Notions of isoclinism for rings, with applications

3 July 2013

Steve Buckley
Dept of Maths and Stats, NUI Maynooth, Ireland

Noncommutative Rings and their Applications
Lens, 1–4 July 2013
What is isoclinism?

- P. Hall introduced **isoclinism** for groups (1940).
- He generalized it to **isologism** wrt a group variety (also 1940).
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- Lie algebras (Moneyhun, 1994).
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- Here we discuss a different, and flexible, concept of isoclinism/isologism.
- Our concept is defined in a universal algebra context but has various applications in combinatorial ring theory.
The formal “noncommutative polynomial” \( f(X, Y) = aXY + bYX, \ a, b \in \mathbb{Z} \), is a symbol of

\[
R \times R \to R,
\]
defined whenever \( R \) is a PN (\( = \) possibly nonassociative) ring. Now let

\[
\text{Pr}_f(R) := \frac{|\{(x, y) \in R \times R : f^R(x, y) = 0\}|}{|R|^2} \quad \text{(if } |R| < \infty\text{)}.
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Associativity and spectra

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The \( f\)-spectrum of a class \( C \) of finite PN rings is now

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\mathcal{S}_f(C) := \{Pr_f(R) \mid R \in C\}.
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Applications of isoclinism

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Associativity makes no difference for any of these spectra!

**Theorem 1 (B.)**

If \( C_1 := \{\text{finite rings}\}, \ C_2 := \{\text{finite PN rings}\}, \) and \( f(X, Y) := aXY + bYX, \ a, b \in \mathbb{Z}, \) then

\[
\mathcal{S}_f(C_1) = \mathcal{S}_f(C_2).
\]
Applications of isoclinism

Spectral containments

We use special names and notation for \( \text{Pr}_f(R) \) and \( \mathcal{S}_f(C) \) in connection with two fundamental functions \( f \) of this type.

- \( f(X, Y) = XY - YX \): commuting probability \( \text{Pr}_c(R) \) and commuting spectrum \( \mathcal{S}_c(C) \);
- \( f(X, Y) = XY \): annihilating probability \( \text{Pr}_\text{ann}(R) \) and annihilating spectrum \( \mathcal{S}_\text{ann}(C) \).

**Theorem 2 (B.)**

If \( C := \{ \text{finite rings} \} \), and \( f(X, Y) := aXY + bYX \), \( a, b \in \mathbb{Z} \), then

\[
\mathcal{S}_f(C) \subseteq \mathcal{S}_\text{ann}(C).
\]
Definitions

The **commuting probability** of a finite ring $R$ is

$$\text{Pr}_c(R) := \frac{|\{(x, y) \in R \times R : xy = yx\}|}{|R|^2}$$

If $G$ is a finite group, similarly define $\text{Pr}_c(G)$. The commuting spectrum $\mathcal{S}_c(C)$ is defined as before.
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- $\mathcal{R} = \mathcal{S}_c\{\text{finite (possibly non-unital) rings}\}$
- $\mathcal{R}_p = \mathcal{S}_c\{\text{finite (possibly non-unital) } p\text{-rings}\}$, $p$ prime.
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- Trivially, $\mathcal{R} \subset (0, 1] \cap \mathbb{Q}$.
- $\Pr(R_1 \oplus R_2) = \Pr(R_1) \Pr(R_2)$. 
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- Much more known about groups e.g. $\mathcal{G} \cap (11/32, 1]$ completely understood (Rusin, 1979).
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- Much more known about groups e.g.
  - $\mathcal{G} \cap (11/32, 1]$ completely understood (Rusin, 1979).
- Semigroups entirely different (MacHale, 1990; Ponomarenko and Selinski, 2012; B. 2013)
R versus G

Values of \( \Pr_c(R) \) in \( \left( \frac{11}{32}, 1 \right) \):
- \( R \) direct sum of \( \mathbb{Z}_p \)-algebras

