On rings with finite number of orbits

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A ring means an associative ring with unit. For a ring $R$, $R^+$ denotes the additive group of $R$. The unit group of $R$ is denoted by $U(R)$. The Jacobson radical of $R$ is denoted by $J(R)$.

Yasuyuki Hirano in [Rings with finitely many orbits under the regular action, Lecture Notes in Pure and Appl. Math. 236, Dekker, New York 2004, 343–347] concentrated on the left regular group action of $U(R)$ on $R^+$ defined by

$$a \rightarrow x = ax$$

for all $a \in U(R), x \in R$. 
Theorem (Hirano). For a ring $R$, the following conditions are equivalent:

1. $R$ has only a finite number of orbits under the left regular group action of $U(R)$ on $R^+$. 

2. $R$ has only a finite number of left ideals.

3. $R$ is the direct sum of a finite ring and a finite number of left uniserial rings (that is, rings which left ideals form a finite chain).

If any of these conditions holds, then $R$ is a left artinian ring. More precisely, $R$ is the direct sum of a finite ring and a finite number of principal left ideal left artinian rings.

We will see later that a ring satisfying conditions of the Hirano Theorem need not be right artinian.
We concentrate on the two-sided regular group action of $U(R) \times U(R)$ on $R^+$ defined by

$$(a, b) \rightarrow x = axb^{-1},$$

for all $a, b \in U(R), x \in R$.

The action (1) induces an action of the group $U(R) \times U(R)$ on each of the following sets: the set of elements of $R$, of principal left ideals of $R$, of left ideals of $R$, and of ideals of $R$, however the action on the latter set is trivial. Orbits under the action (1) are called simply $U$-orbits.
We introduce the following properties:

**FNE** $R$ has only a finite number of $U$-orbits of elements.

**FNPLI** $R$ has only a finite number of $U$-orbits of principal left ideals.

**FNLI** $R$ has only a finite number of $U$-orbits of left ideals.

**FNI** $R$ has only a finite number of $U$-orbits of ideals ($R$ has only a finite number of ideals).

For a ring $R$, the following connections between the above properties holds:

$$FNE \Rightarrow FNPLI \Rightarrow FNI$$

and

$$FNLI \Rightarrow FNPLI \Rightarrow FNI$$ (2)
Theorem. For a commutative ring $R$, the following statements are equivalent:

1. $R$ satisfies each of the properties listed in Formula (2).

2. $R$ satisfies any of the properties listed in Formula (2).

3. $R$ is the direct sum of a finite ring and a finite number of principal ideal local artinian rings.

We will see later that in non-commutative case the converse of the implications listed in Formula (2) is not necessarily true. Although, accordingly to [Jan Okniński, Lex E. Renner, *Algebras with finitely many orbits*, J. Algebra 264 (2003), 479–495], under the assumption on semiperfectness* of a ring, the property FNPLI implies the property FNE.

*A ring $R$ is called semiperfect if $R$ is semilocal (that is, $R/J(R)$ is semisimple artinian), and idempotents of $R/J(R)$ can be lifted to $R$. 
We discuss two questions:

* Must every left and/or right artinian ring satisfy FNE or a similar property?

* Must every ring satisfying FNE or a similar property be left and/or right artinian, or at least semiprimary?

Theorem. Every semisimple artinian ring satisfies all the properties listed in Formula (2).

*A ring $R$ is called semiprimary if $R$ is semilocal, and $J(R)$ is nilpotent.
Example. Let $\mathbb{K}$ be an infinite field, and let $R = \mathbb{K}[x, y]/(x^2, xy, y^2)$ be the homomorphic image of the polynomial ring in commuting variables $x, y$. Then

* $R$ is a 3-dimensional $\mathbb{K}$-algebra.

* $R$ has an infinite number of $U$-orbits of ideals.

* Under which conditions does a left and/or right artinian ring satisfy FNE or a similar property?
Theorem. Assume a ring $R$ satisfies FNI. Then

1. $P(R)$ is nilpotent, where $P(R)$ denotes the prime radical of $R$.

2. If $R$ is left or right noetherian, then $J(R)$ is nilpotent.

3. If every prime image of $R$ is simple artinian, then $R$ is semiprimary.

4. If $R$ satisfies a polynomial identity, then $R$ is semiprimary.

Proof. The statement 1 follows from the definition of the prime radical of $R$ as the sum of a finite number of nilpotent ideals.

The statement 2 follows from the Nakayama Lemma.
3. Let $P_1, P_2, \ldots, P_n$ be all prime ideals of $R$. By assumption, the prime images $R/P_1, R/P_2, \ldots, R/P_n$ of $R$ are simple artinian, and hence $P_1, P_2, \ldots, P_n$ are all maximal ideals of $R$. According to the Chinese Remainder Theorem for rings, $R/P(R) \cong R/P_1 \times R/P_2 \times \ldots \times R/P_n$ is a semisimple artinian ring. Thus $J(R) = P(R)$ is nilpotent, and $R/J(R)$ is semisimple artinian.

