dots(t - y_1) \text{ for } i \in \{1,\dots,n\} \quad (A)$$

We define $(-1)^k\Lambda_k^i(X)$, $0 \leq k \leq i$, to be the coefficient of degree $i - k$ of $p_i(t)$ i.e. $p_i(t) = \sum_{k=0}^{i}(-1)^k\Lambda_k^i t^{i-k}$. As in example 2.2, introducing the roots of p_i in different orders give different factorizations of this polynomial and this shows that the coefficients $\Lambda_k^i(X)$ are symmetric with respect to permutations of X. In terms of the y_j's they can be written as :
$\Lambda_0^1 = 1$.
$\Lambda_1^1 = y_1$.

$\Lambda_0^2 = 1.$
$\Lambda_1^2 = y_2 + S(y_1).$
$\Lambda_2^2 = y_2 y_1 - D(y_1).$

Assume that the $\Lambda_0^i, \dots, \Lambda_i^i$ have been defined (with $i < n$). Remarking that $p_{i+1}(t) = (t - y_{i+1})p_i(t)$ we have:

$$
\begin{aligned}
\Lambda_0^{i+1} &= 1.\\
\Lambda_1^{i+1} &= y_{i+1} + S(\Lambda_1^i).\\
\Lambda_2^{i+1} &= y_{i+1}\Lambda_1^i + S(\Lambda_2^i) - D(\Lambda_1^i).\\
\vdots \quad & \quad \vdots\\
\Lambda_k^{i+1} &= y_{i+1}\Lambda_{k-1}^i + S(\Lambda_k^i) - D(\Lambda_{k-1}^i).\\
\vdots \quad & \quad \vdots\\
\Lambda_{i+1}^{i+1} &= y_{i+1}\Lambda_i^i - D(\Lambda_{i-1}^i).
\end{aligned}
$$

Remark 2.5. a) In the classical case ($S = Id.$, $D = 0$) one can easily describe the Λ's in terms of the y_i's. From (A) above one gets :

$$
\begin{aligned}
\Lambda_0^n &= 1.\\
\Lambda_1^n(X) &= y_1 + \dots y_n.\\
\Lambda_2^n(X) &= \textstyle\sum_{i<j} y_j y_i.\\
\Lambda_3^n(X) &= \textstyle\sum_{i<j<k} y_k y_j y_i.\\
\vdots \quad & \quad \vdots\\
\Lambda_n^n(X) &= y_n y_{n-1} \dots y_1..
\end{aligned}
$$

b) Let us notice that, if $x_1, \dots, x_n \in K$, an algorithm can be written to check P-independency of these elements and in this case to compute the elements $y_1, \dots, y_n$ and the Λ_k^i.

c) In their works on noncommutative symmetric functions Gelfand, Rethak and Wilson used quasideterminants to obtain (in case when $S = Id.$ and $D = 0$) the above symmetric functions. With this point of view the symmetry of these functions was not easy to prove and somewhat surprising (see comments in the introduction of [5]). With the above approach the symmetry is clear even in the (S, D)-setting.

Let us end this section with an important example and result extracted from [25].

Example 2.6. Let $A := k < x_1, \dots, x_n >$ be a free algebra over a commutative field k in n noncommutative free variables. Let us denote by K the universal field of fractions of A (aka. the free field in n noncommutative variables) and consider the (usual) polynomial ring $K[t]$. The least left common multiple $p(t) := [t - x_1, \dots, t - x_n]_l$ factorizes in $K[t]$ as above : $p(t) = (t - y_n) \cdots (t - y_n)$. In this context an analogue of the classical fundamental theorem on symmetric functions was given in [25] : If a polynomial $f \in k[y_1, \dots, y_n]$ is symmetric with respect to the variables $x_1, \ldots, x_n$, then $f \in k[\Lambda_1^n(X), \ldots, \Lambda_n^n(X)]$, where for $1 \leq i \leq n$, the elements $\Lambda_i^n(X)$ are described in Remark 2.5 a).

3 Viète, Bezout and Miura decompositions

Suppose $\{x_1, x_2, \ldots, x_n\} \subseteq K$ is a P-independent set. With the above notations, we have :

$$[t - x_i \,|\, i = 1, \ldots, n]_l = p_n(t) = \sum_{i=0}^{n} (-1)^i \Lambda_i^n t^{n-i} \qquad (B)$$

This is the Viète formula expressing the coefficients of a polynomial in terms of its roots, assumed to be P-independent. This condition will be studied in section 5. In case when $S = Id.$ and $D = 0$ this expression was obtained by Gelfand and Retakh (Cf. [3]) using quasideterminants techniques.

If $z \in K$ is not a root of p_n, we get from equation (A) in section 2, $p_n(z) = ((t - y_n)p_{n-1})(z) = (z^{p_{n-1}(z)} - y_n)p_{n-1}(z)$. Putting $z_i = z^{p_{i-1}(z)}$ for $1 \leq i \leq n$ ($p_0 := 1$), an easy induction leads to:

$$p_n(z) = (z_n - y_n)(z_{n-1} - y_{n-1}) \ldots (z_1 - y_1) \qquad (C)$$

This is sometimes called the Bezout decomposition (Cf. [3]).
For $a \in K$, the left R-module $R/R(t - a)$ induces a left R-module structure on K. In particular, since for $x \in K$, we have $tx = S(x)t + D(x) = S(x)(t - a) + S(x)a + D(x)$ and we conclude that the action of $t \in R$ on $x \in K$ is given by $t.x = S(x)a + D(x)$. The action of t on K will be denoted by T_a; we thus have $T_a(x) = S(x)a + D(x)$. In general the structure of ${}_RK$ is given by computing remainders of right division by $t - a$. We thus have for $f(t) \in R$ and $x \in K$, $f(t).x = f(T_a)(x)$. For easy reference we give the following explicit definitions :

Definitions 3.1.

a) $T_a : K \longrightarrow K : x \mapsto S(x)a + D(x)$.

b) For $a \in K$ we define the (S, D)-conjugacy class of a by $\Delta(a) := \{a^x \,|\, x \in K \setminus \{0\}\}$.

c) For $a \in K$, $C^{S,D}(a) := \{x \in K \setminus \{0\} \,|\, a^x = a\} \cup \{0\}$ stands for the (S, D)-centralizer of a.

d) For $a \in K$ and $i \geq 0$, we put $N_i(a) = t^i(a)$.

In the next lemma we collect some properties of the map T_a.

Lemma 3.2. *Let $f(t) = \sum_{i=0}^{n} a_i t^i$ be any element of $R = K[t; S, D]$ and $a \in K$. Then*

1. *$f(a) = \sum_{i=0}^{n} a_i N_i(a)$.*
2. *$f(T_a)(x) = f(a^x)x$.*
3. *$C^{S,D}(a)$ is a subdivision ring of K.*
4. *$f(T_a)$ is a right $C := C^{S,D}(a)$-linear map i.e. $f(T_a) \in Hom_C(K_C)$.*
5. *$Ker(f(T_a)) = \{y \in K \setminus \{0\} \mid f(a^y) = 0\} \cup \{0\}$ is a right $C := C^{S,D}(a)$ vector space denoted $E(f, a)$.*

Proof. 1. $f(a) = \sum_{i=0}^{n} a_i t^i(a) = \sum_{i=0}^{n} a_i N_i(a)$.
2. By the paragraph preceding the definitions in 3.1 we have, for $f(t) \in R$ and $x \in K$, $f(t).x = f(T_a)(x)$ where the left hand side refers to the action of $f(t)$ on x considered as an element of $R/R(t-a)$; i.e. $f(t).x$ stands for the remainder of $f(t)x$ after right division by $t - a$. In other words, $f(t).x = (f(t)x)(a) = f(a^x)x$.
3. This is easy to check.
4. We also leave this to the reader.
5. It is clear from 4. above that $E(f, a)$ is a right C-vector space. Later (Cf.5.4) we will show that $dim_C(E(f, a)) \leq deg(f)$. □

The map T_a is in fact an (S, D)-pseudo-linear transformation. For more information on these maps we refer to ([17]). Notice also that in earlier papers the map $f(T_a)$ above was denoted $\lambda_{f,a}$ (Cf. e.g. [14]). Remark also that the last statement in 3.2 says that the nonzero solutions of the operator equation $f(T_a) = 0$ are the roots of $f(t)$ belonging to $\Delta(a)$. It is thus natural to look at the special case when all the roots belong to one singular (S, D)-class $\Delta(a) := \{a^x \mid x \in K \setminus \{0\}\}$. In particular, $\Delta(0) = \{D(y)y^{-1} \mid y \in K \setminus \{0\}\}$ is the set of so called logarithmic derivatives and the nonzero solutions of a differential polynomial $f = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0$ coincide with the roots of $f(t)$ belonging to $\Delta(0)$ (Cf. [18]).