Values of \( \Pr_c(G) \) in \( \left( \frac{11}{32}, 1 \right) \):
- \( G \) nilpotent (class 2)
Values of $Pr_c(R)$ in $[11/32, 1]$ \textbf{(B.-MacHale-Ní Shé)}

$$\left\{ \frac{2^{2k} + 1}{2^{2k+1}} \mid k \in \mathbb{N} \right\} \cup \left\{ 1, \frac{7}{16}, \frac{11}{27}, \frac{25}{64}, \frac{11}{32} \right\}$$
**$\mathcal{R}$ versus $\mathcal{G}$**

Here Be Dragons

<table>
<thead>
<tr>
<th></th>
<th>11/32</th>
<th>25/64</th>
<th>7/16</th>
<th>1/2</th>
<th>17/32</th>
<th>5/8</th>
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</tr>
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\[
\begin{align*}
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Values of $\Pr_c(R)$ in $[11/32, 1]$: $R$ direct sum of $\mathbb{Z}_p$-algebras (B.-MacHale-Ní Shé)

$\Pr_c(G)$ in $[11/32, 1]$: $G$ nilpotent (class 2)
R versus G

Values of $\Pr_c(G)$ in $[11/32, 1]$
$ \mathbb{R} \text{ versus } G$

Values of $Pr_c(G)$ in $[11/32, 1]: G$ nilpotent (class 2)
Large values of the commuting probability

We define

\[ \alpha_p = \frac{p^2 + p - 1}{p^3}, \quad \beta_p = \frac{2p^2 - 1}{p^4}, \quad \text{and} \quad \gamma_p = \frac{p^3 + p^2 - 1}{p^5}. \]

Note that

\[ \gamma_p < \alpha_p^2 < \beta_p < \frac{1}{p} < \alpha_p, \quad p \geq 2. \]
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**Theorem 3 (B.-MacHale-Ní Shé)**

\[ \mathcal{R}_p \cap [\gamma_p, 1] = \{1\} \cup \left\{ \frac{p^{2k} + p - 1}{p^{2k+1}} \right\} \bigg| k \in \mathbb{N} \bigg\} \cup \{\beta_p, \alpha_p^2, \gamma_p\}. \]

\[ \mathcal{R} \cap [\gamma_2, 1] = (\mathcal{R}_2 \cap [\gamma_2, 1]) \cup \{\alpha_3\} \]

\[ = \{1\} \cup \left\{ \frac{2^{2k} + 1}{2^{2k+1}} \right\} \bigg| k \in \mathbb{N} \bigg\} \cup \left\{ \frac{7}{16}, \frac{11}{27}, \frac{25}{64}, \frac{11}{32} \right\}. \]
Commuting probability and Z-isoclinism

**Theorem 4 (B.-MacHale-Ní Shé)**

Pr($R$) = $t$ uniquely determines Z-isoclinism type of $R \in C$ if:

- $t \in \mathcal{R}_p \cap (\gamma_p, 1]$, $C = \{p$-rings\}.
- $t \in \mathcal{R} \cap (\gamma_2, 1]$, $C = \{finite$ $rings\}$. 

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**Theorem 5 (B.-MacHale-Ní Shé)**

\( p\text{-rings } R \text{ with } \Pr(R) = \gamma_p \text{ yield exactly:} \)
Commuting probability and $Z$-isoclinism

**Theorem 4 (B.-MacHale-Ní Shé)**

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**Theorem 5 (B.-MacHale-Ní Shé)**

$p$-rings $R$ with $Pr(R) = \gamma_p$ yield exactly:

- five $Z$-isoclinism types;
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**Theorem 5 (B.-MacHale-Ní Shé)**

p-rings \(R\) with \(\text{Pr}(R) = \gamma_p\) yield exactly:

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- four \(R/Z(R)\) group isomorphism types;
Commuting probability and Z-isoclinism

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p-rings \( R \) with \( \Pr(R) = \gamma_p \) yield exactly:

- five Z-isoclinism types;
- four \( R/Z(R) \) group isomorphism types;
- three \([R, R]\) group isomorphism types.
First steps

\[ \Pr(R) = \frac{1}{|R|} \sum_{x \in R} \frac{1}{|R/C(x)|} = \frac{1}{|R|^2} \sum_{x \in R} |C(x)| \]
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\]

