Quantizations of Kac-Moody algebras

V.K. Kharchenko

FES-C UNAM MEXICO

1–4 of July 2013
Noncommutative Rings and their Applications
LENS, France

Quantizations

- Quantized enveloping algebras: Drinfel'd (1985), Jimbo (1985).

Quantizations

- Quantized enveloping algebras:
  Drinfel'd (1985), Jimbo (1985).

- Multiparameter version:
  N. Reshetikhin (1990);
  P. Cotta-Ramusino, M. Rinaldi (1991);
  C. De Concini, V.G. Kac, C. Procesi (1992);
  M. Constantini, M. Varagnolo (1994).

Quantizations

- Quantized enveloping algebras:
  Drinfel'd (1985), Jimbo (1985).

- Multiparameter version:
  N. Reshetikhin (1990);
  P. Cotta-Ramusino, M. Rinaldi (1991);
  C. De Concini, V.G. Kac, C. Procesi (1992);
  M. Constantini, M. Varagnolo (1994).

- Deformations of Hall algebras: C. M. Ringel (1996).

Quantizations

- Quantized enveloping algebras:
  Drinfel'd (1985), Jimbo (1985).

- Multiparameter version:
  N. Reshetikhin (1990);
  P. Cotta-Ramusino, M. Rinaldi (1991);
  C. De Concini, V.G. Kac, C. Procesi (1992);
  M. Constantini, M. Varagnolo (1994).

- Deformations of Hall algebras: C. M. Ringel (1996).

- Two-parameter quantum groups:
  G. Benkart, S. Witherspoon (2004);
  N. Bergeron, Y. Gao, N. Hu (2005).

Hopf algebras

- Algebras and coalgebras:

$$m : H \otimes H \longrightarrow H \qquad \text{product;}$$

$$\Delta : H \longrightarrow H \otimes H \qquad \text{coproduct;}$$

Hopf algebras

- Algebras and coalgebras:

$$m : H \otimes H \longrightarrow H \qquad \text{product;}$$

$$\Delta : H \longrightarrow H \otimes H \qquad \text{coproduct;}$$

- Examples:

$$\mathbf{k}[G], \quad \Delta(g) = g \otimes g, \quad g \in G;$$

$$U(L), \quad \Delta(l) = l \otimes 1 + 1 \otimes l, \quad l \in L.$$

Hopf algebras

- Algebras and coalgebras:

$$m : H \otimes H \longrightarrow H \qquad \text{product;}$$

$$\Delta : H \longrightarrow H \otimes H \qquad \text{coproduct;}$$

- Examples:

$$\mathbf{k}[G], \quad \Delta(g) = g \otimes g, \quad g \in G;$$

$$U(L), \quad \Delta(l) = l \otimes 1 + 1 \otimes l, \quad l \in L.$$

- Both are cocommutative:

$$\Delta(a) = \sum a^{(1)} \otimes a^{(2)} = \sum a^{(2)} \otimes a^{(1)}.$$

Hopf algebras

- Algebras and coalgebras:

$$m : H \otimes H \longrightarrow H \qquad \text{product;}$$

$$\Delta : H \longrightarrow H \otimes H \qquad \text{coproduct;}$$

- Examples:

$$\mathbf{k}[G], \quad \Delta(g) = g \otimes g, \quad g \in G;$$
$$U(L), \quad \Delta(l) = l \otimes 1 + 1 \otimes l, \quad l \in L.$$

- Both are cocommutative:

$$\Delta(a) = \sum a^{(1)} \otimes a^{(2)} = \sum a^{(2)} \otimes a^{(1)}.$$

- Thm. (Cartier–Kostant). *Every cocommutative Hopf algebra (over* $\mathbf{C}$*) has the form* $\mathbf{C}[G]\#U(L)$.