Assume that $\{x_1 = a^{u_1}, \ldots, x_n = a^{u_n}\}$ are P-independent we then have : $y_i = x_i^{p_{i-1}(x_i)} = (a^{u_i})^{p_{i-1}(a^{u_i})} = a^{p_{i-1}(a^{u_i})u_i} = a^{w_i}$ where, as above, $p_{i-1} = [t - a^{u_j} \mid 1 \leq j \leq i-1]_l$ and

$$w_i = p_{i-1}(a^{u_i})u_i = p_{i-1}(T_a)(u_i)$$

.

Equation (A) in section 2 takes now the form

$$p_n(t) = (t - a^{w_n}) \ldots (t - a^{w_1}) \qquad (D)$$

Applying this decomposition to the pseudo-linear transformation T_a we get :

$$p_n(T_a) = (T_a - a^{w_n}) \dots (T_a - a^{w_1}) \qquad (E)$$

This is the Miura decomposition. This was obtained in the case when $a = 0$ (i.e. for $T_0 = D$) and $S = Id.$ in [5], section 4 using quasideterminants techniques.

Although the equations we obtained are independent of the quasideterminants it is worth to show how we can make them appear. This is one of the objectives of the next section.

4 Quasideterminants, (S,D)-Vandermonde and Wronskian matrices

For a matrix $A = (a_{ij}) \in M_n(R)$, where R is a ring, there might be up to n^2 quasideterminants denoted by $|A|_{ij}$. $|A|_{ij}$ is defined when the matrix A^{ij}, obtained from A by deleting the ith row and the jth column, is invertible. In this case we have:

$$|A|_{ij} = a_{ij} - r^i_j (A^{ij})^{-1} c^j_i \,,$$

where r^i_j is the ith row of A from which the jth element has been suppressed and the c^j_i is the jth column of A from which the ith element has been suppressed.

We need the following special case of a more general result :

Lemma 4.1. *If $A \in M_n(K)$ is invertible and A^{nn} is invertible then*

$$|A|_{nn} = ((A^{-1})_{nn})^{-1}.$$

Proof. Let us write

$$A = \begin{pmatrix} A^{nn} & c \\ l & a_{nn} \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} B & v \\ u & \alpha \end{pmatrix} \,,$$

where c, v are columns, l, u are lines and $a_{nn}, \alpha \in K$. Comparing the last column on both side of $AA^{-1} = Id.$ we get two equations : $A^{nn}v + c\alpha = 0$ and $l.v + a_{nn}\alpha = 1$. These lead to $(a_{nn} - l(A^{nn})^{-1}c)\alpha = 1$ which shows that $\alpha^{-1} = |A|_{nn}$. □

Since we are working in an (S, D)-setting we introduce a generalized Vandermonde matrix as follows (Cf. [12]): let $\{x_1, \dots, x_n\}$ be any subset of K,

$$V_n^{S,D}(x_1, \dots, x_n) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ N_2(x_1) & N_2(x_2) & \dots & N_2(x_n) \\ \dots & \dots & \dots & \dots \\ N_{n-1}(x_1) & N_{n-1}(x_2) & \dots & N_{n-1}(x_n) \end{pmatrix}.$$

Recall that, for $x \in K$, $N_i(x)$ is the evaluation $t^i(x)$. Of course, the N_i's can be computed independently of the evaluation process i.e. in terms of S and D. Indeed, using Lemma 2.1 for $t^{i+1} = tt^i$, one has $N_{i+1}(x) = x^{t^i(x)}t^i(x) = x^{N_i(x)}N_i(x) = S(N_i(x))x + D(N_i(x))$. This gives a recurrence formula for the computation of $N_i(x)$. In the classical setting ($S = Id.$ and $D = 0$) one has $N_i(x) = x^i$. The evaluation of $f(t) = \sum_{i=0}^{n-1} a_i t^i \in K[t; S, D]$ at $x \in K$ can now be expressed as $f(x) = \sum_{i=0}^{n-1} a_i N_i(x)$. Hence, as in the classical case ($S = Id., D = 0$), we have, for $\{x_1, \dots, x_n\} \subseteq K$,

$$(f(x_1), f(x_2), \dots, f(x_n)) = (a_0, a_1, \dots, a_{n-1})V(x_1, \dots, x_n).$$

In [12] we also computed the inverse of an (invertible !) generalized Vandermonde matrix. Let us recall this :
for $1 \le i \le n$, define $g_i(t) := [t - x_j \,|\, 1 \le j \le n, j \ne i]_l$. Since, for every $1 \le i \le n$, $deg(g_i(t)) \le n - 1$ one can write $g_i(t) = \sum_{j=0}^{n-1} c_{ij} t^j$. Consider the matrix $C = (c_{ij})$. Since $g_i(x_j) = 0$ if $i \ne j$, one has

$$CV_n^{S,D}(x_1, \dots, x_n) = diag(g_1(x_1), \dots, g_n(x_n)) \qquad (F)$$

Proposition 4.2. *Suppose that the elements $x_1, \dots, x_n$ are P-independent. Then the Vandermonde matrix $V = V_n^{S,D}(x_1, \dots, x_n)$ is invertible and $|V|_{nn} = g_n(x_n)$, where $g_n = [t - x_i \,|\, i = 1, \dots, n-1]_l$.*

Proof. Let us recall,from section 2, that $p_n(t) := [t - x_i \,|_l\, i = 1, \dots, n]$. Assuming that the elements $x_1, \dots, x_n$ are P-independent, we have that $n = deg(p_n(t)) = deg[t - x_i, g_i(t)]$, for $1 \le i \le n$. This implies that $deg(g_i(t)) = n-1$ and $g_i(x_i) \ne 0$. Hence,

$$(V(x_1, \dots, x_n))^{-1} = diag(g_1(x_1)^{-1}, \dots, g_n(x_n)^{-1})C.$$

Since $g_n(t)$ is a monic polynomial we have $C_{n,n-1} = 1$ and so $(V^{-1})_{n,n} = g(x_n)^{-1}$. Lemma 4.1 shows that $|V|_{nn} = g_n(x_n)$. □

Let us now consider the case of generalized Wronskian matrices. Let us recall that, for $x \in K$, $T_a(x) = S(x)a + D(x)$ (Cf. 3.1). For $u_1, \dots, u_n \in K$, we put :

$$W_{n,a}^{S,D}(u_1, \dots, u_n) = \begin{pmatrix} u_1 & \dots & u_n \\ T_a(u_1) & \dots & T_a(u_n) \\ T_a^2(u_1) & \dots & T_a^2(u_n) \\ \dots & \dots & \dots \\ T_a^{n-1}(u_1) & \dots & T_a^{n-1}(u_n) \end{pmatrix}.$$

Example 4.3. It is worth to mention the special form of the Wronskian matrix when $a = 0,\ S = Id.$. The corresponding Wronskian matrix is of the form :

$$W_{n,0}^{Id.,D}(u_1,\dots,u_n) = \begin{pmatrix} u_1 & \dots & u_n \\ D(u_1) & \dots & D(u_n) \\ D^2(u_1) & \dots & D^2(u_n) \\ \dots & \dots & \dots \\ D^{n-1}(u_1) & \dots & D^{n-1}(u_n) \end{pmatrix},$$

and when $a = 1,\ D = 0$. The corresponding Wronskian matrix is of the form :

$$W_{n,1}^{S,0}(u_1,\dots,u_n) = \begin{pmatrix} u_1 & \dots & u_n \\ S(u_1) & \dots & S(u_n) \\ S^2(u_1) & \dots & S^2(u_n) \\ \dots & \dots & \dots \\ S^{n-1}(u_1) & \dots & S^{n-1}(u_n) \end{pmatrix}.$$

The next lemma establishes connections between the Vandermonde and Wronskian matrices and compute a quasideterminant of this Wronskian.