(sum over one representative of each coset)
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**Observation**

For \( x \in R \), additive groups \( R/C(x) \) and \([x, R]\) are isomorphic. In particular, if \( R \) is a \( \mathbb{Z}_p \)-algebra, \( \dim R/C(x) = \dim[x, R] \).
Z-Isoclinism

Definition
Rings $R$ and $S$ are 	extit{Z-isoclinic} if there are additive \textbf{group isomorphisms} $\phi : R/Z(R) \rightarrow S/Z(S)$ and $\psi : [R, R] \rightarrow [S, S]$ such that

$$\psi([u, v]) = [u', v']$$

whenever

$$\phi(u + Z(R)) = u' + Z(S) \quad \text{and} \quad \phi(v + Z(R)) = v' + Z(S).$$
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Kruse and Price’s and Moneyhun’s notions of isoclinism for rings and Lie algebras involve **ring isomorphisms**.
Isoclinism properties

- Z-isoclinism is an equivalence relation.
- Isomorphic $\Rightarrow$ Z-isoclinic; converse false.
- Z-isoclinism class determines gp isomorphism classes of $R/Z(R)$ and $[R, R]$.
- Z-isoclinism induces group isomorphisms of $[x, R]$ subgroups.
- **If $R$ and $S$ are Z-isoclinic, then $Pr(R) = Pr(S)$**.
Universal algebras: definition

If $S$ is any set, $S \times 0 := \{\emptyset\}$, and $S \times m$ is the cartesian product of $m$ copies of $S$, $m \in \mathbb{N}$. 
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If $S$ is any set, $S^{\times 0} := \{\emptyset\}$, and $S^{\times m}$ is the cartesian product of $m$ copies of $S$, $m \in \mathbb{N}$.

An algebra $A$ consists of an underlying set, also denoted $A$, with associated fundamental operations $g^A : A^{\times n} \to A$. $n \geq 0$ is the arity of $g^A$. 
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We speak of **nullary**, **unary**, or **binary** operations if $n = 0$, $n = 1$, or $n = 2$, respectively; a nullary operation is a significant constant e.g. 0 or 1 in a unital ring.
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We speak of **nullary**, **unary**, or **binary** operations if $n = 0$, $n = 1$, or $n = 2$, respectively; a nullary operation is a significant constant e.g. 0 or 1 in a unital ring.

If $g^A$ is $n$-ary, write $g^A(\underline{x})$ to mean $g^A(x_1, \ldots, x_n)$. Each $x_i$ is a **coordinate** of $\underline{x}$. The **coordinate set of $\underline{x}$** is $CS(\underline{x}) = \{x_1, \ldots, x_n\}$. 
Suppose \((A, +)\) an abelian group, and \(g^A\) is \(n\)-ary, \(n \in \mathbb{N}\). 
\(g^A\) is **distributive over addition** if 
\[
g^A(z) = g^A(x) + g^A(y)
\]
whenever \(z_j = x_j + y_j\) for some \(j\), and \(z_k = x_k = y_k\) for \(k \neq j\).
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**Definition**

Suppose \(I\) is an index set and \(\rho : I \rightarrow \mathbb{N}\).

An \((I, \rho)\)-**algebra** is an abelian group \((A, +)\) with \(\rho(i)\)-ary operations \(g^A_i\) on \(A, i \in I\), that are distributive over + whenever \(\rho(i) > 0\); \(A\) has **type** \((I, \rho)\).

A **distributive algebra** is an \((I, \rho)\)-algebra for some type \((I, \rho)\).
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If $|I|$ is small, convenient to let $I = \{1, \ldots, k\}$ and write the type as $[\rho(1), \ldots, \rho(k)]$, so:

- PN rings and $[2]$-algebras coincide;
- a unital PN ring is a special kind of $[2, 0]$-algebra.
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- PN rings and \([2]\)-algebras coincide;
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The **reduced index set** \(I_0\) consists of all \(i \in I\) such that \(\rho(i) > 0\), and \(\rho_0 := \rho|_{I_0}\). \((I_0, \rho_0)\) is the **reduced type corresponding to the type** \((I, \rho)\).
Definition

An **ideal in an \((I, \rho)\)-algebra** \(A\) is a subgp \(J\) of \((A, +)\) such that \(g_i^A(\vec{x}) \in J\) whenever \(i \in I_0\), \(\vec{x} \in A^{\times \rho(i)}\), and \(\text{CS}(\vec{x}) \cap J\) is nonempty. We write \(J \trianglelefteq A\) or \(A \trianglerighteq J\).