Hopf algebras

- Algebras and coalgebras:

$$m : H \otimes H \longrightarrow H \qquad \text{product;}$$

$$\Delta : H \longrightarrow H \otimes H \qquad \text{coproduct;}$$

- Examples:

$$\mathbf{k}[G], \quad \Delta(g) = g \otimes g, \quad g \in G;$$
$$U(L), \quad \Delta(l) = l \otimes 1 + 1 \otimes l, \quad l \in L.$$

- Both are cocommutative:

$$\Delta(a) = \sum a^{(1)} \otimes a^{(2)} = \sum a^{(2)} \otimes a^{(1)}.$$

- Thm. (Cartier–Kostant). *Every cocommutative Hopf algebra (over* $\mathbf{C}$*) has the form* $\mathbf{C}[G]\#U(L)$.
- **A quantum group** = a Hopf algebra which is neither commutative nor cocommutative, but it is close to.

Quantizations

- In what extend a quantum universal enveloping algebra of a Kac-Moody algebra $\mathfrak{g}$ is defined by multi-degrees of its defining relations?

Quantizations

- In what extend a quantum universal enveloping algebra of a Kac-Moody algebra $\mathfrak{g}$ is defined by multi-degrees of its defining relations?

- $\mathfrak{K}_{\mathfrak{g}}$ is a class of character Hopf algebras defined by the same number of defining relations of the same degrees as $\mathfrak{g}$ is.

Gabber-Kac representation

- Cartan matrix $A = ||a_{ij}||$: an integral $n \times n$ matrix
  - $a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$,
  - $a_{ij} = 0$ implies $a_{ji} = 0$.

Gabber-Kac representation

- Cartan matrix $A = ||a_{ij}||$: an integral $n \times n$ matrix
  - $a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$,
  - $a_{ij} = 0$ implies $a_{ji} = 0$.
- Generators: $\mathbf{e}_i, \mathbf{f}_i, \mathbf{h}_i$, $1 \leq i \leq n$.

Gabber-Kac representation

- Cartan matrix $A = ||a_{ij}||$: an integral $n \times n$ matrix
  - $a_{ii} = 2$, $a_{ij} \le 0$ for $i \ne j$,
  - $a_{ij} = 0$ implies $a_{ji} = 0$.
- Generators: $\mathbf{e}_i, \mathbf{f}_i, \mathbf{h}_i$, $1 \le i \le n$.
- Relations: $[\mathbf{h}_i, \mathbf{h}_j] = 0, \quad [\mathbf{h}_i, \mathbf{e}_j] = a_{ij}\mathbf{e}_j, \quad [\mathbf{h}_i, \mathbf{f}_j] = -a_{ij}\mathbf{f}_j;$

$$[\mathbf{e}_i, \mathbf{f}_j] = 0 \text{ if } i \ne j, \quad [\mathbf{e}_i, \mathbf{f}_i] = \mathbf{h}_i;$$

$$(\mathrm{ad}\,\mathbf{e}_i)^{1-a_{ij}}\mathbf{e}_j = 0, \quad (\mathrm{ad}\,\mathbf{f}_i)^{1-a_{ij}}\mathbf{f}_j = 0 \text{ if } i \ne j,$$

- $(\mathrm{ad}\,a)^m b = [\ldots[[b, \underbrace{a], a], \ldots, a]}_{m}.$

Quantification of Cartan subalgebra

- $h \longrightarrow \exp(h) = 1 + h + \frac{h^2}{2} + \frac{h^3}{3!} + \frac{h^4}{4!} + \cdots$

  $\Delta(h) = 1 \otimes h + h \otimes 1; \qquad \Delta(\exp(h)) = \exp(h) \otimes \exp(h).$

Quantification of Cartan subalgebra

- $h \longrightarrow \exp(h) = 1 + h + \frac{h^2}{2} + \frac{h^3}{3!} + \frac{h^4}{4!} + \cdots$

$$\Delta(h) = 1 \otimes h + h \otimes 1; \qquad \Delta(\exp(h)) = \exp(h) \otimes \exp(h).$$

$[\mathbf{h}_i, \mathbf{h}_j] = 0$ $\quad G = \exp(H)$ is a commutative group;
$[\mathbf{h}_i, \mathbf{e}_j] = a_{ij}\mathbf{e}_j,$ $\quad G$ acts on $x_j = q(\mathbf{e}_j)$ by a character χ^j;
$[\mathbf{h}_i, \mathbf{f}_j] = -a_{ij}\mathbf{f}_j,$ $\quad G$ acts on $x_j^- = q(\mathbf{f}_j)$ by a character χ^j_-.