Proposition 4.4. *Let $u_1,\dots,u_n$ be nonzero elements in K. For any $a \in K$ one has*

1. $V_n^{S,D}(a^{u_1},\dots,a^{u_n})diag(u_1,\dots,u_n) = W_{n,a}^{S,D}(u_1,\dots,u_n)$.
2. $V_n^{S,D}(a^{u_1},\dots,a^{u_n})$ *is invertible if and only if* $W_{n,a}^{S,D}(u_1,\dots,u_n)$ *is invertible.*
3. *If* $W := W_{n,a}^{S,D}(u_1,\dots,u_n)$ *is invertible then we have*
$$|W|_{n,n} = g_n(a^{u_n})u_n = g_n(T_a)(u_n)$$
where $g_n = [t - a^{u_1},\dots,t - a^{u_{n-1}}]_l$.

Proof. 1) Recall that for $f(t) \in K[t;S,D]$ we have proved in 3.2 that $f(T_a)(x) = f(a^x)x$. In particular, $N_i(a^x)x = t^i(a^x)x = T_a^i(x)$. From this, one gets easily the equation relating Vandermonde and Wronskian matrices.
2) This is an obvious consequence of 1)
3) From 1) we obviously get that $(W^{-1})_{nn} = u_n^{-1}(V_n^{S,D}(a^{u_1},\dots,a^{u_n})^{-1})_{nn}$. Lemma 4.1 yields the result. □

These propositions show that the form of Viète, Bezout and Miura decompositions obtained in [3], [4], [5] are special cases (S=Id. D=0) of the one obtained in section 3. Indeed, it suffices to replace the evaluations of least left common multiples by the corresponding quasideterminants to find back the formulas from the papers mentioned above. To be more precise, let us write $V_i := |V_i^{S,D}(x_1,\dots,x_i)|_{i,i}$ then, thanks to Proposition 4.2, the y_i's obtained in equation (A) can be written $y_i = x_i^{V_i}$. Similarly writing $Z_i := |V_i^{S,D}(x_1,\dots,x_{i-1},z)|_{ii}$ the z_i's appearing in (C) are given by $z_i = z^{Z_i}$. Finally, we remark that the w_i's in (D) are equal to $w_i = |W_{i,a}^{S,D}(u_1,\dots,u_i)|_{i,i}$.

Remark 4.5. The fact that the Λ_i^k are symmetric in $x_1, \dots, x_n$ is somewhat surprising when the y_i's are expressed with quasideterminants (i.e. $y_i = x_i^{V_i}$, as above). This fact is obvious from the point of view of least left common multiple as developed in section 2.

We end this section with two easy propositions giving LU-decomposition (this corresponds also to the strict Bruhat normal form) of invertible Vandermonde and Wronskian matrices. Let us first recall that a matrix $A \in GL_n(K)$ can be written in the form $A = LDPU$ where L is lower unitriangular D is diagonal P is a permutation matrix and U is an upper unitriangular matrix. (see [2] p. 128, for details). This form is unique and has some importance nowadays due to its usage in computer packages for solving linear systems of equations. It will be easier for the notations to index the rows of the matrices starting with 0. Notice that, in this case, the diagonal elements of a matrix $a_{i,j}$ are $a_{0,1}, \dots, a_{n-1,n}$. Let us first state, without proof, the following easy lemma :

Lemma 4.6. *Let $\{x_1, \dots, x_n, a\}$ be a subset of K and $\{u_1, \dots, u_n\}$ be a subset of $K \setminus \{0\}$. Let $l \geq 0$ be an integer and $A = (a_{ij}) \in M_{(l+1)\times n}(K)$. Then*

$$AV_n^{S,D}(x_1, \dots, x_n) = (f_i(x_j)) \quad \text{and} \quad AW_{n,a}^{S,D}(u_1, \dots, u_n) = (f_i(T_a)(u_j))$$

where for $0 \leq i \leq l$, $f_i(t) := \sum_{j=1}^{n} a_{ij} t^{j-1}$.

Let $\{x_1, \dots, x_n\} \subset K$ be a P-independent subset of K. Recall from section 2 that $p_i(t) = [t - x_j \,|\, j \leq i]_l = \sum_{k=0}^{i} (-1)^k \Lambda_k^i t^{i-k}$. We define $\Lambda \in M_n(K)$ a lower unitriangular matrix via

$$\Lambda = \begin{pmatrix} 1 = \Lambda_0^0 & 0 & 0 & \dots & 0 \\ -\Lambda_1^1 & 1 = \Lambda_0^1 & 0 & \dots & 0 \\ \Lambda_2^2 & -\Lambda_1^2 & 1 = \Lambda_0^2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ (-1)^{n-1}\Lambda_{n-1}^{n-1} & (-1)^{n-2}\Lambda_{n-2}^{n-1} & (-1)^{n-3}\Lambda_{n-3}^{n-1} & \dots & 1 \end{pmatrix}$$

In the following proposition it is important to notice that the rows of the matrix U are indexed starting with 0 i.e. the first row of U is $(U_{01}, \dots, U_{0n})$.

Proposition 4.7. *Let $\{x_1, \dots, x_n\} \subset K$ be a P-independent set. With the above notations we have:*

1. $\Lambda V = U$ *where* $U_{ij} = \begin{cases} 0 & if \quad i \geq j \\ p_i(x_j) & if \quad i < j \end{cases}$

2. $V = \Lambda^{-1}U = \Lambda^{-1} diag(1, p_1(x_2), \dots, p_{n-1}(x_n))U'$ *where U' is an upper unitriangular matrix.*

Proof. 1. This is obvious since the elements of the i^{th}-row of the matrix Λ are the coefficients of the polynomial $p_i(t) = [t - x_j \,|\, j \leq i]_l$ for $i = 0, \dots, n-1$ ($p_0 = 1$,

see section 2 or equation (B) in section 3). The above Lemma 4.6 then yields the result.
2. Obviously Λ is invertible and the P-independence of $x_1, \ldots, x_n$ shows that $p_i(x_{i+1}) \neq 0$. We can thus define $U' = diag(1, p_1(x_2)^{-1}, \ldots, p_{n-1}(x_{n+1})^{-1})U$ and the last equality follows.

□

Since the Vandermonde and Wronskian matrices are so closely related (Cf. 4.4) it is not surprising that we get a similar result for Wronskian matrices.

Proposition 4.8. *Let $\{u_1, \ldots, u_n\}$ be a subset of nonzero elements in K which are right independent over $C^{S,D}(a)$ for some $a \in K$. Then*

1. $\Lambda W = Z$ *where* $Z_{ij} = \begin{cases} 0 & if \quad i > j \\ p_i(T_a)(u_j) & if \quad i \leq j \end{cases}$
2. $W = (\Lambda)^{-1}Z = \Lambda^{-1}diag(1, p_1(T_a)(u_2), \ldots, p_{n-1}(T_a)(u_n))Z'$ *where* Z' *is an upper unitriangular matrix.*

Proof. The proof, based on the above lemma 4.6, is similar to the proof of the previous proposition. □

Remark 4.9. a) In fact one can get an LU decomposition of a Vandermonde (resp. Wronskian) matrix without assuming that the set $\{x_1, \ldots, x_n\}$ is P-independent (resp. $\{u_1, \ldots, u_n\}$ is right $C^{S,D}(a)$ linearly independent). Indeed there always exist monic polynomials $q_i = \sum_j q_{ij}t^j$ of degree i such that $q_i(x_j) = 0$ for $1 \leq j \leq i$. The matrix $L = (q_{ij})$ of the coefficients of these polynomials will give the invertible lower unitriangular matrix L and $U = LV$ will be an upper triangular matrix.

b) The matrix U can be algorithmically computed and offers a way for testing the invertibility of a Vandermonde matrix. This will be explained at the end of section 5.

5 P-independence and W-polynomials

Let us start this section with the formal definition of a Wedderburn polynomial.

Definition 5.1. A monic polynomial $f(t) \in R = K[t; S, D]$ is a Weddderburn polynomial if there exists $A \subseteq K$ such that $Rf(t) = \cap_{a \in A} R(t - a)$.