4. If $R$ is a prime PI-ring satisfying FNI, then $R$ is a central prime PI-algebra, and according to the Kaplansky Theorem, $R$ is a simple artinian ring. Let $R$ be now an arbitrary PI-ring satisfying FNI. From the statement 3, $R$ is a semiprimary ring. \[
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Lemma. Assume a ring $R$ satisfies FNPLI. Then every one-sided nil-ideal of $R$ is nilpotent of nilpotency index not greater than $n+1$, where $n$ denotes the number of $U$-orbits of principal left ideals of $R$.

Proof. Let $I$ be a one-sided nil-ideal of $R$. Suppose that $x_1x_2\cdots x_{n+1} \neq 0$ for some $x_1, x_2, \ldots, x_{n+1} \in I$. Out of all the left ideals $Rx_1, Rx_1x_2, \ldots, Rx_1x_2\cdots x_{n+1}$ of $R$ at least two belong to the same $U$-orbit, say $Rx_1x_2\cdots x_i$ and $Rx_1x_2\cdots x_ix_{i+1}\cdots x_j$. Set $x = x_1x_2\cdots x_i$ and $y = x_{i+1}\cdots x_j$. Then $x = rxyb^{-1}$ for some $r \in R$, $b \in U(R)$. By induction on $m \geq 1$, $x = r^mx(yb^{-1})^m$. But $(yb^{-1})^m = 0$ for any sufficiently large $m$, which contradicts $x \neq 0$. \qed
Theorem. Assume a ring $R$ satisfies FNPLI. Then $R$ is semiprimary provided at least one of the following conditions is fulfilled:

1. $R$ is semilocal, and $J(R)$ is nil.

2. $R$ satisfies ACC or DCC on principal left ideals.

Proof. The statement 1 follows from the previous lemma.

2. If $R$ does not satisfy DCC on principal left ideals, then there exist left ideals $Rx \subsetneq Ry$ of $R$ belonging to the same $U$-orbit, hence $Rxb^{-1} = Ry$ for some $b \in U(R)$, and thus $Rx \subsetneq Rxb^{-1} \subsetneq Rxb^{-2} \subsetneq \ldots$, which means that $R$ does not satisfy ACC on principal left ideals, contrary to assumption. Thereby $R$ must satisfy DCC on principal left ideals. According to the Bass Theorem, $R$ is right perfect*. In particular, $R$ is semilocal, and $J(R)$ is nil. From the statement 1, $R$ is a semiprimary ring.

*A ring $R$ is called right perfect if $R$ is semilocal and $J(R)$ is right T-nilpotent.
Jan Okniński and Lex E. Renner in [Algebras with finitely many orbits, J. Algebra 264 (2003), 479–495] conjectured that every ring satisfying FNLI is semiprimary.

Theorem. Assume a ring $R$ satisfies FNLI. Then $R$ is semilocal. Moreover, $R$ is semiprimary provided at least one of the following conditions is fulfilled:

1. $J(R)$ is nil.

2. Every prime image of $R$ is left bounded* (such a ring $R$ is called left fully bounded).

*A ring $R$ is called left bounded if every essential left ideal of $R$ contains a non-zero ideal of $R$. 
Proof. The semilocalness of $R$ was in fact proved by Jan Okniński and Lex E. Renner. Suppose that there exists a strictly decreasing sequence of left ideals $M_1 \supsetneq M_1 \cap M_2 \supsetneq M_1 \cap M_2 \cap M_3 \supsetneq \ldots$ for some maximal left ideals $M_1, M_2, \ldots$ of $R$. From $(M_1 \cap M_2 \cap \ldots \cap M_k)/(M_1 \cap M_2 \cap \ldots \cap M_{k+1}) \cong R/M_{k+1}$ we see that $R/(M_1 \cap M_2 \cap \ldots \cap M_n)$ is a left $R$-module of length $n$, for every $n \geq 1$. On the other hand, there exist $m \neq n$ for which left ideals $M_1 \cap M_2 \cap \ldots \cap M_m$ and $M_1 \cap M_2 \cap \ldots \cap M_n$ belong to the same $U$-orbit, and hence $R/(M_1 \cap M_2 \cap \ldots \cap M_m) \cong R/(M_1 \cap M_2 \cap \ldots \cap M_n)$ as left $R$-modules, contrary to the Jordan-Hölder Theorem. Thereby $J(R) = M_1 \cap M_2 \cap \ldots \cap M_n$ for some $n \geq 1$, and thus $R/J(R)$ is a semisimple artinian ring.
The statement 1 follows from the previous lemma.

