# Characterizations of left orders in left Artinian rings

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$R$ is a ring with 1,

$\mathcal{C} = \mathcal{C}_R$ is the set of regular elements of $R$,

$Q = Q_{l,cl}(R) := \mathcal{C}^{-1}R$ is the **left quotient ring** (the **classical left ring of fractions**) of $R$ (if it exists),

$\mathfrak{n}$ is a prime radical of $R$ and $\nu$ is its nilpotency degree ($\mathfrak{n}^\nu \neq 0$ but $\mathfrak{n}^{\nu+1} = 0$),

$\overline{R} := R/\mathfrak{n}$ and $\pi : R \to \overline{R}$, $r \mapsto \overline{r} = r + \mathfrak{n}$,

$\overline{\mathcal{C}} := \mathcal{C}_{\overline{R}}$ is the set of regular elements of $\overline{R}$,

$\overline{Q} := \overline{\mathcal{C}}^{-1}\overline{R}$ is its left quotient ring,

$\mathcal{C}' := \pi^{-1}(\overline{\mathcal{C}}) := \{c \in R \mid c + \mathfrak{n} \in \overline{\mathcal{C}}\}$,

$Q' := \mathcal{C}'^{-1}R$.

A ring $R$ is a **left Goldie ring** if

(i) $R$ satisfies ACC for left annihilators,

(ii) $R$ contains no infinite direct sums of left ideals.

**Thm (Goldie, 1958, 1960)**. *A ring $R$ is a semiprime left Goldie ring iff it has an Artinian left quotient ring which is semi-simple.*

Lessieur and Croisot (1959): prime case.

**Question**: *When $Q$ does exist and is a left Artinian ring?*

Answer: Small (1966), Robson (1967), Tachikawa (1971), Hajarnavis (1972) and Bavula (2012).

In all the proofs of the criteria above Goldie's Thm is used.

**Theorem**. *Let $A$ be a left Artinian ring and $\mathfrak{r}$ be its radical. Then*

*1. The radical $\mathfrak{r}$ of $A$ is a nilpotent ideal.*

*2. The factor ring $A/\mathfrak{r}$ is semi-simple.*

*3. An $A$-module $M$ is semi-simple iff $\mathfrak{r}M = 0$.*

*4. There is only finite number of non-isomorphic simple $A$-modules.*

*5. The ring $A$ is a left noetherian ring.*

## **Robson's Criterion**.

Let $W$ be the sum of all the nilpotent ideals of the ring $R$.

**Theorem (Robson, 1967)**. *TFAE*

*1. The ring $R$ has a left Artinian left quotient ring $Q$.*

*2. (a) The ring $R$ is $W$-reflective,*

*(b) the ring $R$ is $W$-quorite,*

*(c) $R/W$ is a left Goldie ring,*

*(d) $W$ is a nilpotent ideal of $R$, and*

*(e) the ring $R$ satisfies ACC on $\mathcal{C}$-closed left ideals.*

$R$ is $W$-**reflective** if, for $c \in R$, then $c \in \mathcal{C}$ iff $c + W \in \mathcal{C}_{R/W}$ ($\Leftrightarrow \mathcal{C}' = \mathcal{C}$).

$R$ is $W$-**quorite** if, given $w \in W$ and $c \in \mathcal{C}$, there exist $c' \in \mathcal{C}$ and $w' \in W$ s.t. $c'w = w'c$.

A l.ideal $I$ of $R$ is $\mathcal{C}$-**closed** if $cr \in I$, where $c \in \mathcal{C}$ and $r \in R$, then $r \in I$.

Robson's Criterion is based on the work of Feller and Swokowski (1961, 1961, 1961) and Talintyre (1963).

**Thm (Small's Criterion, 1966, 1966)** *TFAE*

*1. $R$ has a left Artinian left quotient ring $Q$.*

*2. (a) $R$ is a left Goldie ring,*

*(b) $W$ is a nilpotent ideal of $R$,*

*(c) for all $k \geq 1$, $R/(r(W^k) \cap W)$ is a left Goldie ring,*

*(d) $r + W \in \mathcal{C}_{R/W} \implies r \in \mathcal{C}$ (i.e. $\mathcal{C}' \subseteq \mathcal{C}$).*

**Thm (Hajarnavis, 1972)** *TFAE*

*1. $R$ has a left Artinian left quotient ring $Q$.*

*2. (a) $R$ and $R/W$ are left Goldie rings,*

*(b) $W$ is a nilpotent ideal of $R$,*

*(c) for all $k \geq 1$, $R/r(W^k)$ has finite left uniform dimension,*

*(d) $r + W \in \mathcal{C}_{R/W} \implies r \in \mathcal{C}$ (i.e. $\mathcal{C}' \subseteq \mathcal{C}$).*

His approach is very close to Small's but improvement has been done by using some of the results of Goldie and Talintyre.

