Ideals of the enveloping algebra of $U(sl_3)$

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   - Enveloping Algebras.
   - Classical Results on Prime/Primitive Ideals
   - The Enveloping Algebra $U(sl_3)$.
   - Prime and Primitive Ideals of $U(sl_3)$
   - $U(sl_3)$-Results
2. Motivation
   - Work of A. Joseph
   - Connection to Previous Work
3. Details
   - General Setting
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   - Main Theorem

Enveloping Algebras.

Let L be a finite dimensional semisimple Lie algebra over field $k = \mathbb{C}$. Any Cartan subalgebra $\mathfrak{h}$ of L gives rise to a unique triangular decomposition

$$L = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

Disjoint bases of $\mathfrak{n}^+$, $\mathfrak{n}^-$, and $\mathfrak{h}$ consisting of elements labeled respectively as $e's$, $f's$, and $h's$ add up to a basis $\mathcal{B}$ of L. The universal enveloping algebra of L is the algebra

$$U = U(L) = k < \mathcal{B} > /(xy - yx - [x, y] : x, y \in \mathcal{B}).$$

It has a PBW-basis

$$U = \text{Span}\{f^a h^b e^c \mid a, c \in \mathbb{N}^{|\mathfrak{n}^\pm|}, b \in \mathbb{N}^{|\mathfrak{h}|=n=\text{rank}(L)}\}.$$

It is a Noetherian domain with center $Z \cong S(\mathfrak{h})$.

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Classical Results on Prime/Primitive Ideals

1. $P \in \text{Spec}(U) \Rightarrow P \cap Z \in \text{Spec}(Z)$.(Dixmier, 70's)
2. $J \in \text{Spec}(Z) \Rightarrow JU \in \text{Spec}(U)$. (Dixmier, 70's)
3. $\text{cl}K\dim Z = n$ (Chevalley/Gelfand), $\text{cl}K\dim U = 2n$. (open)
4. $P \in \text{Spec}(U)$ is primitive iff $P \cap Z \in \text{Max}(Z)$. (Dix., 70's)
5. Conversely, $J \in \text{Max}(Z) \Rightarrow \mathcal{L}_J(U) = \{I \lhd U \mid I \cap Z = J\}$ is a finite set. (Dixmier) These are the annihilators of the simple subquotients of the principal series. Their description given by A. Joseph for sl_3 and sp_4, open for higher rank.
6. $\mathcal{L}_J(U) \cap \text{Prim}(U)$ described by Borho and Jantzen (1977) using orbit method.
7. Theory extended to the quantum case in the 90's by A. Joseph and many others.

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The Enveloping Algebra $U(sl_3)$.

Lie algebra $L = sl_3$ over field $k = \mathbb{C}$ has a basis

$$\mathcal{B} = \{e_{12}, e_{23}, e_{13}, e_{11} - e_{22}, e_{22} - e_{33}, e_{21}, e_{32}, e_{31}\}$$

and a triangular decomposition $L = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where

$$\begin{aligned} \mathfrak{n}^+ &= \text{Span}\{e_1 = e_{12}, e_2 = e_{23}, e_3 = e_{13}\} \\ \mathfrak{n}^- &= \text{Span}\{f_1 = e_{21}, f_2 = e_{32}, f_3 = e_{31}\} \\ \mathfrak{h} &= \text{Span}\{h_1 = e_{11} - e_{22}, h_2 = e_{22} - e_{33}\}. \end{aligned}$$

The center Z of its universal enveloping algebra $U = U(sl_3)$ is the polynomial algebra $Z = k[z_1, z_2]$, where

$$\begin{aligned} z_1 &= -h_1^2 - h_1h_2 - h_2^2 - 3(h_1 + h_2) - 3(f_1e_1 + f_2e_2 + f_3e_3) \\ z_2 &= -(-2h_1 - h_2 - 3)(h_1 - h_2)(h_1 + 2h_2 + 3) + \text{ other terms.} \end{aligned}$$

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Prime and Primitive Ideals of $U(sl_3)$

Consider the central subset $\Omega = \{\Omega_m \mid m \geq 1\}$, where

$$\Omega_m = z_2^2 + (z_1 + 3m^2 - 3)^2(4z_1 + 3m^2 - 12).$$

As $\mathrm{Spec}(U) = \bigcup_{J \in \mathrm{Spec}(Z)} \mathrm{Spec}_J(U)$, the prime ideals of U classify according to the following possible cases:

- **Prime non-primitive ideals of** U $(J \in \mathrm{Spec}(Z) - \mathrm{Max}(Z))$

$$\mathrm{Spec}_J(U) = \begin{cases} \{JU\} & \text{, if } J \cap \Omega = \varnothing \\ \{JU, I_{\Omega_m}\} & \text{, if } J \cap \Omega = \{\Omega_m\}. \end{cases}$$

- **Primitive ideals of** U $(J \in \mathrm{Max}(Z))$

$$\mathrm{Spec}_J(U) = \begin{cases} \{JU\} & \text{, if } J \cap \Omega = \varnothing \\ \{JU, I_{\Omega_m} + JU\} & \text{, if } J \cap \Omega = \{\Omega_m\} \\ \{JU, A, B, A + B\} & \text{, if } J \cap \Omega = \{\Omega_m, \Omega_n, \Omega_{m+n}\}. \end{cases}$$

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$U(sl_3)$-Results

1. Completely prime primitive ideals were listed by J. Dixmier in 1975.
2. Primitive ideals were listed by W. Borho and J. C. Jantzen (1977) as part of the general semi-simple case, and independently by A. Joseph for sl_3.
3. Lattice Krull dimension of $U(sl_3)$ computed by T. Levasseur in 1985.
4. Prime ideals I_{Ω_m} we listed by W. Soergel in 1990, using the orbit method.
5. All prime ideals of U were listed by generators as both ideals and adjoint modules in [C], (2000).

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Work of A. Joseph

Let V be a finite dimensional irreducible L-module. Then

- $J = \operatorname{ann}_U(V) \cap Z \in \operatorname{Max}(Z)$ (Dixmier, 1970)
- Lattice $\mathcal{L}_J(U) = \{I \lhd U \mid I \cap Z = J\}$ is finite. (Dixmier)
- For $L = sl_3$, the lattices $\mathcal{L}_J(U)$ were described by A. Joseph in 1977 as follows:

$$\mathcal{L}_J(U) = \begin{cases} \operatorname{Spec}_J(U), & |J \cap \Omega| < 2 \\ \operatorname{Spec}_J(U) \cup \{(A \cap B)^2, AB, BA, A \cap B\}, & |J \cap \Omega| \geq 2. \end{cases}$$

The elements of $\mathcal{L}_J(U)$ are the annihilators of the simple subquotients of the principal series.

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Connection to Previous Work

- With the adjoint structure of elements of $\text{Spec}_J(U)$ being described in our previous paper (C, 2000), the goal is to give a similar description for the remaining ideals

$$(A \cap B)^2, AB, BA, A \cap B \text{ when } |J \cap \Omega| = 2, 3,$$

that is, when $A + B$ is the annihilator of a finite dimensional irreducible module.

- The work for AB and BA is non-trivial. Giving their adjoint structure amounts to splitting tensor products of irreducible sl_3-modules as sums of irreducibles.

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General Setting

- The general theory looks at the relationship

$$\text{"Ideals of } U\text{" vs. "Ideals of } Z = \mathbb{C}_U(L) = \mathbb{C}_U(\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+)\text{"}.$$

- We look at the relationship

$$\text{"Ideals of } U\text{" vs. "Ideals of } R = \mathbb{C}_U(\mathfrak{n}^+)\text{"}.$$

This algebra R has two important properties:

- R is the linear span of all highest weight elements of U under the adjoint action.
- Any ideal I of U is an adjoint submodule, hence generated as a module by its highest weight elements. Write this as

$$I = [I \cap R].$$

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Generators and Relations for R

$$R = k[z_1, z_2, x_1, x_2][\zeta, \overline{\zeta}, \sigma, \sigma^{-1}]/(rels),$$

where

$$x_1 = e_3\ ,\ x_2 = e_3(h_1 - h_2) - 3e_1e_2,$$
$$\zeta = e_3^2 f_2 - e_1^2 e_2 + e_1 e_3 h_2,$$
$$\overline{\zeta} = e_3^2 f_1 - e_1 e_2^2 - e_2 e_3 (h_1 - 2).$$

Variables $\zeta, \overline{\zeta}$ commute with x_1 and skew-commute with x_2 as

$$\zeta x_2 = (x_2 + 3x_1)\zeta\ ,\ \overline{\zeta} x_2 = (x_2 - 3x_1)\overline{\zeta}.$$

If $D = k[z_1, z_2, x_1, x_2]$, then $\sigma \in \mathrm{Aut}(D)$ is identity on z_1, z_2, x_1 and $\sigma(x_2) = x_2 + 3x_1$.