Quantification of Cartan subalgebra

- $h \longrightarrow \exp(h) = 1 + h + \frac{h^2}{2} + \frac{h^3}{3!} + \frac{h^4}{4!} + \cdots$

$$\Delta(h) = 1 \otimes h + h \otimes 1; \qquad \Delta(\exp(h)) = \exp(h) \otimes \exp(h).$$

$$\begin{array}{ll} [\mathbf{h}_i, \mathbf{h}_j] = 0 & G = \exp(H) \text{ is a commutative group;} \\ [\mathbf{h}_i, \mathbf{e}_j] = a_{ij}\mathbf{e}_j, & G \text{ acts on } x_j = q(\mathbf{e}_j) \text{ by a character } \chi^j; \\ [\mathbf{h}_i, \mathbf{f}_j] = -a_{ij}\mathbf{f}_j, & G \text{ acts on } x_j^- = q(\mathbf{f}_j) \text{ by a character } \chi_-^j. \end{array}$$

- $$g^{-1} y_j g = \chi^j(g) y_j, \qquad g^{-1} y_j^- g = \chi_-^j(g) y_j^-,$$
$$\chi_-^j = (\chi^j)^{-1}.$$

Hopf algebra structure

- Group-like element: $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$.
  The grouplikes span a Hopf subalgebra $\cong \mathbf{k}\,[G]$.

Hopf algebra structure

- Group-like element: $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$.
  The grouplikes span a Hopf subalgebra $\cong \mathbf{k}\,[G]$.
- A skew-primitive element:
  $\Delta(a) = a \otimes h + f \otimes a, \quad \text{with} \quad f, h \in G.$

Hopf algebra structure

- Group-like element: $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$.
  The grouplikes span a Hopf subalgebra $\cong \mathbf{k}\,[G]$.
- A skew-primitive element:
  $\Delta(a) = a \otimes h + f \otimes a$, with $f, h \in G$.
- Coproduct:

$$\Delta(y_i^{\pm}) = y_i^{\pm} \otimes h_i^{\pm} + f_i^{\pm} \otimes y_i^{\pm}, \qquad h_i^{\pm},\ f_i^{\pm} \in G.$$

Hopf algebra structure

- Group-like element: $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$.
  The grouplikes span a Hopf subalgebra $\cong \mathbf{k}\,[G]$.
- A skew-primitive element:
  $\Delta(a) = a \otimes h + f \otimes a$, with $f, h \in G$.
- Coproduct:

$$\Delta(y_i^{\pm}) = y_i^{\pm} \otimes h_i^{\pm} + f_i^{\pm} \otimes y_i^{\pm}, \qquad h_i^{\pm},\ f_i^{\pm} \in G.$$

- $g^{-1} y_j^{\pm} g = \chi_{\pm}^{j}(g) y_j, \quad \chi_{-}^{j} = (\chi^{j})^{-1}, \ g \in G.$

Second and third groups of relations

- $[\mathbf{e}_i, \mathbf{f}_j] = 0$ if $i \neq j$, $\quad [\mathbf{e}_i, \mathbf{f}_i] = \mathbf{h}_i$;

$$R_{ij} \stackrel{\mathrm{df}}{=} \alpha_{ij}\, y_i y_j^- + \beta_{ij}\, y_j^- y_i = 0,$$

$$R_{ii}(y_i, y_i^-) \stackrel{\mathrm{df}}{=} \alpha_{ii}\, y_i y_i^- + \beta_{ii}\, y_i^- y_i - \sum_k \mu_{ik}\, g_k = 0, \quad g_k \in G.$$

Second and third groups of relations

- $[\mathbf{e}_i, \mathbf{f}_j] = 0$ if $i \neq j$, $\quad [\mathbf{e}_i, \mathbf{f}_i] = \mathbf{h}_i$;

$$R_{ij} \stackrel{\mathrm{df}}{=} \alpha_{ij}\, y_i y_j^- + \beta_{ij}\, y_j^- y_i = 0,$$

$$R_{ii}(y_i, y_i^-) \stackrel{\mathrm{df}}{=} \alpha_{ii}\, y_i y_i^- + \beta_{ii}\, y_i^- y_i - \sum_k \mu_{ik}\, g_k = 0, \quad g_k \in G.$$