Suppose that $\overline{R} := R/\mathfrak{n}$ is a *(semiprime) left Goldie ring*.

By Goldie's Thm, $\overline{Q} := \overline{\mathcal{C}}^{-1}\overline{R}$ is a semi-simple (Artinian) ring.

The $\mathfrak{n}$-*adic filtration*: $R \supset \mathfrak{n} \supset \cdots \supset \mathfrak{n}^i \supset \cdots$

$\mathrm{gr}\, R = \overline{R} \oplus \mathfrak{n}/\mathfrak{n}^2 \oplus \cdots \oplus \mathfrak{n}^i/\mathfrak{n}^{i+1} \oplus \cdots$

For $i \geq 1$, $\tau_i := \mathrm{tor}_{\overline{\mathcal{C}}}(\mathfrak{n}^i/\mathfrak{n}^{i+1}) := \{u \in \mathfrak{n}^i/\mathfrak{n}^{i+1} \mid \bar{c}u = 0 \text{ for some } \bar{c} \in \overline{\mathcal{C}}\}$ is the $\overline{\mathcal{C}}$-*torsion submodule* of the *left* $\overline{R}$-module $\mathfrak{n}^i/\mathfrak{n}^{i+1}$.

$\tau_i$ is an $\overline{R}$-bimodule. Then the $\overline{R}$-bimodule $\mathfrak{f}_i := (\mathfrak{n}^i/\mathfrak{n}^{i+1})/\tau_i$ is a $\overline{\mathcal{C}}$-*torsion free*, left $\overline{R}$-module.

There is a unique ideal $\mathfrak{t}_i$ of $R$ s. t. $\mathfrak{n}^{i+1} \subseteq \mathfrak{t}_i \subseteq \mathfrak{n}^i$ and $\mathfrak{t}_i/\mathfrak{n}^{i+1} = \tau_i$. Clearly, $\mathfrak{f}_i \simeq \mathfrak{n}^i/\mathfrak{t}_i$.

**Thm (B., 2012)** *TFAE*

*1. The ring $R$ has a left Artinian left quotient ring $Q$.*

*2. (a) The ring $\overline{R}$ is a left Goldie ring,*

*(b) $\mathfrak{n}$ is a nilpotent ideal,*

*(c) $\mathcal{C}' \subseteq \mathcal{C}$,*

*(d) the left $\overline{R}$-modules $\mathfrak{f}_i$, where $i \geq 1$, contain no infinite direct sums of nonzero submodules, and*

*(e) for every element $\overline{c} \in \overline{\mathcal{C}}$, the map $\cdot\overline{c} : \mathfrak{f}_i \to \mathfrak{f}_i$, $f \mapsto f\overline{c}$, is an injection.*

*If one of the equivalent conditions holds then $\mathcal{C} = \mathcal{C}'$, $\mathcal{C}^{-1}\mathfrak{n}$ is the prime radical of the ring $Q$ which is a nilpotent ideal of nilpotency degree $\nu$, and the map $Q/\mathcal{C}^{-1}\mathfrak{n} \to \overline{Q}$, $c^{-1}r \mapsto \overline{c}^{-1}\overline{r}$, is a ring isomorphism with the inverse $\overline{c}^{-1}\overline{r} \mapsto c^{-1}r$.*

**Corollary**. *Let $R$ be a left Noetherian ring. TFAE*

*1. $R$ has a left Artinian left quotient ring.*

*2. $\mathcal{C}' \subseteq \mathcal{C}$.*

*3. For each element $\alpha \in \overline{\mathcal{C}}$, there exists an element $c = c(\alpha) \in \mathcal{C}$ such that $\alpha = c + \mathfrak{n}$.*

$1 \Leftrightarrow 2$ is due to Small (1966).

