

Second Modules over Noncommutative Rings

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July 2013

Joint Work With: S.CEKEN- P.F. SMITH
Throughout all rings have identity elements and all modules are unital.

Definitions

i) A right $R$-module $M$ is called prime in case $M \neq 0$ and $\text{ann}_R(M) = \text{ann}_R(N)$ for every non-zero submodule $N$ of $M$.

ii) A right $R$-module $M$ will be called a second module provided $M \neq 0$ and $\text{ann}_R(M) = \text{ann}_R(M/N)$ for every proper submodule $N$ of $M$.

- By a prime submodule of $M$, we mean a submodule $P$ such that the module $M/P$ is prime.
- By a second submodule of $M$, we mean a submodule which is also a second module.
- In [S. Annin Attached primes over noncommutative rings, J. Pure Appl. Algebra 212 (2008), 510-521.] second modules are called coprime.
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Prime submodules

Prime modules and prime submodules of modules have been studied by various authors over the past 30 years

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Let $R$ be a commutative ring and let $M$ be a non-zero $R$-module. Given any element $r \in R$, let $\mu_r : M \to M$ denote the endomorphism of $M$ defined by $\mu_r(m) = rm \ (m \in M)$.

- $M$ is prime if and only if for each $r \in R$ either $\mu_r$ is zero or a monomorphism.

- $M$ is prime if and only if for any $r$ in $R$ and $m$ in $M$, $rm = 0$ implies that $m = 0$ or $rM = 0$.

- $M$ is second if and only if for each $r \in R$ either $\mu_r$ is zero or an epimorphism.

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If \( R \) is any ring and \( M \) is a second \( R \)-module then \( P = \text{ann}_R(M) \) is a prime ideal of \( R \) because if \( MAB = 0 \), for some ideals \( A \) and \( B \) of \( R \), and \( 0 \neq MA \) then we get that \( M = MA \) and so \( MB = 0 \).

In this case, \( M \) is called a \( P \)-second module. Clearly a simple modules are both prime and second modules.
More generally, a homogeneous semisimple modules are both prime and second.
If $R$ is a simple ring then every non-zero module is a prime second module.
Conversely, every ring $R$ such that the right $R$-module $R$ is a second module is simple.
Clearly every non-zero submodule of a prime module is prime and every non-zero homomorphic image of a second module is second.
**Lemma**

Let $R$ be a ring such that every prime ideal is maximal. Then a right $R$-module $M$ is prime if and only if $M$ is second. Moreover, if $R$ is commutative then the module $M$ is second if and only if $M$ is homogeneous semisimple.

**Proof.**

Suppose first that $M$ is prime. Then $M \neq 0$ and $P = \text{ann}_R(M)$ is a prime, and hence maximal ideal of $R$. Let $N$ be any proper submodule of $M$. Then $P \subseteq \text{ann}_R(M/N) \subset R$, so that $P = \text{ann}_R(M/N)$. It follows that $M$ is a second module.

Conversely, if $M$ is a second module then again $P = \text{ann}_R(M)$ is a maximal ideal of $R$. For each non-zero submodule $L$ of $M$ we have $P \subseteq \text{ann}_R(L) \subset R$ and hence $P = \text{ann}_R(L)$. Thus $M$ is a prime module.

Now suppose that $R$ is commutative. If $M$ is a second module then $MP = 0$ for some maximal ideal $P$ of $R$ so that $M$ is homogeneous semisimple.
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Now suppose that $R$ is commutative. If $M$ is a second module then $MP = 0$ for some maximal ideal $P$ of $R$ so that $M$ is homogeneous semisimple.
Corollary

Let $R$ be either a commutative von Neumann regular ring or a right perfect ring. Then a non-zero module $M$ is second if and only if $M$ is homogeneous semisimple.

Lemma

Let $R$ be a ring such that $R/P$ is right Artinian for every right primitive ideal $P$. Then the following statements are equivalent for a module $M$.

1. $M$ is a prime module which contains a simple submodule.
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The following statements are equivalent for a non-zero module $M$.

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2. For every ideal $A$ of $R$ either $MA = 0$ or $M = MA$.
3. $M = MA$ for every ideal $A$ of $R$ not contained in $\text{ann}_R(M)$.
4. $M = MA$ for every ideal $A$ of $R$ properly containing $\text{ann}_R(M)$.

Proof.

$(i) \Rightarrow (ii)$ Suppose that $M \neq MA$ for any ideal $A$ of $R$. Then $MA$ is a proper submodule. If $B = \text{ann}_R(M/MA)$ then $(i)$ gives that $MB = 0$. But we know that $A \subseteq B$ and hence $MA = 0$.

$(ii) \Rightarrow (iii) \Rightarrow (iv)$ Clear.

$(iv) \Rightarrow (i)$ Let $N$ be a proper submodule and let $C = \text{ann}_R(M/N)$. Then $\text{ann}_R(M) \subseteq C$ and $MC \subseteq N \neq M$ so that $C = \text{ann}_R(M)$ and $MC = 0$. Thus $\text{ann}_R(M) = \text{ann}_R(M/N)$ and hence $M$ is second. \qed
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(i) $\Rightarrow$ (ii) Suppose that $M \neq MA$ for any ideal $A$ of $R$. Then $MA$ is a proper submodule. If $B = \text{ann}_R(M/MA)$ then (i) gives that $MB = 0$. But we know that $A \subseteq B$ and hence $MA = 0$.