- $(\mathrm{ad}\, \mathbf{e}_i)^{1-a_{ij}} \mathbf{e}_j = 0, \quad (\mathrm{ad}\, \mathbf{f}_i)^{1-a_{ij}} \mathbf{f}_j = 0$ if $i \neq j$:

$$S_{ij}(y_i, y_j) \stackrel{\mathrm{df}}{=} \sum_{k=0}^{1-a_{ij}} \gamma_{ijk}\, y_i^k y_j y_i^{1-a_{ij}-k} = 0, \quad i \neq j;$$

$$S_{ij}^-(y_i^-, y_j^-) \stackrel{\mathrm{df}}{=} \sum_{k=0}^{1-a_{ij}} \delta_{ijk}\, (y_i^-)^k y_j^- (y_i^-)^{1-a_{ij}-k} = 0, \quad i \neq j.$$

Second and third groups of relations

- $H = G\langle y_i, y_i^- || R_{ij} = 0, S_{ij}^{\pm} = 0\rangle,\quad 2n^3 + 3n^2$ parameters

$$\alpha_{ij}, \beta_{ij}, \mu_{ik}, \gamma_{ijk}, \delta_{ijk}.$$

Second and third groups of relations

- $H = G\langle y_i, y_i^- \| R_{ij} = 0, S_{ij}^{\pm} = 0 \rangle, \quad 2n^3 + 3n^2$ parameters

$$\alpha_{ij}, \beta_{ij}, \mu_{ik}, \gamma_{ijk}, \delta_{ijk}.$$

- **Theorem 1.** *The only way to keep the Hopf algebra structure on H is to replace the Lie operation in Gabber-Kac relations with a skew commutator:*

$$[u, v] = \chi^v(h_u)uv - \chi^u(f_v)vu.$$

Second and third groups of relations

- $H = G\langle y_i, y_i^- || R_{ij} = 0, S_{ij}^{\pm} = 0\rangle, \quad 2n^3 + 3n^2$ parameters

$$\alpha_{ij}, \beta_{ij}, \mu_{ik}, \gamma_{ijk}, \delta_{ijk}.$$

- **Theorem 1.** *The only way to keep the Hopf algebra structure on H is to replace the Lie operation in Gabber-Kac relations with a skew commutator:*

$$[u, v] = \chi^v(h_u)uv - \chi^u(f_v)vu.$$

- Remains just n^2 parameters, $p_{ij} = \chi^i_-(h_j)\chi^i(f_j)$, of $2n^3 + 3n^2$.

Indecomposable Cartan matrices $A = ||a_{ij}||$.

- **Definition.** A quantification H is *regular* if

$$p_{ij}p_{ji} = p_{ii}^{a_{ij}}.$$

Indecomposable Cartan matrices $A = ||a_{ij}||$.

- **Definition.** A quantification H is *regular* if

$$p_{ij}p_{ji} = p_{ii}^{a_{ij}}.$$

- All known quantizations are regular.

Indecomposable Cartan matrices $A = ||a_{ij}||$.

- **Definition.** A quantification H is *regular* if

$$p_{ij}p_{ji} = p_{ii}^{a_{ij}}.$$

- All known quantizations are regular.
- **Theorem 2.** *There exists just a finite number of algebra structures for exceptional quantizations of a given Kac-Moody algebra.*

Indecomposable Cartan matrices $A = ||a_{ij}||$.

- **Definition.** A quantification H is *regular* if

$$p_{ij}p_{ji} = p_{ii}^{a_{ij}}.$$

- All known quantizations are regular.
- **Theorem 2.** *There exists just a finite number of algebra structures for exceptional quantizations of a given Kac-Moody algebra.*
- **Theorem 3.** *If one of the parameters* $p_{ij}p_{ji}$, $1 \leq i, j \leq n$ *is not a root of 1 then the quantification* H *is regular.*

Indecomposable Cartan matrices $A = ||a_{ij}||$.

