Characterizations of left orders in left Artinian rings

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*talk-genGoldie-12.tex
$R$ is a ring with 1,

$\mathcal{C} = \mathcal{C}_R$ is the set of regular elements of $R$,

$Q = Q_{l,cl}(R) := \mathcal{C}^{-1}R$ is the left quotient ring (the classical left ring of fractions) of $R$ (if it exists),

$n$ is a prime radical of $R$ and $\nu$ is its nilpotency degree ($n^\nu \neq 0$ but $n^{\nu+1} = 0$),

$\overline{R} := R/n$ and $\pi : R \to \overline{R}$, $r \mapsto \overline{r} = r + n$,

$\overline{\mathcal{C}} := \mathcal{C}_{\overline{R}}$ is the set of regular elements of $\overline{R}$,

$\overline{Q} := \overline{\mathcal{C}}^{-1}\overline{R}$ is its left quotient ring,

$\mathcal{C}' := \pi^{-1}(\overline{\mathcal{C}}) := \{c \in R \mid c + n \in \overline{\mathcal{C}}\}$,

$Q' := \mathcal{C}'^{-1}R$. 

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A ring $R$ is a left Goldie ring if

(i) $R$ satisfies ACC for left annihilators,

(ii) $R$ contains no infinite direct sums of left ideals.

**Thm (Goldie, 1958, 1960).** A ring $R$ is a semiprime left Goldie ring iff it has an Artinian left quotient ring which is semi-simple.


**Question:** When $Q$ does exist and is a left Artinian ring?


In all the proofs of the criteria above Goldie’s Thm is used.
**Theorem.** Let $A$ be a left Artinian ring and $\tau$ be its radical. Then

1. The radical $\tau$ of $A$ is a nilpotent ideal.

2. The factor ring $A/\tau$ is semi-simple.

3. An $A$-module $M$ is semi-simple iff $\tau M = 0$.

4. There is only finite number of non-isomorphic simple $A$-modules.

5. The ring $A$ is a left noetherian ring.
Robson’s Criterion.

Let \( W \) be the sum of all the nilpotent ideals of the ring \( R \).

**Theorem (Robson, 1967).** \( TFAE \)

1. The ring \( R \) has a left Artinian left quotient ring \( Q \).

2. (a) The ring \( R \) is \( W \)-reflective,

   (b) the ring \( R \) is \( W \)-quorite,

   (c) \( R/W \) is a left Goldie ring,

   (d) \( W \) is a nilpotent ideal of \( R \), and

   (e) the ring \( R \) satisfies ACC on \( C \)-closed left ideals.
$R$ is $W$-reflective if, for $c \in R$, then $c \in C$ iff $c + W \in C_{R/W}$ ($\Leftrightarrow C' = C$).

$R$ is $W$-quorite if, given $w \in W$ and $c \in C$, there exist $c' \in C$ and $w' \in W$ s.t. $c'w = w'c$.

A l.ideal $I$ of $R$ is $C$-closed if $cr \in I$, where $c \in C$ and $r \in R$, then $r \in I$.

Thm (Small’s Criterion, 1966, 1966) \( TFAE \)

1. \( R \) has a left Artinian left quotient ring \( Q \).

2. (a) \( R \) is a left Goldie ring,

(b) \( W \) is a nilpotent ideal of \( R \),

(c) for all \( k \geq 1 \), \( R/(r(W^k) \cap W) \) is a left Goldie ring,

(d) \( r + W \in \mathcal{C}_{R/W} \implies r \in \mathcal{C} \) (i.e. \( \mathcal{C}' \subseteq \mathcal{C} \)).
Thm (Hajarnavis, 1972) TFAE

1. $R$ has a left Artinian left quotient ring $Q$.

2. (a) $R$ and $R/W$ are left Goldie rings,
   
   (b) $W$ is a nilpotent ideal of $R$,

   (c) for all $k \geq 1$, $R/r(W^k)$ has finite left uniform dimension,

   (d) $r + W \in C_{R/W} \implies r \in C$ (i.e. $C' \subseteq C$).

His approach is very close to Small’s but improvement has been done by using some of the results of Goldie and Talintyre.
Suppose that \( \overline{R} := R/n \) is a (semiprime) left Goldie ring.

By Goldie’s Thm, \( \overline{Q} := \overline{C}^{-1} \overline{R} \) is a semi-simple (Artinian) ring.

The \( n \)-adic filtration: \( R \supset n \supset \cdots \supset n^i \supset \cdots \)

\( \text{gr } R = \overline{R} \oplus n/n^2 \oplus \cdots \oplus n^i/n^{i+1} \oplus \cdots \)

For \( i \geq 1 \), \( \tau_i := \text{tor}_C(n^i/n^{i+1}) := \{u \in n^i/n^{i+1} | \overline{c}u = 0 \text{ for some } \overline{c} \in \overline{C}\} \) is the \( \overline{C} \)-torsion submodule of the left \( \overline{R} \)-module \( n^i/n^{i+1} \).

\( \tau_i \) is an \( \overline{R} \)-bimodule. Then the \( \overline{R} \)-bimodule \( f_i := (n^i/n^{i+1})/\tau_i \) is a \( \overline{C} \)-torsion free, left \( \overline{R} \)-module.