An ideal in an \((I, \rho)\)-algebra is an \((I_0, \rho_0)\)-algebra.

Lemma

*If* \(J \trianglelefteq A\), *then* \(A/J\) **naturally has same type as** \(A\), *with natural maps* \(g_i^A/J\).
Annihilators and product ideals

**Definition**

The **annihilator** of \( A \) is \( \text{Ann}(A) = \bigcap_{i \in l_0} \text{Ann}(A; i) \), where

\[
\text{Ann}(A; i) = \{ a \in A \mid \forall x \in A^{\times \rho(i)} : a \in \text{CS}(x) \Rightarrow g_i^A(x) = 0 \}, \quad i \in l_0.
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$$\text{Ann}(A; i) = \{ a \in A \mid \forall \bar{x} \in A^{\times \rho(i)} : a \in \text{CS}(\bar{x}) \Rightarrow g^A_i(\bar{x}) = 0 \}, \quad i \in I_0.$$  

**Definition**
The **product ideal** of $A$, $\pi(A)$, is the subgroup of $(A, +)$ generated by elements of $\pi(A; i), \ i \in I_0$, where $\pi(A, i)$ is the subgroup of $(A, +)$ generated by $g^A_i(\bar{x}), \ \bar{x} \in A^{\times \rho(i)}$. 

Annihilators and product ideals

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**Null algebra**: $\text{Ann}(A) = A$, or equivalently $\pi(A) = 0$.

**Remark**

$g^A_i / \text{Ann}(A)$ factors through $A$ to give natural map $\tilde{g}^A_i : (A / \text{Ann}(A))^{\times n} \rightarrow A$. 
Annihilator series

**Definition**

A finite sequence of ideals \((A_j)_{j=0}^m, m \geq 0\), in an \((I, \rho)\)-algebra \(A\) is an **annihilator series (of length \(m\))** if \(A_0 = A\), \(A_m = 0\),

\[
A_0 \supset A_1 \supset \cdots A_m
\]

and \(A_{j-1}/A_j \leq \text{Ann}(A/A_j)\) for \(1 \leq i \leq m\).

\(A\) is **nilpotent** if it has an annihilator series.
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We can define **upper** and **lower annihilator series**, as done by Kruse and Price for rings.
Isoclinism

**Definition**

An **isoclinism** from one \((I, \rho)\)-algebra \(A\) to another one \(B\) consists of a pair of additive group isomorphisms

\[
\phi : A / \text{Ann}(A) \rightarrow B / \text{Ann}(B) \quad \text{and} \quad \psi : \pi(A) \rightarrow \pi(B)
\]
such that if \(i \in I_0\), \(\phi(x_j + \text{Ann}(A)) = y_j + \text{Ann}(B), j = 1, \ldots, \rho(i),\) then

\[
\psi(g_i^A(x)) = g_i^B(y).
\]

(As usual, \(I_0\) is the reduced index set.)
**Isoclinism**

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An **isoclinism** from one \((I, \rho)\)-algebra \(A\) to another one \(B\) consists of a pair of additive group isomorphisms

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such that if \(i \in l_0\), \(\phi(x_j + \text{Ann}(A)) = y_j + \text{Ann}(B)\), \(j = 1, \ldots, \rho(i)\), then

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\psi(g_i^A(x)) = g_i^B(y).
\]

(As usual, \(l_0\) is the reduced index set.)