The W-polynomials were introduced in [14] and studied in depth in [15] and [16]. They are special kind of fully reducible polynomials (also called completely reducible polynomials by Ore) Cf.[1], [19] for more details. From the definition above, it is clear that the W-polynomials are exactly the polynomials we have been dealing with from the beginning of this paper. In particular, these are exactly the polynomials for which we have presented the symmetric functions an the Viète formulas in section 2 and 3. The different factorizations of these polynomials will be presented in later sections. In the above mentioned papers the second author in collaboration with Lam and Ozturk studied the roots, the factorizations, the products of W -polynomials. It is also clear that these polynomials are related to (S, D)-algebraic sets and a lot of information contained in related works are relevant to W-polynomials. The interested reader can refer to the bibliography mentioned in [14] [15], or [19]. Let us recall, without proofs, a few characterizations of these polynomials :

Proposition 5.2. *For a monic polynomial f of degree n in $R = K[t; S, D]$, the following are equivalent :*

i) f is a W-polynomial.

ii) There exists a subset $\{a_1, \ldots, a_n\}$ in K such that $Rf = \cap_{i=1}^{n} R(t - a_i)$.

iii) There exists a subset $\{a_1, \ldots, a_n\}$ in K such that $V = V_n^{S,D}(a_1, \ldots, a_n)$ is invertible and

$$C_f V = S(V) diag(a_1, \ldots, a_n) + D(V).$$

iv) The companion matrix C_f is (S, D)-diagonalisable i.e. there exists an invertible matrix P in $M_n(K)$ and a diagonal matrix Δ such that $S(P)C_f P^{-1} + D(P)P^{-1} = \Delta$.

Obviously the W-polynomials are strongly related to the notion of P-independency as defined in section 2 definition 2.3. We give now a few characterizations of this notion. Let us recall, from definitions 3.1, that $C^{S,D}(a) = \{0 \neq x \in K \mid a^x = a\} \cup \{0\}$ is a subdivision ring of K called the (S, D)-centralizer of a.

Theorem 5.3. *Let $\{x_1, \ldots, x_n\}$ be a subset of K. The following are equivalent :*

i) $\{x_1, \ldots, x_n\}$ is a P-independent set.

i') For every subset $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$, $\{x_{i_1}, \ldots, x_{i_r}\}$ is P-independent.

ii) $V(x_1, \ldots, x_n)$ is invertible.

ii') For every subset $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$, $V(x_{i_1}, \ldots, x_{i_r})$ is invertible.

If there exists $a \in K$ such that $\{x_1, \ldots, x_n\} \subseteq \Delta(a)$ say $x_i = a^{u_i}$ for $i = 1, \ldots, n$ and $\{u_1, \ldots, u_n\} \subseteq K \setminus \{0\}$. Then the above statements are also equivalent to the following ones :

iii) $\{u_1, \ldots, u_n\}$ is right $C^{S,D}(a)$-independent.

iii') For every subset $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$, $\{u_{i_1}, \ldots, u_{i_r})$ is right $C^{S,D}(a)$-independent.

iv) $W_a^{S,D}(u_1,\dots,u_n)$ is invertible.

iv') For every subset $\{i_1,\dots,i_r\}\subseteq\{1,\dots,n\}$, $W_a^{S,D}(u_{i_1},\dots,u_{i_r})$ is invertible.

Proof. $i) \Longrightarrow ii)$. This was already proved in 4.2.
$ii) \Longrightarrow i)$. If $V_n^{S,D}(x_1,\dots,x_n)$ is invertible but $deg([t-x_i\,|\,i=1,\dots,n]_l)<n$ then we claim that there exists $i\in\{1,\dots,n\}$ such that $g_i(x_i)=0$ (let us recall that $g_i(t):=[t-x_j\,|\,j\neq i]_l$). Indeed, assuming that the $g_1(x_1)\neq 0,\dots,g_n(x_n)\neq 0$, we get that the matrix C in formula (F)(just before 4.2) is invertible. On the other hand, for any $i=1,\dots,n$ $g_i(x_i)\neq 0$ implies that $deg(g_i)+1=deg([g_i(t),t-x_i]_l)=deg([t-x_1,\dots,t-x_n]_l)<n$, by our assumption. Hence $deg(g_i)\leq n-2$, this implies that the last column of the matrix C is zero. This contradicts the invertibility of C and proves the claim. Assume thus that there exists $1\leq i\leq n$ such that $g_i(x_i)=0$. Comparing the ith row on both sides of the equation (F) we get $(c_{i,0}\dots,c_{i,n-1})V_n^{S,D}(x_1,\dots,x_n)=(0,\dots,0)$. Since $g_i\neq 0$ and $V_n^{S,D}(x_1,\dots,x_n)$ is invertible this gives a contradiction.
$i)\Leftrightarrow i')$. It is clear that $i')$ implies $i)$. For the converse implication, let us put $p_n(t):=[t-x_i\,|\,1\leq i\leq n]_l$ and suppose that $deg(p_n)=n$. Define $A:=\{i_1,\dots,i_r\}\subseteq\{1,\dots,n\}$, $A':=\{1,\dots,n\}\setminus A$, $p_A(t):=[t-x_i\,|\,i\in A]_l$ and $p_{A'}(t):=[t-x_j\,|\,j\in A']_l$. We then have $p_n(t)=[t-x_i\,|\,i\in A\cup A']_l=[p_A(t),p_{A'}(t)]_l$ and since $deg(p_n(t))=n$, we easily get that $deg(p_A(t))=|A|=r$, as desired.
$i')\Leftrightarrow ii')$. This is clear from $i)\Leftrightarrow ii)$.
Let us now suppose that $x_1=a^{u_1},\dots,x_n=a^{u_n}$.
$i) \Longrightarrow iii)$. Assume $i)$ holds but $u_1,\dots,u_n$ are right $C^{S,D}(a)$ linearly dependent. Without loss of generality we may assume that $u_n=\sum_{i=1}^{n-1}u_ic_i$ where $c_1,\dots,c_{n-1}\in C^{S,D}(a)$. Let us put $p_{n-1}(t)=[t-x_i\,|\,i=1,\dots,n-1]_l$. Using Lemma 3.2 we obtain $p_{n-1}(a^{u_n})=p_{n-1}(T_a)(u_n)u_n^{-1}=p_{n-1}(T_a)(\sum_{i=1}^{n-1}u_ic_i)u_n^{-1}=\sum_{i=1}^{n-1}p_{n-1}(T_a)(u_i)c_iu_n^{-1}=\sum_{i=1}^{n-1}p_{n-1}(a^{u_i})u_i^{-1}c_iu_n^{-1}=0$ where the last equality comes from the definition of $p_{n-1}(t)$. This shows that $p_{n-1}(x_n)=0$ and contradicts $i)$.
$iii) \Longrightarrow i)$. We prove, by induction on n, that $Ker(p_n(T_a))=\sum_{i=1}^{n}u_iC^{S,D}(a)$ and that $deg(p_n(t))=n$. If $n=1$, $p_1(t)=t-a^{u_1}$ is of degree 1 and $p_1(T_a)(v)=0$ implies that either $v=0$ or $T_a(v)=a^{u_1}v$ which leads to $a^{u_1}=a^v$ and hence $v\in u_1C^{S,D}(a)$.
Suppose $n>1$ and assume, by induction, that $Ker(p_{n-1}(T_a))=\sum_{i=1}^{n-1}u_iC^{S,D}(a)$ and $deg(p_{n-1}(t))=n-1$. If $p_n(t)=p_{n-1}(t)$ we get $u_n\in Ker(p_n(T_a))=Ker(p_{n-1}(T_a))=\sum_{i=1}^{n-1}u_iC^{S,D}$, a contradiction. Hence we must have $p_n(t)\neq p_{n-1}(t)$. Using lemma 2.1 2. and 3.2 one can write $p_n(t)=[t-x_n,p_{n-1}(t)]_l=(t-x_n^{p_{n-1}(x_n)})p_{n-1}(t)=(t-a^{w_n})p_{n-1}(t)$, where $w_n=p_{n-1}(a^{u_n})u_n=(p_{n-1}(T_a))(u_n)$. In particular, $degp_n(t)=n$. The induction hypothesis shows that $Ker(p_{n-1}(T_a))=\sum_{i=1}^{n_1}u_iC^{S,D}(a)\subseteq\sum_{i=1}^{n}u_iC^{S,D}(a)\subseteq Ker(p_n(Ta))$. Moreover, if $v\in Ker(p_n(T_a))\setminus Ker(p_{n-1}(T_a))=\sum_{i=1}^{n-1}u_iC^{S,D}(a)$, we have $T_a(p_{n-1}(T_a)(v))=a^{w_n}p_n(T_a)(v)$. This

leads to $p_{n-1}(T_a)(v) = w_n c = p_{n-1}(T_a)(u_n c)$. Hence $v - u_n c \in Ker(p_{n-1}(T_a)) = \sum_{i=1}^{n-1} u_i C^{S,D}(a)$. This yields the conclusion.
$ii) \Leftrightarrow iv)$. This is extracted from Proposition 4.4.
The other implications are clear. □