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A Generalized Weyl Algebra

In addition,

$$27\zeta\overline{\zeta} = x_1^3 z_2 + 3x_1^2 x_2(z_1 - 3) + x_2^3 = a \in D,$$
$$27\overline{\zeta}\zeta = x_1^3(z_2 + 9z_1) + 3x_1^2 x_2(z_1 + 6) + 9x_1 x_2^2 + x_2^3 = \sigma(a)$$

This makes R into a degree 1 generalized Weyl algebra.

- Generalized Weyl algebras were introduced by V. V. Bavula in 1992. It is a larger class of algebras that include $U(sl_2)$ and $U_q(sl_2)$, down-up algebras of G. Benkart and T. Roby, some ambiskew polynomial algebras of D. A. Jordan, Woronowicz and Witten algebras, etc.. They were extended to the twisted case by V. Mazorchuk and L. Turowska.
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Adjoint Generating Ideals

If $I \lhd U$, then $I \cap R \lhd R$. And since R is a GWA, we have

$$I \cap R = \left(\oplus \sum_{t=0}^{\infty} \zeta^t I_t \right) \oplus \left(\oplus \sum_{t=1}^{\infty} \bar{I}_t \bar{\zeta}^t \right),$$

where

$$\ldots \bar{I}_2 \supseteq \bar{I}_1 \supseteq I_0 \subseteq I_1 \subseteq I_2 \ldots$$

are ideals of $D = k[z_1, z_2, x_1, x_2]$. In our case, $I \cap Z = J$ is maximal in Z. We mod out by J, and without loss, we may assume that the above is a chain of ideals in $S = k[x_1, x_2]$.

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Notation

Let V_{pq} be the f. d. irred. adjoint module of weight (p,q). Then

- $\mathrm{Ann}_U(V_{pq}) = A + B$.
- If $p \leq q$, then $m = p + 1$ and $n = q - p$, and $(A + B) \cap Z = J = (z_1 + 3m^2 + 3mn + n^2 - 3, z_2 - n(3m + n)(3m + 2n))$.
- Define $\lambda = -3m - n + 3$, $\mu = -n + 3$, and polynomials

$$P_s^r = \begin{cases} \prod_{i=0}^{s-1} [x_2 + (r + 3i)x_1], & s > 0 \\ 1, & s \leq 0. \end{cases}$$

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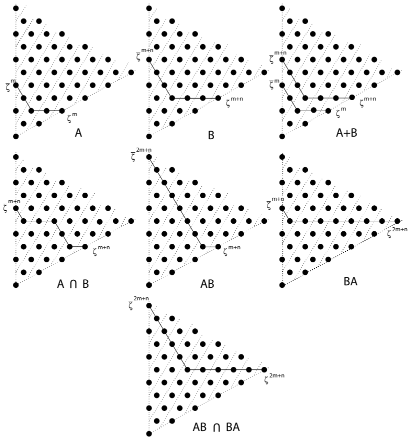
Main Theorem

Theorem

With the above notation and $C = (A \cap B)^2 = AB \cap BA$, for $t \geq 0$, we have

- $I_t(A) = \bar{I}_t(A) = P^{\lambda}_{m-t}S.$
- $I_t(B) = \bar{I}_t(B) = P^{\mu}_{m+n-t}S.$
- $I_t(A+B) = \bar{I}_t(A+B) = P^{\lambda}_{m-t}S + P^{\mu}_{m+n-t}S.$
- $I_t(A \cap B) = \bar{I}_t(A \cap B) = P^{\lambda}_{m-t}P^{\mu}_{m+n-t}S.$
- $I_t(C) = \bar{I}_t(C) = P^{\mu}_{2m+n-t}S.$
- $I_t(AB) = I_t(A \cap B)$ *and* $\bar{I}_t(AB) = \bar{I}_t(C).$
- $I_t(BA) = I_t(C)$, *and* $\bar{I}_t(BA) = \bar{I}_t(A \cap B)$

Picture



THANK YOU !