There is a unique ideal \( t_i \) of \( R \) s. t. \( n^{i+1} \subseteq t_i \subseteq n^i \) and \( t_i/n^{i+1} \equiv \tau_i \). Clearly, \( f_i \simeq n^i/t_i \).
Thm (B., 2012) TFAE

1. The ring \( R \) has a left Artinian left quotient ring \( Q \).

2. (a) The ring \( \overline{R} \) is a left Goldie ring,
(b) \( n \) is a nilpotent ideal,
(c) \( C' \subseteq C \),
(d) the left \( \overline{R} \)-modules \( f_i \), where \( i \geq 1 \), contain no infinite direct sums of nonzero submodules, and
(e) for every element \( c \in \overline{C} \), the map \( \cdot c : f_i \rightarrow f_i, f \mapsto f \overline{c} \), is an injection.

If one of the equivalent conditions holds then \( C = C' \), \( C^{-1}n \) is the prime radical of the ring \( Q \) which is a nilpotent ideal of nilpotency degree \( \nu \), and the map \( Q/C^{-1}n \rightarrow \overline{Q}, c^{-1}r \mapsto \overline{c^{-1}r} \), is a ring isomorphism with the inverse \( \overline{c^{-1}r} \mapsto c^{-1}r \).
Corollary. Let $R$ be a left Noetherian ring. \textit{TFAE}

1. $R$ has a left Artinian left quotient ring.

2. $C' \subseteq C$.

3. For each element $\alpha \in \overline{C}$, there exists an element $c = c(\alpha) \in C$ such that $\alpha = c + n$.

$1 \Leftrightarrow 2$ is due to Small (1966).
**Corollary.** Let $R$ be a commutative ring. TFAE

1. The ring $R$ has an Artinian quotient ring.

2. (a) The ring $\bar{R}$ is a Goldie ring.
   (b) $n$ is a nilpotent ideal.
   (c) $C' \subseteq C$.
   (d) The $\bar{R}$-modules $f_i$, $1 \leq i \leq \nu$, contain no infinite direct sums of nonzero submodules.

3. (a) The ring $\bar{R}$ is a Goldie ring.
   (b) $n$ is a nilpotent ideal.
   (c) For each element $\alpha \in \bar{C}$, there exists an element $c = c(\alpha) \in C$ such that $\alpha = c + n$.
   (d) The $\bar{R}$-modules $f_i$, $1 \leq i \leq \nu$, contain no infinite direct sums of nonzero submodules.
4. \( R \) is a Goldie ring and \( C' \subseteq C \).

5. \( R \) is a Goldie ring and, for each element \( \alpha \in \overline{C} \), there exists an element \( c = c(\alpha) \in C \) such that \( \alpha = c + n \).

1 \( \Leftrightarrow \) 4 P. F. Smith (1972).
Theorem (B., 2012) Let $R$ be a ring. TFAE

1. The ring $R$ has a left Artinian ring left quotient ring $Q$.

2. The set $\overline{C}$ is a left denominator set in the ring $\text{gr } R$, $\overline{C}^{-1}\text{gr } R$ is a left Artinian ring, $\mathfrak{n}$ is a nilpotent ideal and $C' \subseteq C$.

3. The set $\overline{C}$ is a left denominator set in the ring $\text{gr } R$, the left quotient ring $Q(\text{gr } R/\tau)$ of the ring $\text{gr } R/\tau$ is a left Artinian ring, $\mathfrak{n}$ is a nilpotent ideal and $C' \subseteq C$.

If one of the equivalent conditions holds then $\text{gr } Q \simeq Q(\text{gr } R/\tau) \simeq \overline{C}^{-1}\text{gr } R$ where $\text{gr } Q$ is the associated graded ring with respect to the prime radical filtration.
Criteria similar to Robson’s criterion

Theorem (B., 2012) Let $R$ be a ring. TFAE

1. The ring $R$ has a left Artinian left quotient ring $Q$.

2. (a) The ring $\overline{R}$ is a left Goldie ring.

   (b) $n$ is a nilpotent ideal.

   (c) $C' \subseteq C$.

   (d) If $c \in C'$ and $n \in n$ then there exist elements $c_1 \in C'$ and $n_1 \in n$ such that $c_1 n = n_1 c$.

   (e) The ring $R$ satisfies ACC for $C'$-closed left ideals.
A left quotient ring of a factor ring

Theorem (B., 2012) Let $R$ be a ring with a left Artinian left quotient ring $Q$, and $I$ be a $C$-closed ideal of $R$ such that $I \subseteq \mathfrak{n}$. Then

1. The set $C_{R/I}$ of regular elements of the ring $R/I$ is equal to the set $\{c + I \mid c \in C\}$.

2. The ring $R/I$ has a left Artinian left quotient ring $Q(R/I)$ and the map $Q/C^{-1}I \to Q(R/I)$, $c^{-1}r + C^{-1}I \mapsto (c + I)^{-1}(r + I)$, is a ring isomorphism with the inverse $(c + I)^{-1}(r + I) \mapsto c^{-1}r + C^{-1}I$. 