Let us recall that for $f \in R = K[t; S, D]$ and $a \in K$, $E(f, a)$ stands for the set $\{x \in K \mid f(a^x) = 0\} \cup \{0\}$. Lemma 3.2 shows that $E(f, a) = Ker(f(T_a))$.

Corollary 5.4. *1. For $f \in R$ and $a \in K$, we have $dim_{C(a)} E(f, a) \leq deg(f)$, where $C(a)$ stands for $C^{S,D}(a)$.*

2. If $u_1, \ldots, u_n \in K$ are right $C^{S,D}(a)$-linearly independent and $p_n(t) = [t - a^{u_i} \mid 1 \leq i \leq n]_l$ then $Ker(p_n(T_a)) = \sum_{i=1}^{n} u_i C^{S,D}(a)$.

Proof. 1. Assume, at the contrary, that $dim_{C(a)} E(f, a) > deg(f) =: n$. This means that there exist $u_1, \ldots, u_{n+1} \in E(f, a)$ which are linearly independent over $C(a)$. Let us put $g := [t - a^{u_1}, \ldots, t - a^{u_{n+1}}]_l$; g is a W-polynomial of degree $n+1$. Since $f(a^{u_i}) = 0$ for $i = 1, \ldots, n+1$, we have $Rf \subseteq \cap_{i=1}^{n+1} R(t - a^{u_i}) = Rg$. Hence $deg(f) \geq n + 1$. This contradiction yields the lemma.

2. This has been shown in the proof of the implication $iii) \Longrightarrow i)$ in the above Theorem 5.3. □

Remarks 5.5. a) The notion of P-independence comes from an abstract dependence relation. Let us recall the definition :

Definition 5.6. Let S be a set. A dependence relation on S is a rule which associates with each finite subset X of S certain elements of S, said to be dependent on X. The following conditions must be satisfied :

(a) If $X = \{x_1, \ldots, x_n\}$, then each x_i is dependent on X.

(b) If z is dependent on $\{y_1, \ldots, y_n\}$ and each y_i is dependent on $\{x_1, \ldots, x_n\}$, then z is dependent on $\{x_1, \ldots, x_n\}$.

(c) If y is dependent on $\{x_1, \ldots, x_n\}$, but not on $\{x_2, \ldots, x_n\}$, then x_1 is dependent on $\{y, x_2, \ldots, x_n\}$.

In our situation an element $y \in K$ is P-dependent on a finite subset X of the division ring K if $[t - x \mid x \in X]_l(y) = 0$. We leave to the reader the easy proof that this defines indeed a dependence relation on K. Actually these notions can be put in the more general frame of 2-firs (Cf. [19]).

b) In [5] section 4.1 the Miura decomposition of differential operators $L(D) = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0$ with coefficients a_i in a division ring K was given. In this paper the authors assumed that :

(a) There exist n independent solutions say $u_1, \ldots, u_n$ of the equation $l(D) = 0$ (over the subdivision ring $KerD$)).

(b) They also assumed that for any subset $\{\imath_1, \dots, i_r\} \subseteq \{1, \dots, n\}$ the Wronskian matrix $W(u_{i_1}, \dots, u_{i_r})$ is invertible.

Lemma 3.2 shows that for $0 \neq x \in K$ and $L(t) \in K[t; Id.D]$, $L(D)(x) = 0$ if and only is $L(0^x) = 0$. Notice also that $KerD = C^{Id.,D}(0)$. The equivalence between $iii)$ and $iv')$ shows that this second hypothesis is uncessary.

As mentioned in remark 4.9 b), it is worth to notice that the matrix U in 4.7 can be algorithmically computed. This is the aim of the next proposition. Moreover it offers a way of testing if a given set $\{x_1, \dots, x_n\} \subseteq K$ is P-independent. The diagonal elements of the matrix U also gives an algorithm for computing a linear factorization of a W-polynomial $[t - x_j \,|\, j = 1, \dots, n]$.

Proposition 5.7. *Let $x_1, \dots, x_n$ be any finite set of elements of K and define inductively the elements $u_{ij} \in K$ for $i = 0, \dots, n-1$ and $j = 1, \dots, n$ as follows: $u_{0j} = 1$ for $j = 1, \dots, n$ and assuming that $u_{i,1}, \dots, u_{i,n}$ have been defined we put*

$$u_{i+1,j} = \begin{cases} 0 & if \quad u_{i,j}u_{i,i+1} = 0 \\ (x_j^{u_{ij}} - x_{i+1}^{u_{i,i+1}})u_{ij} & if \quad u_{ij} \neq 0 \neq u_{i,i+1}. \end{cases} .$$

The following are equivalent :

a) The set $\{x_1, \dots, x_n\}$ is P-independent.

b) $u_{ij} \neq 0$ for all $(i,j) \in \mathbb{N}$ such that $0 \leq i < j \leq n$.

c) $u_{n-1,n} \neq 0$.

In this case, the matrix $U = (u_{ij})$ is the one obtained in Proposition 4.7 and one has : $[t - x_j \,|\, j = 1, \dots, n]_l = (t - x_n^{u_{n-1,n}}) \cdots (t - x_2^{u_{12}})(t - x_1^{u_{01}})$.

Proof. Let us define, for $1 \leq i \leq n$, $p_i(t) := [t - x_j \,|\, j = 1, \dots, i]_l$ and $p_o(t) = 1$.
a) $\Longrightarrow$ *b)*. Suppose that $\{x_1, \dots, x_n\} \subseteq K$ is a P-independent set, then $p_{i+1}(t) = [t - x_{i+1}, p_i(t)]_l = (t - x_{i+1}^{p_i(x_{i+1})})p_i(t)$. Using this formula it is easy to prove, by induction on i $0 \leq i \leq n-1$, that $u_{ij} = p_i(x_j)$. The independence of the set $\{x_1, x_2 \dots, x_n\}$ yields that $u_{ij} \neq 0$ for $j > i$.
b) $\Longrightarrow$ c) This is clear.
c) $\Longrightarrow$ a) Suppose that $u_{n-1,n} \neq 0$. The definition of this element shows that $u_{n-2,n} \neq 0$ and $u_{n-2,n-1} \neq 0$. Continuing this process "backwards" we get that $u_{ij} \neq 0$ for all (i,j) such that $0 \leq i < j \leq n$. Now let us show, by induction on i, that for $0 \leq i < j \leq n$ we have $u_{ij} = p_i(x_j)$. For $i = 0$, we have $u_{0j} = 1 = p_0(x_j)$ (recall that $P_0(t) = 1$). Assume we have proved $u_{ij} = p_i(x_j)$ for all $j > i$ and let us consider $u_{i+1,j}$ for $j > i+1$. We have :
$u_{i+1,j} = (x_j^{u_{ij}} - x_{i+1}^{u_{i,i+1}})u_{ij} = (x_j^{p_i(x_j)} - x_{i+1}^{p_i(x_{i+1})})p_i(x_j) = ((t - x_{i+1}^{p_i(x_{i+1})})p_i(t))(x_j)$ $= [t - x_{i+1}, p_i(t)]_l(x_j) = p_{i+1}(x_j)$. This ends the induction and shows that $p_{i+1}(x_j) = u_{i+1,j} \neq 0$, for $j > i+1$. We conclude that $p_i(x_j) \neq 0$ for $0 \leq i < j \leq n$. In particular, $\{x_1, \dots, x_n\}$ is a P-independent set.