**Alternative definition:**

\[
\begin{array}{ccc}
(A/ \text{Ann}(A)) \otimes^n & \xrightarrow{\phi \otimes^n} & (B/ \text{Ann}(B)) \otimes^n \\
\downarrow g_i^A \otimes & \cong & \downarrow g_i^B \otimes \\
\pi(A; i) & \xrightarrow{\psi|_{\pi(A,i)}} & \pi(B; i)
\end{array}
\]

\(A\) and \(B\) are isoclinic via \((\phi, \psi)\) if and only if the above diagram is commutative for each \(i \in l_0\) and \(n := \rho(i)\).
Isoclinism: basics

Theorem

- *Isoclinism is an equivalence relation on distributive algebras of any given type; equivalence classes are called **isoclinism families**.*
- All null algebras of a given type are isoclinic.
- If \((\phi_j, \psi_j)\) is an isoclinism from one \((I, \phi)\)-algebra \(A_j\) to another one \(B_j\), for all \(j \in J \neq \emptyset\), then \(\prod_{j \in J} A_j\) is isoclinic to \(\prod_{j \in J} B_j\), and \(\bigoplus_{j \in J} A_j\) is isoclinic to \(\bigoplus_{j \in J} B_j\).
- Isomorphic algebras are isoclinic.
Isoclinism

Canonical form

Definition
A distributive algebra $A$ has **canonical form** if:

1. $(A, +)$ is the internal direct sum of subgroups $A_1$ and $A_2$.
2. $\pi(A) = \text{Ann}(A) = A_2$.

A canonical form member of an isoclinism family is called a **canonical relative** of all algebras in that family.

Theorem
Canonical relatives exist and are unique (up to isomorphism).

Distributive algebras $A$ and $B$ are isoclinic if and only if their canonical relatives $\text{Can}(A)$ and $\text{Can}(B)$ are isomorphic.

A canonical form distributive algebra is nilpotent of exponent $\leq 2$.

Nilpotency is not an isoclinism invariant.
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Invariant probability functions

Let

$$\Pr(A; g_i^A, n) := \frac{|\{a \in A^\times n : g_i^A(a) = 0\}|}{|A|^n}.$$
Invariant probability functions

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\[ Pr(A; g^A_i, n) := \frac{\{ a \in A^{\times n} : g^A_i(a) = 0 \}}{|A|^n} \].

**Lemma**

Suppose \((\phi, \psi)\) is an isoclinism from one finite \((I, \rho)\)-algebra \(A\) to another \(B\). Then \(Pr(A; g^A_i, n) = Pr(B; g^B_i, n)\) for all \(i \in I\) and \(n := \rho(i)\).
Invariant probability functions

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**Lemma**

Suppose \((\phi, \psi)\) is an isoclinism from one finite \((I, \rho)\)-algebra \(A\) to another \(B\). Then \(\Pr(A; g_i^A, n) = \Pr(B; g_i^B, n)\) for all \(i \in I\) and \(n := \rho(i)\).

The argument in the above lemma can be generalized. In particular, we can replace \(g_i^A\) by \(f^A : A^{\times m} \to A\), where

\[ f^A(x_1, \ldots, x_m) := g_i^A(\sum_{j=1}^{m} a_1 j x_j, \ldots, \sum_{j=1}^{m} a_n j x_j), \]

One simple example is \(f^A(x) = g^A(x, x)\) in a PN ring \(A\) where \(g^A(x, y) = xy\). So the proportion of **dinilpotents** (elements satisfying \(x^2 = 0\)) is an isoclinism invariant for PN rings.
Spectral identity

**Theorem**

Let $C_0$, $C_1$, and $C_2$ be the classes of all finite nilpotent rings of exponent at most 2, all finite rings, and all finite PN rings, respectively. Then for all $f(X, Y) := aXY + bYX$, $a, b \in \mathbb{Z}$, and all classes $C$ such that $C_0 \subseteq C \subseteq C_2$,

$$\mathcal{S}_f(C) = \mathcal{S}_f(C_1).$$

Above theorem works for other function symbols $f$ such as $f(X) = X^2$, so the sets of possible dinilpotent proportions in finite rings and in finite PN rings coincide. However idempotent proportion is not an isoclinism invariant and the sets of possible idempotent proportions in finite rings and in finite PN rings do not coincide (B.–Yu. Zelenyuk; work in progress).
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