- **Definition.** A matrix A is *symmetrizable* if there exists natural $d_1, d_2, \ldots, d_n$ such that, $d_i a_{ij} = d_j a_{ji}$, $1 \leq i, j \leq n$. An n-tuple $(m_1, m_2, \ldots, m_n)$ is a *modular symmetrization* if

$$m_i a_{ij} \equiv m_j a_{ji} (\mathrm{mod}\, N), \quad 0 \leq m_i,\ m_j < N, \quad 1 \leq i, j \leq n.$$

Indecomposable Cartan matrices $A = ||a_{ij}||$.

- **Definition.** A matrix A is *symmetrizable* if there exists natural $d_1, d_2, \ldots, d_n$ such that, $d_i a_{ij} = d_j a_{ji}$, $1 \le i, j \le n$. An n-tuple $(m_1, m_2, \ldots, m_n)$ is a *modular symmetrization* if

$$m_i a_{ij} \equiv m_j a_{ji} (\mathrm{mod}\, N), \quad 0 \le m_i,\ m_j < N, \quad 1 \le i,\ j \le n.$$

- **Theorem 4.** *The algebraic structure of a regular quantification is defined by two independent parameters $q \in \bar{\mathbf{k}}^*$ and $\mathfrak{m} \in \mathfrak{M}$ so that*

$$p_{ii} = q^{d_i} \xi^{m_i}, \qquad \mathfrak{m} = (m_1, m_2, \ldots, m_n),$$

where ξ is a fixed Nth primitive root of 1.

Indecomposable Cartan matrices $A = ||a_{ij}||$.

- **Definition.** A matrix A is *symmetrizable* if there exists natural $d_1, d_2, \dots, d_n$ such that, $d_i a_{ij} = d_j a_{ji}$, $1 \le i, j \le n$. An n-tuple $(m_1, m_2, \dots, m_n)$ is a *modular symmetrization* if

$$m_i a_{ij} \equiv m_j a_{ji} (\mathrm{mod}\, N), \quad 0 \le m_i,\ m_j < N, \quad 1 \le i,\, j \le n.$$

- **Theorem 4.** *The algebraic structure of a regular quantification is defined by two independent parameters $q \in \bar{\mathbf{k}}^*$ and $\mathfrak{m} \in \mathfrak{M}$ so that*

$$p_{ii} = q^{d_i} \xi^{m_i}, \qquad \mathfrak{m} = (m_1, m_2, \dots, m_n),$$

where ξ is a fixed Nth primitive root of 1.

- In all known quantizations, $\mathfrak{m} = (0, 0, \dots, 0)$.

The number of exceptional quantizations

Here φ_i, $i \geq 0$ is the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$.

A_n	$\varphi_{2n} - 2$	G_2	12	$E_7^{(1)}$	632
B_n	$2\varphi_{2n} - 2$	$A_1^{(1)}$	6	$E_8^{(1)}$	4344
C_n	$2\varphi_{2n-2} + 4$	$A_l^{(1)}$	$\varphi_{2n} + \varphi_{2n-2} - 2$	$A_2^{(2)}$	24
E_6	240	$C_l^{(1)}$	$4\varphi_{2n-4} + 28$	$A_{2l}^{(2)}$	$4\varphi_{2n-2} + 16$
E_7	632	$G_2^{(1)}$	38	$D_{l+1}^{(2)}$	$4\varphi_{2n} + 2$
E_8	1658	$F_4^{(1)}$	91	$E_6^{(2)}$	80
F_4	40	$E_6^{(1)}$	635	$D_4^{(3)}$	19

D_n	$\varphi_{2n} + \varphi_{2n-7} - 2$	$D_l^{(1)}$	$15\,\varphi_{2n-6} + 11\,\varphi_{2n-8} - 2$
$A_{2l-1}^{(2)}$	$2\varphi_{2n-2} + 2\varphi_{2n-9} + 2$	$B_l^{(1)}$	$\varphi_{2n+1} + 5\varphi_{2n-5} - 2$

THE END

THANK YOU