The other statements are now clear. □

6 Linear factorizations of W-polynomials.

Let us introduce some notations and a definition:
For $f \in R = K[t; S, D]$ we denote by $V(f)$ the set of right roots of f i.e. $V(f) := \{a \in K \mid f(a) = 0\} = \{a \in K \mid f(t) \in R(t-a)\}$. In case $S = Id.$ and $D = 0$ it was proved by Gordon and Motzkin (Cf. [6]) that $V(f)$ intersects a finite number of conjugacy classes. This is also true in an (S, D)-setting (Cf. [16]). We can thus write $V(f) = \cup_{i=1}^{r} V_i$ where $V_i = V(f) \cap \Delta^{S,D}(a_i)$. For $i = 1, \ldots, r$ the set $E(f, a_i) := \{x \in K \setminus 0 \mid a_i^x \in V_i\} \cup \{0\}$ is in fact a right vector space over the division ring $C_i = C^{S,D}(a_i) := \{x \in K \setminus \{0\} \mid a_i^x = a_i\} \cup \{0\}$. It was proved in [16] that :

$$\sum_{i=1}^{r} dim_{C_i} E(f, a_i) \leq deg(f)$$

Moreover, in this formula the equality holds if and only if f is a W-polynomial.

It is easy to remark that for any $f \in R$, there exists a W-polynomial $g \in R$ such that $V(f) = V(g)$. For the rest of this section f will stand for a W-polynomial of degree n.

Definition 6.1. Let $f \in R = K[t; S, D]$ be a W-polynomial of degree n. A P-basis for $V(f) := \{x \in K \mid f(x) = 0\}$ is a set $\{x_1, \ldots, x_n\} \subseteq V(f)$ such that $[t - x_i \mid i = 1, \ldots, n]_l = f$.

We are interested in describing all the different linear factorizations of f. Let us first consider the case when all the roots of f are in a single conjugacy class : $\Delta^{S,D}(a)$. If $x_1 = a^{u_1}, \ldots, x_n = a^{u_n}$ is a P-basis for $V(f)$, Theorem 5.3 shows that all the P-bases of f are of the form $a^{v_1}, \ldots, a^{v_n}$ where $(v_1, \ldots, v_n) = (u_1, \ldots, u_n)A$ for some matrix $A \in GL_n(C^{S,D}(a))$. To every ordered P-basis $(x_1, \ldots, x_n)$ we can associate, as in section 2, a factorization of f given by $f(t) = (t - x_n^{z_n}) \cdots (t - x_i^{z_i}) \cdots (t - z_1)$ where $z_i = p_{i-1}(x_i)$ and $p_i = [t - x_1, \ldots, t - x_i]$ for $i = 1, \ldots, n-1$. But different ordered P-bases can lead to the same factorization. Let us give an easy example :

Example 6.2. Let K be a division ring $S = Id.$ and $D = 0$. Suppose that $a \in K$ is such that $dimK_{C(a)} \geq 3$ where $C(a)$ denotes the usual centralizer of a. Let u_1, u_2, u_3 be three elements in K which are right linearly independent over $C(a)$ and consider $f = [t - a^{u_i} \mid i = 1, 2, 3]_l$. The elements $v_1 = u_1, v_2 = u_1 + u_2, v_3 = u_3$ form another right basis of K over $C(a)$. The two ordered bases $(a^{u_1}, a^{u_2}, a^{u_3})$ and $(a^{v_1}, a^{v_2}, a^{v_3})$ are different but give rise to the same factorization. Indeed, putting $x_i := a^{u_i}$ and $y_i := a^{v_i}$ for $1 \leq i \leq 3$, we have $p_1(t) = t - x_1 = t - y_1$, $x_2^{p_1(x_2)} = a^{p_1(x_2)u_2}$ and $y_2^{p_1(y_2)} = a^{p_1(y_2)v_2}$; but $p_1(y_2)v_2 = p_1(a^{v_2})v_2 = p_1(T_a)(v_2) = p_1(T_a)(u_1 + u_2) = p_1(T_a)(u_2) = p_1(a^{u_2})u_2 = p_1(x_2)u_2$ and we conclude that $x_2^{p_1(x_2)} = y_2^{p_1(y_2)}$. It is then easy to check that the two factorizations given by the different ordered P-bases are the same. Details are left to the reader.

The next lemma will be very useful.

Lemma 6.3.

a) $a^{p(T_a)(v)} = a^{p(T_a)(u)}$ *if and only if* $Kerp(T_a) + vC = Kerp(T_a) + uC$.

b) Let $g \in R$ *be any polynomial and* $a, b \in K$ *be distinct elements of* K *such that* $g(a) \neq 0$ *and* $g(b) \neq 0$. *Then* $a^{g(a)} = b^{g(b)}$ *if and only if there exist* $u \in K \setminus C^{S,D}(a)$ *and* $c \in C^{S,D}(a)$ *such that* $b = a^u$ *and* $g(a^{u-c}) = 0$. *In particular, in these circumstances we have* $V(g) \cap \Delta(a) \neq \emptyset$.

Proof. a) We have $a^{p(T_a)(v)} = a^{p(T_a)(u)}$ if and only if there exists $c \in C^{S,D}(a)$ such that $p(T_a)(v) = p(T_a)(u)c$. Since $p(T_a)$ is a right $C^{S,D}(a)$-linear map this is equivalent to $p(T_a)(v - uc) = 0$. This easily gives the result.
b) Since $a^{g(a)} = b^{g(b)}$, we can write $b = a^u$ for some $u \in K \setminus C^{S,D}(a)$. We then have $a^{g(T_a)(1)} = a^{g(a)} = b^{g(b)} = a^{g(T_a)(u)}$, hence there exists $c \in C^{S,D}(a)$ such that $g(T_a)(u) = g(T_a)(1)c = g(T_a)(c)$ and $u - c \in Ker(g(T_a))$ i.e. $g(a^{u-c}) = 0$. The converse implication is left to the reader. □

Theorem 6.4. *Let* $f \in R = K[t; S, D]$ *be a W-polynomial of degree* n *such that* $V(f) \subseteq \Delta^{S,D}(a)$. *The linear factorizations of* f *are in bijection with the complete flags of the right* $C^{S,D}(a)$ *vector space* $E(f, a)$.

Proof. Let $f(t) = (t-a_n)\cdots(t-a_1)$ be a linear factorization of the W-polynomial f. The polynomial $p_i(t) := (t-a_i)(t-a_{i-1})\ldots(t-a_1)$ is also a W-polynomial and the fact that $V(f) \subseteq \Delta^{S,D}(a)$ implies that we can write $Kerp_i(T_a) = \sum u_j C^{S,D}(a)$ for some right $C^{S,D}(a)$-independent elements $u_1, \ldots, u_i$ in $E(p_i, a) \subseteq E(f, a)$ (Cf. Corollary 5.4 and 5.2).
Let ψ be the map from the set of factorizations of f to the set of flags in $E(f, a)$ defined by associating the flag $Ker(p_1(T_a)) \subset Ker(p_2(T_a)) \subset \cdots \subset Ker(p_n(T_a)) = E(f, a)$ to the factorization $f = (t - a_n)\cdots(t - a_1)$.
Let us show that ψ is injective : if $f(t) = (t-a_n)\cdots(t-a_1) = (t-a'_n)\cdots(t-a'_1)$ are two different factorizations of f, we define $l := min\{i \mid a_i \neq a'_i\}$. We thus have $p_i(t) = (t-a_i)\cdots(t-a_1) = (t-a'_i)\cdots(t-a'_1)$ for any $i < l$, $p_l(t) = (t-a_l)p_{l-1}(t)$ and $p'_l(t) = (t - a'_l)p_{l-1}(t)$ (if $l = 1$, $p_{l-1}(t) = p_o(t) = 1$). Notice that $p_l(t)$ and $p'_l(t)$ are W-polynomials. Hence one can write $p_l(t) = [t - a^u, p_{l-1}(t)]$ and $p'_l(t) = [t - a^v, p_{l-1}(t)]$ for some $u, v \in E(f, a)$. This gives $a_l = a^{p_{l-1}(T_a)(u)}$ and $a'_l = a^{p_{l-1}(T_a)(v)}$. Lemma 6.3 a) shows that the flags associated with these two factorizations will be different.
Let us show that ψ is onto. Consider a complete flag $u_1C \subset u_1C + u_2C \subset \cdots \subset \sum_{i=1}^n u_iC = E(f, a)$. We build successively the following right factors of f : $p_o(t) = 1, p_1(t) = t - a_1, \ldots, p_i(t) = [t - a^{u_i}, p_{i-1}(t)]_l$. This will give a factorization of $f = p_n$: $f(t) = (t - a^{p_{n-1}(T_a)(u_n)}(t - a^{p_{n-2}(T_a)(u_{n-1})})\ldots(t - a^{u_1})$. It is then easy to check that ψ maps this factorization on the complete flag we started with.
□