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Let $P$ be a prime ideal of a ring $R$ and let $N$ be a submodule of a module $M$ such that the modules $N$ and $M/N$ are both $P$-second. Then $M$ is $P$-second if and only if $MP = 0$.

Let $M$ be a $P$-second module for some prime ideal $P$ of $R$. Then every non-zero pure submodule of $M$ is $P$-second.

Let $A$ be an ideal of a ring $R$ and let $M$ be a $R$-module such that $MA = 0$. Then the $R$-module $M$ is a second module if and only if the $(R/A)$-module $M$ is a second module.

Let $P$ be a prime ideal of a commutative ring $R$. Then the sum of any non-empty collection of $P$-second submodules of a $R$-module $X$ is also a $P$-second submodule of $X$. 

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Lemma

Let $R$ be a prime right Goldie ring. Then

1. every non-zero divisible right $R$-module is a second module.
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Let $P$ be a prime ideal of a ring $R$ such that the ring $R/P$ is right Goldie and let $X$ be a non-zero injective right $R$-module. Then $X$ contains a $P$-second submodule if and only if $xP = 0$ for some $0 \neq x \in X$. 
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The Theorems

- Let $R$ be a ring such that $R/P$ is a left bounded left Goldie ring for every prime ideal $P$ of $R$. Then
  
  1. a module $M$ is a second module if and only if $Q = \text{ann}_R(M)$ is a prime ideal of $R$ and $M$ is a divisible right $(R/Q)$-module.
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  3. Let $M$ be a second $R$-module such that every homomorphic image of $M$ is a flat module. Then $M$ is semisimple.
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For an arbitrary ring $R$, let $M$ be a Bass $R$-module, (i.e, every proper submodule is contained in a maximal submodule) Let $P$ be an attached prime of $M$. There exists a proper submodule $N$ of $M$ such that $M/N$ is $P$-second.

Let $L$ be a maximal submodule of $M$ such that $N \subseteq L$. Then $P = \text{ann}_R(M/N) = \text{ann}_R(M/L)$ and hence $P$ is a right primitive ideal of $R$. Thus every attached prime ideal of a Bass module is right primitive.
Propositions

- Let $R$ be a semilocal ring. Then every Bass $R$-module has a finite number of attached prime ideals.

- Let $M$ be a non-zero $R$-module such that there exists an ideal $P$ of $R$ maximal in the collection of ideals $A$ of $R$ such that $M \neq MA$. Then $P$ is an attached prime ideal of $M$ and $M/MP$ is a $P$-second module.

- Let $M$ be a non-zero $R$-module. Then there exists a proper submodule $N$ of $M$ such that $M/N$ is a second module if and only if there exist a proper submodule $L$ of $M$ and a prime ideal $P$ of $R$ such that $P$ is maximal in the collection of ideals $A$ of $R$ such that $M \neq MA + L$. 
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A second submodule $L$ of a module $M$ is called a maximal second submodule if $L$ is not contained in another second submodule of $M$.

- Let $N_i (i \in I)$ be chain of second submodules of a right modules $M$. Then $N = \bigcup_{i \in I} N_i$ is a second submodule of $M$.

- Then every second submodule of a nonzero module $M$ is contained in a maximal second submodule of $M$.

- Every non-zero Artinian module contains a maximal second submodules.
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### Theorem

*Every non-zero Artinian module contains only a finite number of maximal second submodules.*

### Proof.

Suppose the result is false.  
Let $M$ be a non-zero Artinian right $R$-module such that $M$ does not contain a finite number of maximal second submodules.  
Let $N$ be a non-zero submodule of $M$ minimal with respect to the property that $N$ does not contain a finite number of maximal second submodules.  
Clearly $N$ is not a second module.  
Then there exists an ideal $A$ of $R$ such that $NA \neq 0$ and $N \neq NA$.  
Let $L = \{ x \in N : xA = 0 \}$. Then $L$ is a submodule of $N$ such that $LA = 0$ and hence $L \neq N$. 

Proof.

Suppose that $L \neq 0$. By the choice of $N$, $L$ contains only a finite number of maximal second submodules $L_i \ (1 \leq i \leq n)$, for some positive integer $n$, and $NA$ contains only a finite number of maximal second submodules $K_j \ (1 \leq j \leq t)$, for some positive integer $t$.

Let $H$ be a maximal second submodule of $N$. Then we get that either $HA = 0$ or $H = HA$.

If $HA = 0$ then $H \subseteq L$ and hence $H \subseteq L_i$ for some $1 \leq i \leq n$ and it follows that $H = L_i$.

If $H = HA$ then $H \subseteq NA$ so that $H \subseteq K_j$ for some $1 \leq j \leq t$. In this case, $H = K_j$.

Thus every maximal second submodule of $N$ belongs to the list $L_1, \ldots, L_n, K_1, \ldots, K_t$ of submodules of $N$.

Thus $N$ has at most $n + t$ maximal second submodules, a contradiction.

Now suppose that $L = 0$. In this case, $H = K_j$ for some $1 \leq j \leq t$ and again $N$ has at most a finite number of maximal second submodules. The result follows.


Thank you for your attentions