On lattices of annihilators

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(with Małgorzata Jastrzębska)

1 Introduction

In this notes $\mathbb{K}$ will be a field. All algebras are associative $\mathbb{K}$-algebras, with $1 \neq 0$. If A is an algebra, then A^{op} denotes the algebra with the same linear structure over $\mathbb{K}$ but with the opposite multiplication.

All lattices have the smallest element ω and the largest element $\Omega \neq \omega$. If L is a lattice then by L^{op} we denote the lattice with the reverse order.

For every algebra A the set $\mathcal{I}_l(A)$ of all left ideals and the set $\mathcal{I}_r(A)$ of all right ideals, ordered by inclusion are complete, modular lattices with operations:

$$I \vee J = I + J \quad \text{and} \quad I \wedge J = I \cap J.$$

In these lattices $\omega = 0$ and $\Omega = A$. Also:

$$\mathcal{I}_l(A^{op}) = \mathcal{I}_r(A) \quad \text{and} \quad \mathcal{I}_r(A^{op}) = \mathcal{I}_l(A).$$

If $X \subseteq A$ is a subset, then let $\mathrm{L}_A(X) = \mathrm{L}(X)$ be the left annihilator of X in A and let $\mathrm{R}_A(X) = \mathrm{R}(X)$ be the right annihilator of X in A :

$$\mathrm{L}(X) = \{a \in A : aX = 0\},$$

$$\mathrm{R}(X) = \{a \in A : Xa = 0\}.$$

Then

$$\mathrm{L}(X) = \mathrm{L}(\mathrm{R}(\mathrm{L}(X))) \quad \text{and} \tag{1}$$

$$\mathrm{R}(X) = \mathrm{R}(\mathrm{L}(\mathrm{R}(X))). \tag{2}$$

Let $\mathcal{A}_l(A)$ be the set of all left annihilators in A and $\mathcal{A}_r(A)$ be the set of all right annihilators in A. Then $\mathcal{A}_l(A) \subseteq \mathcal{I}_l(A)$ is a complete lattice with operations:

$$I \vee J = \mathrm{L}(\mathrm{R}(I) \cap \mathrm{R}(J)) \quad \text{and} \quad I \wedge J = I \cap J,$$

for $I, J \in \mathcal{A}_l(A),\ \omega = 0$ and $\Omega = A$.

Similarly, $\mathcal{A}_r(A) \subseteq \mathcal{I}_r(A)$ is a complete lattice with operations

$$I \vee J = \mathrm{R}(\mathrm{L}(I) \cap \mathrm{L}(J)) \quad \text{and} \quad I \wedge J = I \cap J,$$

for $I, J \in \mathcal{I}_r(A),\ \omega = 0$ and $\Omega = A$.

$$\mathcal{A}_l(A^{op}) = \mathcal{A}_r(A) \quad \text{and} \quad \mathcal{A}_r(A^{op}) = \mathcal{A}_l(A).$$

The operators L and R induce a Galois correspondence between $\mathcal{A}_l(A)$ and $\mathcal{A}_r(A)$, by Formulas (1) and (2):

$$\mathcal{A}_l(A) \overset{\mathrm{R}}{\simeq} (\mathcal{A}_r(A))^{op}, \quad (\mathcal{A}_r(A))^{op} \overset{\mathrm{L}}{\simeq} \mathcal{A}_l(A).$$

Question 1. *Let A be any algebra.*

- *Is $\mathcal{A}_l(A)$ a sublattice of $\mathcal{I}_l(A)$?*
- *Is $\mathcal{A}_l(A)$ modular or satisfies some other identities?*

2 Some results

Theorem 2.1 (see [2, 6])**.** *If A is a semiprime algebra, then the following are equivalent:*

1. *A is semisimple artinian;*
2. *$\mathcal{A}_l(A) = \mathcal{I}_l(A)$;*
3. *$\mathcal{A}_r(A) = \mathcal{I}_r(A)$.*

Left and right artinian algebras with $\mathcal{I}_l(A) = \mathcal{A}_l(A)$ and $\mathcal{I}_r(A) = \mathcal{A}_r(A)$ are QF-algebras.