Example 6.5. Let us describe all the factorizations of $f = [t - a^x, t - a]_l$. These factorizations are in bijection with the complete flags in the two dimensional vector space $E(f, a) = C + xC$ where $C := C^{S,D}(a)$. These flags are of the form $0 \neq yC \subset E(f, a)$. Apart from the flag $0 \subset xC \subset E(f, a)$, they are given by $0 \subset (1 + x\beta)C \subset E(f, a)$, where $\beta \in C^{S,D}(a)$. Hence we get the following factorizations $f = (t - a^{a-a^x})(t - a^x)$ and $(t - a^{a-\gamma})(t - a^{1+x\beta})$, where $\gamma = a - a^{1+x\beta}$.

Let us now describe all linear factorizations of a general W-polynomial f (i.e. without assuming that all its roots are in a single conjugacy class). Before stating our last theorem in this section let us fix some notations. For a W-polynomial f we decompose $V(f)$ with respect to conjugacy classes : $V(f) = \bigcup_{i=1}^r V_i$ where $V_i = V(f) \cap \Delta(a_i)$. Consider the semisimple ring $C := \prod_{i=1}^r C_i$ where, for $i = 1, \dots, r$, $C_i := C^{S,D}(a_i)$. K^r has a natural structure of right C-module. Its submodules are all of the form $U_1 \times \cdots \times U_r$, where, for $i = 1, \dots, r$, $U_i \subseteq K$ is a right C_i vector space. In particular, $E(f) := \prod_{i=1}^r E(f, a_i)$ is a right C-module. Notice that the dimensions $dim_{C_i} E(f, a_i) < \infty$.

Definitions 6.6. *Let f be a W-polynomial in $R = K[t; S, D]$.*

a) For a C-submodule $U = U_1 \times U_2 \times \cdots \times U_r$ of $E(f)$ we define its dimension to be $dim(U) := (dim_{C_1} U_1, \dots, dim_{C_r} U_r)$ and its weight to be $wt(U) := \sum_{i=1}^r dim_{C_i} U_i$.

b) A complete flag in $E(f)$ is a sequence of C-submodules $0 \subsetneqq M_1 \subsetneqq M_2 \subsetneqq \cdots \subsetneqq M_n$ such that $n = deg(f)$.

c) For $a = (a_1, \dots, a_r) \in K^r$ we define $T_a : K^r \longrightarrow K^r : (x_1, \dots, x_r) \mapsto (T_{a_1}(x_1), \dots, T_{a_r}(x_r))$.

Let us notice that, by the remark stated at the end of the first paragraph introducing this section, if f is a W-polynomial the weight of the right C-module $E(f)$ is $wt(E(f)) = deg(f)$. Notice also that a sequence of right C-modules $0 \subsetneqq M_1 \subsetneqq M_2 \subsetneqq \cdots \subsetneqq M_n$ is a complete flag if and only if $wt(M_i) = i$.
With these notations we can state the next result which generalizes Theorem 6.4.

Theorem 6.7. *Let f be a W-polynomial. Then :*
There is a bijection between the set of factorizations of f and the set of complete flags in $E(f) = \prod_{i=1}^r E_i$.

Proof. As in the proof of 6.4 we associate to the factorization $f(t) = (t - b_1) \dots (t - b_n)$ the flag of right C-module given by $0 \subsetneqq Ker(p_1(T_a)) \subsetneqq \cdots \subsetneqq Ker(p_n(T_a))$ where $p_i(t) = (t - b_i)(t - b_{i-1}) \cdots (t - b_1)$. Let us show, by induction on i, that $wt(Ker(p_i(T_a))) = i$. If $i = 1$ we have $(0, \dots, 0) \neq (x_1, \dots, x_n) \in Ker(p_1(T_a))$ if and only if $(0, \dots, 0) = (T_a - b_1)(x_1, \dots, x_n) = (S(x_1)a_1 + D(x_1) - b_1 x_1, \dots, S(x_r)a_r + D(x_r) - b_1 x_r)$. This shows that for $1 \leq i \leq r$, we have

$S(x_i)a_i + D(x_i) = b_1x_i$. We can assume that $b_1 \in \Delta(a_1)$, say $b_1 = a_1^{v_1}$. Now assume that for some $j > 1$, $x_j \neq 0$. we then have $b_1 = S(x_j)a_jx_j^{-1} + D(x_j)x_j^{-1}$. In particular $b_1 \in \Delta^{S,D}(a_j)$. This contradicts the fact that $b_1 \in \Delta(a_1)$. We conclude that $Ker(p_1(T_a)) = v_1C^{S,D}(a_1)$, as desired. Assume that $wt(Ker(p_i(T_a))) = i$ and let us show that $wt(Ker(p_{i+1}(T_a))) = i+1$. We have $p_{i+1} = (t - b_{i+1})p_i(t)$. If $v \in Ker(p_{i+1}(T_a)) \setminus Ker(p_i(T_a))$ and $b_{i+1} \in \Delta(a_j)$, say $b_{i+1} = a_j^{v_j}$, for some $j \in \{1, \dots, r\}$ and some $v_j \in K \setminus \{0\}$. As in case $i = 1$ we easily check that $v = (0, \dots, x_j, 0 \dots, 0)$ where $x_j = v_jC^{s,D}(a_j)$. This shows that $wt(Ker(p_{i+1}(T_a))) = wt(Kerp(p_i(T_a)))+1$. The induction hypothesis implies that $wt(Ker(p_{i+1}(T_a))) = i + 1$, as desired. The rest of the proof is completely similar to the one given in 6.4 abd is left to the reader.

□

Remark 6.8. a) Let us remark that $\Delta(a)$ has a structure of right $C^{S,D}(a)$ projective space which is in fact given by the right $C^{S,D}(a)$ vector space structure of K itself (the map $\phi : \mathbb{P}(K_{C(a)}) \longrightarrow \Delta(a) : x \mapsto a^x$ is a bijection). If $V(f) \subseteq \Delta(a)$ then $V(f)$ is a projective subspace of $\mathbb{P}(K)$, the associated vector space being $E(f, a)$.

b) Gelfand, Retakh and Wilson introduced and studied in details an algebra Q_n associated with factorizations of certain polynomials $f(t)$ in the universal field of quotients K (aka. free field) of the free algebra $k < x_1, x_2, \dots, x_n >$ over a commutative field k. The aim is to replace te free field by some "smaller" algebra in which all the factorizations of $f(t)$ already take place. Using the least left common multiple, the introduction and the description of this algebra are very natural. In this language, the polynomial of which we study the factorization is $f(t) = [t - x_i \,|\, i\{1, \dots, n\}]_l \in K[t]$. Let us first fix some notations. For $A \subseteq \{1, \dots, n\}$ we denote $p_A := [t - x_i \,|\, i \in A]_l \in K[t]$. Let $i, j \in \{1, \dots, n\} \setminus A$ we have $p_{A\cup\{i,j\}}(t) = [t - x_j, [t - x_i, p_A(t)]_l]_l = [t - x_i, [t - x_j, p_A(t)]_l]_l$. Putting $x_{A,i} := x_i^{p_A(x_i)}$ and using 2.1 2. we obtain : $(t - x_{A\cup\{i\},j})(t - x_{A,i})p_A(t) = (t - x_{A\cup\{j\},i})(t - x_{A,i})p_A(t)$. This leads to the quadratic relations :

$$x_{A\cup\{j\},i} + x_{A,j} = x_{A\cup\{i\},j} + x_{A,i} \quad \text{and} \quad x_{A\cup\{j\},i}.x_{A,j} = x_{A\cup\{i\},j}.x_{A,i}.$$

The algebra is then describe by $Q_n := k < z_{A,i} \,|\, A \subseteq \{1, \dots, n\} > /I$, where I is the ideal generated by the analogue of the quadratic relations obtained above. All the factorizations we have obtained are also obtainable in $Q_n[t]$. This algebra Q_n have been studied in depth by Gelfand, Retakh and Wilson (Cf.[5]). Notice also that it is possible to formally introduce more generally such an algebra in an (S, D)-setting. This might be an interesting way of looking to factorizations of differential operators.