**Corollary**. *Let $R$ be a commutative ring. TFAE*

*1. The ring $R$ has an Artinian quotient ring.*

*2. (a) The ring $\overline{R}$ is a Goldie ring.*

*(b) $\mathfrak{n}$ is a nilpotent ideal.*

*(c) $\mathcal{C}' \subseteq \mathcal{C}$.*

*(d) The $\overline{R}$-modules $\mathfrak{f}_i$, $1 \leq i \leq \nu$, contain no infinite direct sums of nonzero submodules.*

*3. (a) The ring $\overline{R}$ is a Goldie ring.*

*(b) $\mathfrak{n}$ is a nilpotent ideal.*

*(c) For each element $\alpha \in \overline{\mathcal{C}}$, there exists an element $c = c(\alpha) \in \mathcal{C}$ such that $\alpha = c + \mathfrak{n}$.*

*(d) The $\overline{R}$-modules $\mathfrak{f}_i$, $1 \leq i \leq \nu$, contain no infinite direct sums of nonzero submodules.*

*4. $R$ is a Goldie ring and $\mathcal{C}' \subseteq \mathcal{C}$.*

*5. $R$ is a Goldie ring and, for each element $\alpha \in \overline{\mathcal{C}}$, there exists an element $c = c(\alpha) \in \mathcal{C}$ such that $\alpha = c + \mathfrak{n}$.*

$1 \Leftrightarrow 4$ P. F. Smith (1972).

# Associated graded ring

**Theorem (B., 2012)** *Let $R$ be a ring. TFAE*

*1. The ring $R$ has a left Artinian ring left quotient ring $Q$.*

*2. The set $\overline{\mathcal{C}}$ is a left denominator set in the ring* $\mathrm{gr}\, R$, $\overline{\mathcal{C}}^{-1}\mathrm{gr}\, R$ *is a left Artinian ring, $\mathfrak{n}$ is a nilpotent ideal and $\mathcal{C}' \subseteq \mathcal{C}$.*

*3. The set $\overline{\mathcal{C}}$ is a left denominator set in the ring* $\mathrm{gr}\, R$, *the left quotient ring* $Q(\mathrm{gr}\, R/\tau)$ *of the ring* $\mathrm{gr}\, R/\tau$ *is a left Artinian ring, $\mathfrak{n}$ is a nilpotent ideal and $\mathcal{C}' \subseteq \mathcal{C}$.*

*If one of the equivalent conditions holds then* $\mathrm{gr}\, Q \simeq Q(\mathrm{gr}\, R/\tau) \simeq \overline{\mathcal{C}}^{-1}\mathrm{gr}\, R$ *where* $\mathrm{gr}\, Q$ *is the associated graded ring with respect to the prime radical filtration.*

# Criteria similar to Robson's criterion

**Theorem (B., 2012)** *Let $R$ be a ring. TFAE*

*1. The ring $R$ has a left Artinian left quotient ring $Q$.*

*2. (a) The ring $\overline{R}$ is a left Goldie ring.*

*(b) $\mathfrak{n}$ is a nilpotent ideal.*

*(c) $\mathcal{C}' \subseteq \mathcal{C}$.*

*(d) If $c \in \mathcal{C}'$ and $n \in \mathfrak{n}$ then there exist elements $c_1 \in \mathcal{C}'$ and $n_1 \in \mathfrak{n}$ such that $c_1 n = n_1 c$.*

*(e) The ring $R$ satisfies ACC for $\mathcal{C}'$-closed left ideals.*

# A left quotient ring of a factor ring

**Theorem (B., 2012)** *Let $R$ be a ring with a left Artinian left quotient ring $Q$, and $I$ be a $\mathcal{C}$-closed ideal of $R$ such that $I \subseteq \mathfrak{n}$. Then*

*1. The set $\mathcal{C}_{R/I}$ of regular elements of the ring $R/I$ is equal to the set $\{c+I \mid c \in \mathcal{C}\}$.*

*2. The ring $R/I$ has a left Artinian left quotient ring $Q(R/I)$ and the map $Q/\mathcal{C}^{-1}I \to Q(R/I)$, $c^{-1}r+\mathcal{C}^{-1}I \mapsto (c+I)^{-1}(r+I)$, is a ring isomorphism with the inverse $(c+I)^{-1}(r+I) \mapsto c^{-1}r+\mathcal{C}^{-1}I$.*