2. Let $R$ be a prime left bounded ring satisfying FNLI. Let $I$ be a minimal ideal of $R$, and let $0 \neq a \in I$. Let $M$ be a left ideal of $R$ maximal with respect to $a \notin M$. Since every non-zero submodule of $R/M$ contains $a+M$, it follows that $(M+Ra)/M$ is a simple left $R$-module. On the other hand, by assumption, $M$ is a non-essential left ideal of $R$, hence $M \nsubseteq M \oplus N$ for some left ideal $N$ of $R$, and by maximality of $M$, $M + Ra \subseteq M \oplus N$. Thus $(M + Ra)/M \subseteq (M \oplus N)/M \cong N \subseteq R$, which means that $R$ contains simple submodules (minimal left ideals), say $L_1, L_2, \ldots$. Out of all the left ideals $L_1, L_1 \oplus L_2, \ldots$ of $R$ none two of them belong to the same $U$-orbit. Thereby $R$ has only a finite number of minimal left ideals, and in consequence, is a simple artinian ring. Let $R$ be now an arbitrary left fully bounded ring satisfying FNLI. From one of the previous theorem, $R$ is a semiprimary ring. \qed
Theorem. Let $R$ be a semiprimary ring. Then $R$ is left artinian provided at least one of the following conditions is fulfilled:

1. $R$ satisfies FNLI.

2. $J(R)$ is a finitely generated $R$-module.

3. $R$ satisfies FNI, and is a finitely generated PI-algebra over its center.

In particular, every ring satisfying both FNLI and ACC or DCC on principal left ideals is left artinian.
Example. Let $\mathbb{L}$ be a field, let $F = \mathbb{L}(x)$ be the field of rational functions in one variable $x$, and let $\sigma : F \to F$ be the $\mathbb{L}$-endomorphism defined by $\sigma(x) = x^n$ for some positive integer $n \geq 2$. Let $R = F[y; \sigma]/(y^2)$ be the homomorphic image of the skew polynomial ring in one variable $y$. Then

* $R$ is both left and right artinian.

* $R$ has exactly three $U$-orbits of elements, of principal left ideals, of left ideals, and of principal right ideals.

* If $n = 2$ (respectively, $n = 3$), then $R$ has exactly four (five) $U$-orbits of right ideals.

* If $n \geq 4$, then $R$ has an infinite number of $U$-orbits of right ideals.
Example. Let $\mathbb{L}$ be a field, let $\mathbb{F} = \mathbb{L}(x_1, x_2, \ldots)$ be the field of rational functions in infinitely many variables $x_1, x_2, \ldots$, and let $\sigma: \mathbb{F} \to \mathbb{F}$ be the $\mathbb{L}$-endomorphism defined by $\sigma(x_i) = x_i^2$ for every $i \geq 1$. Let $R = \mathbb{F}[y; \sigma]/(y^2)$. Then

* $R$ is left, but not right artinian.

* $R$ has exactly three $U$-orbits of elements, of principal left ideals, of left ideals, and of principal right ideals.

* $R$ has an infinite number of $U$-orbits of right ideals.
Example. Let $R$ be the same as in the previous example, let $R^{op}$ be the opposite ring, and let $S = R \times R^{op}$. Then

* $S$ is semiprimary, but neither left nor right noetherian.

* $S$ has exactly nine $U$-orbits of elements, of principal left ideals, and of principal right ideals.

* $S$ has an infinite number of $U$-orbits both left and right ideals.
Example. Let $K$ be an infinite subfield of a field $F$, and let $R = \begin{bmatrix} F & F \\ 0 & K \end{bmatrix}$ be the ring of matrices of the form $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$, where $x, y \in F$ and $z \in K$, with formal matrix multiplication. Then

* $R$ has exactly five $U$-orbits of elements, of principal left ideals, and of principal right ideals.

* $R$ has exactly six $U$-orbits of left ideals.

* If $[F : K] = 1$, then $R$ has exactly six $U$-orbits of right ideals.

* If $[F : K] = 2$ (respectively, $[F : K] = 3$), then $R$ has exactly eight (ten) $U$-orbits of right ideals.

* If $[F : K] \geq 4$, then $R$ has an infinite number of $U$-orbits of right ideals.
Example. Let $\mathbb{F}$ be an infinite field, let $D = \text{diag}(\mathbb{F}, \mathbb{F})$ be the $2 \times 2$ diagonal matrix ring, let $M = M_2(\mathbb{F})$ be the $2 \times 2$ matrix ring, and let $R = \begin{bmatrix} D & M \\ 0 & D \end{bmatrix}$. Then

* $R$ is an 8-dimensional $\mathbb{F}$-algebra.

* $R$ has a finite number of $U$-orbits of ideals.

* $R$ has an infinite number of $U$-orbits of principal left ideals.