Maximal isotropic subgroups

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Then the form

$$\begin{aligned} \alpha_\pi : V \times V &\to F \\ (w_1 + u_1, w_2 + u_2) &\mapsto \langle \pi(u_1), w_2 \rangle - \langle \pi(u_2), w_1 \rangle. \end{aligned}$$

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is indeed alternating, with W isotropic, that is $\alpha_\pi|_{W \times W} = 0$. More generally, for any alternating form $\alpha' : U \times U \to F$ on U, the form

$$(w_1 + u_1, w_2 + u_2) \mapsto \alpha_\pi(w_1 + u_1, w_2 + u_2) + \alpha'(u_1, u_2)$$

also satisfies the above requirements.

The following properties are easily verified:

1. W is a maximal isotropic subspace with respect to any of the above alternating forms if and only if π is injective.
2. in case $\dim_F W = \dim_F U$, then none of the above forms admits a nontrivial radical if and only if π is bijective.

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In other words, for any subspace $W \subset V$ with $\dim_F W = \frac{1}{2}\dim_F V$, the above describes a method to construct symplectic forms on V such that W is a *Lagrangian* with respect to this form.

Conversely, let $\alpha : V \times V \to F$ be an alternating form with $W \subset V$ isotropic. Define a linear map

$$\begin{aligned} \pi_\alpha : U &\to W^* \\ \langle \pi_\alpha(u), w \rangle &:= \alpha(u, w). \end{aligned}$$

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Then

1. W is maximal isotropic with respect to the form α if and only if π_α is injective.
2. α is symplectic if and only if π_α is bijective, in particular, $\dim_F W = \frac{1}{2}\dim_F V$.

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The maps $\pi \mapsto \alpha_\pi$ and $\alpha \mapsto \pi_\alpha$ may be considered as mutually inverse in the sense that $\pi_{\alpha_\pi} = \pi$, and that α_{π_α} differs from α by an alternating form which is inflated from U.

An analog is established for forms on groups. A G-form over an abelian group M is a map

$$\alpha : G \times G \to M$$

such that $\forall g \in G$, both $\mathrm{res}|^{G}_{C_G(g)}\alpha(g,-)$ and $\mathrm{res}|^{G}_{C_G(g)}\alpha(-,g)$ are group homomorphisms from the centralizer $C_G(g)$ to M.

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- A subgroup $H < G$ is **isotropic** with respect to a form α if the restrictions $\mathrm{res}|^{G}_{C_H(h)}\alpha(h,-)$ and $\mathrm{res}|^{G}_{C_H(h)}\alpha(-,h)$ to the homomorphisms from $C_H(h)$ to M are trivial for any $h \in H$.

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- A G-form $\alpha : G \times G \to M$ is **symplectic** if, in addition,
  1. $\alpha(g, h) = -\alpha(h, g)$ for every $(g, h) \in G \times G$ such that g and h commute (α is **alternating**), and
  2. $\text{res}|^G_{C_G(g)}\alpha(g, -) = 0$ if and only if $g = e$ (α is **non-degenerate**).

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Alternating forms on groups are naturally obtained from 2-cocycles. Let $c \in Z^2(G, M)$ be a 2-cocycle with values in a trivial G-module M. Then

$$\begin{array}{rcl} \alpha_c : G \times G & \to & M \\ (g, h) & \mapsto & c(h, g) - c(g, h). \end{array}$$

is an alternating form on G, called the alternating form **associated** to c. It is easy to show that if g and h commute, then $\alpha_c(g, h)$ depends only on the cohomology class of c, and not on the particular representative.

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From now onwards our discussion is over 2-cocycles (or over cohomology classes) with values in the trivial G-module $\mathbb{C}^*$ rather than over arbitrary alternating G-forms.

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In this regard, we say that a subgroup $H < G$ is isotropic with respect to $c \in Z^2(G, \mathbb{C}^*)$ if $\text{res}_H^G[c] = 1 \in H^2(H, \mathbb{C}^*)$.

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Proposition

Let $H < G$ be an isotropic with respect to a non-degenerate cocycle $c \in Z^2(G, \mathbb{C}^*)$. Then $|H|$ divides $\sqrt{|G|}$.

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By a deep result of R. Howlett and I. Isaacs (1982), based on the classification of finite simple groups, it is known that all such groups are solvable.