Example 2.2. Let A be an algebra with the base $\{x_\alpha : 0 \leq \alpha \leq 1\}$ indexed by numbers from the interval $[0, 1] \subset \mathbb{R}$, with multiplication given by $x_\alpha x_\beta = x_{\alpha+\beta}$ if $\alpha + \beta \leq 1$ and 0 otherwise. Then A is a commutative local algebra and

$$\mathcal{I}_l(A) = \mathcal{A}_l(A) = \mathcal{A}_r(A) = \mathcal{I}_r(A).$$

However, A is not artinian.

Lemma 2.3. *Let* $A \subseteq B$ *be algebras and let* $\mu : \mathcal{A}_l(A) \longrightarrow \mathcal{A}_l(B)$ *be given by:*

$$\mu(I) = \mathrm{L}_B(\mathrm{R}_A(I)) \quad \textit{for} \quad I \in \mathcal{A}_l(A).$$

Then μ *is an embedding of ordered sets.*

Main argument: $\mu(I) \cap A = I$ for $I \in \mathcal{A}_l(A)$.

Corollary 2.4. *Let* A *be a semiprime, left Goldie algebra with the classical left ring of fractions* B*. Then* $\mathcal{A}_l(A)$ *and* $\mathcal{A}_r(A)$ *are modular lattices of finite height.*

Theorem 2.5 ([5]). *There exists a finitely presented algebra A with $\mathcal{A}_l(A)$ not modular. Hence $\mathcal{A}_l(A)$ is not a sublattice of $\mathcal{I}_l(A)$.*

Theorem 2.6 ([3]). *There exists an algebra A with maximum and minimum condition on annihilators, but with no common bound for lengths of chains of annihilators.*

3 Lattices as annihilators

Example 3.1 (Basic)**.** Let L be a lattice, $L_0 = L\setminus\{\omega\}$ and let $\mathbb{K}\langle L\rangle = \mathbb{K}\{L_0\}/I$ where $\mathbb{K}\{L_0\}$ is the free algebra with the set L_0 of free generators, and I is the ideal generated by the following elements: xy for $x \leq y \in L_0$ and xyz for all $x, y, z \in L_0$.

The algebra $\mathbb{K}\langle L\rangle$ is an algebra with gradation given by

$$\mathbb{K}\langle L\rangle = \mathbb{K} \oplus V \oplus V^2, \tag{3}$$

where the natural base of V can be identified with L_0 and the natural base of V^2 consists of all products xy for $x, y \in L_0$, such that $x \not\leq y$.

Our algebra $\mathbb{K}\langle L\rangle$ is a local algebra with the Jacobson radical $J = V \oplus V^2$ and with the residue field $\mathbb{K}\langle L\rangle/J = \mathbb{K}$.

Theorem 3.2. *Let L be a lattice. Under the notation from the above example and $\mathbb{K}\langle L\rangle = A$ let $\phi : L \longrightarrow \mathcal{A}_l(A)$ be given by $\phi(x) = \mathrm{L}_A(x)$ for $x \in L_0$, $\phi(\omega) = V^2$ if L_0 has not the smallest element and $\phi(\omega) = 0$ if L_0 has the smallest element. Then ϕ is a lattice embedding and preserves existing infinite meets and joins. Moreover, if L is complete, then ϕ is an isomorphism of L with the interval $[\phi(\omega),\ J] \subset \mathcal{A}_l(A)$.*

Corollary 3.3. *There is no lattice identity satisfied in every lattice of annihilators.*

Corollary 3.4. *Let L be a lattice with $|L| = n < \infty$. Under the notation from the above theorem, the algebra $\mathbb{K}\langle L\rangle$ is finite dimensional, because*

$$1 + \frac{n(n-1)}{2} \leq \mathrm{Dim}_{\mathbb{K}}(\mathbb{K}\langle L\rangle) \leq n + (n-2)^2.$$

For further information about the case of finite lattices see the talk of Małgorzata Jastrzębska on the conference "Classical aspects of ring theory and module theory", Będlewo, July 14-20 2013.

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