7 Existence of LLCM

The aim of this short section is to have a nice criterion for a ring A to be such that for any finite subset $\{a_1, \ldots, a_n\} \subseteq A$, there exists a monic polynomial $p(t) \in A[t]$, $deg(p(t)) = n$ such that, for any $i \in \{1, \ldots, n\}$, $t - a_i$ divides $p(t)$ on the right. With our standard notation we can then write $p(t) = [t - a_i \,|\, i = 1, \ldots, n]$. Recall that a ring is called left duo if its left ideals are in fact two-sided ideals.

Lemma 7.1. *Let A, S, D be a ring an endomorphism of A and an S-derivation of A, respectively. Assume that for any $a, b \in A$ there exist $c, d \in A$ such that $(t-c)(t-a) = (t-d)(t-b) \in R = A[t; S, D]$ then for any finite subset $\{a_1, \ldots, a_n\} \subseteq A$ there exists a monic polynomial of degree n in $\cap_{i=1}^{n} R(t - a_i)$.*

Proof. We first construct, for any $1 \le l \le n$, elements $a_{i_1 \ldots i_l}$ with $1 \le i_j \le n$ for all $j = 1 \ldots l$ and $i_j \ne i_k$ if $j \ne k$.
If $l = 1$, a_{i_1} is given and belongs to $\{a_1, \ldots a_n\}$.
For $l \ge 2$ we assume that the elements $a_{i_1 \ldots i_{l-1}} \in A$ have been constructed for any set of distinct indexes $\{i_1, \ldots, i_{l-1}\}$ with $1 \le i_j \le n$ for any $1 \le j \le l-1$.
By hypothesis, there exist elements $a_{i_1 \ldots i_{l-1} i_l}$ and $a_{i_1 \ldots i_{l-2} i_l i_{l-1}}$ in A such that :

$$(t - a_{i_1 \ldots i_l})(t - a_{i_1 \ldots i_{l-1}}) = (t - a_{i_1 \ldots i_{l-2} i_l i_{l-1}})(t - a_{i_1 \ldots i_{l-2} i_l}).$$

This defines $a_{i_1 \ldots i_{l-1} i_l}$ for any subset $\{i_1, \ldots, i_l\}$ of $\{1, \ldots, n\}$. Let us put $f_1(t) := t - a_1$ and $f_l(t) := (t - a_{1 \ldots l}) f_{l-1}(t)$, for $1 < l < n$. First let us show, by induction on l, that for any $l < s \le n$ we have $(t - a_{1 \ldots ls}) f_l \in R(t - a_s)$. For $l = 1$, $f_1 = t - a_1$ and we have $(t - a_{1s})(t - a_1) = (t - a_{s1})(t - a_s) \in R(t - a_s)$. For $l > 1$, we have $(t - a_{1 \ldots ls}) f_l = (t - a_{1 \ldots ls})(t - a_{1 \ldots l}) f_{l-1} = (t - a_{1 \ldots (l-1)sl})(t - a_{1 \ldots (l-1)s}) f_{l-1} \in R(t - a_s)$ by the induction hypothesis.
We can now prove easily, by induction on l, that $f_l \in \cap_{i=1}^{l} R(t - a_i)$. This is left to the reader. f_n is then the required monic polynomial of degree n in $\cap_{i=1}^{n} R(t - a_i)$ □

Theorem 7.2. *Let A, S, D be a ring an endomorphism of A and a S-derivation of A, respectively. For $R = A[t; S, D]$ the following are equivalent :*

1. *For any $\{a_1, \ldots, a_n\} \subseteq A$ there exists a monic polynomial of degree n in $\cap_{i=1}^{n} R(t - a_i) \subseteq R[t]$.*

2. *For any $\{a, b\} \subseteq A$ there exists a monic polynomial of degree two in $R(t - a) \cap R(t - b)$.*

3. *For any $r, b \in A$ there exists $c \in A$ such that $cr = S(r)b + D(r)$.*

If $S = Id.$ and $D = 0$ the above statements are also equivalent to A being left duo.

Proof. 1) $\Leftrightarrow$ 2) This is clear thanks to the above lemma 7.1.

2) $\Longrightarrow$ 3) Let $r, b \in A$. By hypothesis we know that there exists $c, d \in A$ such that $(t-c)(t-(r+b)) = (t-d)(t-b) \in R = A[t; S, D]$. Equating coefficients of the same degree gives $c(r+b) - D(r+b) = db - D(b)$ and $c + S(r+b) = d + S(b)$. Replacing d in the first equality we get $cr - D(r) = S(r)b$, as desired.

3) $\Longrightarrow$ 2) Assume that for any $r, b \in A$ there exists $c \in A$ such that $cr - D(r) = S(r)b$ and let $a, b \in A$ be given. There exists $c \in A$ such that $c(a-b) = S(a-b)b + D(a-b)$. Putting $d := c + S(a-b)$ we get that $(t-d)(t-b) = (t-c)(t-a) \in A[t; S, D]$, as required.
Now, if $S = id.$ and $D = 0$, 3. above is equivalent to the fact that, for any $r, b \in A$, $Ab \in Ar$. This means that Ar is a two-sided ideal. This means that every principal left ideal is in fact two-sided i.e. A is left duo. $\square$

Remark 7.3. The last statement of the above theorem shows that in an S, D setting the equality in 7.2 3. could be considered as a definition for an (S, D) left duo ring.

Example 7.4. Let k be a field. Obviously $M_2(k)$ is not left duo. Indeed it is easy to remark that the matrices

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

are such that $t - a$ and $t - b$ have no left monic common multiple of degree two. Of course this also means that there are no matrices $c, d \in M_2(k)$ such that $c + a = d + b$ and $ca = db$. This also gives that there is no solution to the matrix equation $t^2 - (a+b)t + ab = 0$.

We exhibit now an example showing that the left and right existence of a least monic common multiple are different notions.

Example 7.5. Consider A the subring of upper triangular matrices over $\mathbb{Q}(x)$ define as follows :

$$A = \{ \begin{pmatrix} f(x^2) & g(x) \\ 0 & f(x) \end{pmatrix} \mid f(x), g(x) \in \mathbb{Q}(x) \}$$

Let R denote the usual polynomial rings $R = A[t]$. It is easy to check that there is no monic polynomial of degree 2 in $R(t-a) \cap R(t-b)$ where

$$a = \begin{pmatrix} x^2 & x^2 \\ 0 & x \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} x^2 & 0 \\ 0 & x \end{pmatrix}$$

On the other hand we have $(t-a)(t-c) = (t-b)(t-d)$ where

$$c = \begin{pmatrix} x^4 & 0 \\ 0 & x^2 \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} x^4 & x^2 \\ 0 & x^2 \end{pmatrix}$$

The ring A is of course right but not left duo (Cf [11] Exercise 22 4A p.318).

Let us end this paper with a brief account of some recent developments related to duo rings and Ore extensions.

Remark 7.6. Hirano, Hong, Kim and Park proved in [7] that an ordinary polynomial ring is one-sided duo only if it is commutative. Marks in [20] extended this result to Ore extensions, by showing that if a noncommutative Ore extension which is a duo ring on one side exists, then it has to be right duo, σ must be non-injective and $\delta \neq 0$. He also obtained a series of necessary conditions for the Ore extension to be right duo. Matczuk in [21] showed that noncommutative Ore extensions which are right duo rings do exist and that the necessary conditions obtained by Marks are not sufficient for the Ore extension to be right duo.

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