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The linear transformation $\pi : U \to W^*$ is replaced now by a 1-cocycle

$$\pi : Q \to \check{A}.$$

Here $\check{A}$ =Hom$(A, \mathbb{C}^*)$ is endowed with the diagonal Q-action

$$\langle q(\chi), a\rangle = \chi(q^{-1}(a)),$$

for every $q \in Q, \chi \in \check{A}$ and $a \in A$.

Suppose that A admits a complement in G, that is

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P. Etingof and S. Gelaki observed (2000) that any 1-cocycle $\pi \in Z^1(Q, \check{A})$ gives rise to a 2-cocycle

$$\begin{array}{rcl} c_\pi : G \times G & \rightarrow & \mathbb{C}^* \\ (a_1 q_1, a_2 q_2) & \mapsto & \langle \pi(q_1), q_1(a_2) \rangle^{-1} (= \langle \pi(q_1^{-1}), a_2 \rangle) \end{array}$$

with an associated alternating form

$$\alpha_{c_\pi} = \langle \pi(q_1), q_1(a_2) \rangle \cdot \langle \pi(q_2), q_2(a_1) \rangle^{-1}, a \in A, q \in Q.$$

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admitting A as a maximal isotropic subgroup.
Moreover, π is bijective if and only if c_π (or α_{c_π}) is non-degenerate.

Conversely, let $c \in Z^2(G, \mathbb{C}^*)$ with A isotropic. Define

$$\begin{array}{rcl} \pi_c = \pi_{[c]} : Q & \to & \check{A} \\ \langle \pi_c(q), a \rangle & := & \alpha_c(q, a) \end{array} \quad \forall a \in A, \forall q \in Q.$$

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Then π_c is a 1-cocycle.

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Furthermore, c is non-degenerate if and only if π_c is bijective.

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Furthermore, c is non-degenerate if and only if π_c is bijective.
Again, we obtain the mutually inverse property in the sense that $\pi_{c_\pi} = \pi$, and that c_{π_c} differs from c by an alternating form inflated from Q.

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- P. Etingof, T. Schedler and A. Soloviev (1999)
- T. Gateva-Ivanova (2004)
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- E. Jespers and J. Okniński (2005)
- J.H. Lu, M. Yan and Y.C. Zhu (2000)
- W. Rump (2005, 2007)

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Theorem (N. Ben-David, G., 2009)

Let

$$[\beta] : 1 \to A \to G \to Q \to 1, \quad [\beta] \in H^2(Q, A)$$

be an extension of finite groups, where A is abelian. Then there is a 1-1 correspondence between classes $[\pi] \in H^1(Q, \check{A})$ annihilating the cup product with $[\beta]$, that is

$$[\beta] \cup [\pi] = 0 \in H^3(Q, \mathbb{C}^*),$$

and classes in $\ker(res_A^G)$ mod $[\mathrm{im}(\inf_G^Q)]$. *If, additionally, $|A| = |Q| (= \sqrt{|G|})$, then in this way bijective classes correspond to non-degenerate classes.*

When the extension $[\beta] : 1 \to A \to G \to Q \to 1$ splits, then certainly $[\beta] \cup [\pi] = 0$ for every $[\pi] \in H^1(Q, A^*)$, and the correspondence in the theorem amounts to the one described above.

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Corollary

Let A be a finite abelian group, Q a finite group acting on A and $[\pi] \in H^1(Q, \check{A})$ a bijective class (in particular $|A| = |Q|$). Then for every $[\beta] \in H^2(Q, A)$ such that $[\beta] \cup [\pi] = 0$, the group G determined by the extension $[\beta] : 1 \to A \to G \to Q \to 1$ is of central type.

An intriguing question arises following the theorem:

Question

Let $[c] \in H^2(G, \mathbb{C}^*)$ be a non-degenerate class. Does $[c]$ admit a normal maximal isotropic (and hence abelian) subgroup $A \lhd G$ of order $\sqrt{|G|}$?

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If a non-degenerate class $[c] \in H^2(G, \mathbb{C}^*)$ gives an affirmative answer to this question, then by the above theorem, the corresponding quotient G/A is an IYB group, admitting a bijective 1-cocycle datum determined by $[c]$.

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If the answer to the question is always positive, then all groups of central type are obtained from such data.

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- Nilpotent groups of central type are not necessarily obtained from bijective data in general.
- However, non-degenerate classes over nilpotent groups G, whose orders are free of eighth powers always admit normal Lagrangians of order $\sqrt{|G|}$.
- Relaxing the normality demand, we have that non-degenerate classes over nilpotent groups G do admit a Lagrangian of order $\sqrt{|G|}$.