Jordan nilpotency in Group Rings

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Introduction

Notation

Let A be an associative ring. The **Lie bracket** of two elements $a, b \in A$ is given by:

$$[a, b] = ab - ba.$$

With usual addition and the Lie bracket as a multiplication, A becomes a **Lie Algebra** in the sense that it satisfies the **Jacobi Identity**:

$$[[x, y], z] + [[z, x], y] + [[y, z], x] = 0 \quad \forall x, y, z \in A.$$

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The ring is said to be **Lie nilpotent**, of index n, if $[\dots[[x_1, x_2], x_3], \dots, x_n] = 0$ for all choices of elements $x_1, x_2, \dots, x_n \in A$.

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$$a \circ b = ab + ba.$$

With usual addition and this operation as a multiplication, A becomes a **Jordan Algebra** in the sense that it satisfies the **Jordan Identity**:

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Theorem (Passi, Passman and Sehgal 1973)

Let K be a field with $char(K) = p \geq 0$ and G a group. The group algebra KG is Lie nilpotent if and only if G is nilpotent and p-abelian.

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Recall: G is p-abelian if G' is a finite p group (when $p > 0$) or G is an abelian group (when p=0).

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Definition

Let $*$ be an involution on a group ring RG. We consider:

$$\begin{aligned} RG^+ &= \{\alpha \in RG \mid \alpha^* = \alpha\} \\ RG^- &= \{\alpha \in RG \mid \alpha^* = -\alpha\} \end{aligned}$$

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the sets of **symmetric** and **skew-symmetric** elements of RG, respectively.

Notice:

- RG^- is a **Lie subalgebra** of RG.
- RG^+ is a **Jordan subalgebra** of RG.

Theorem (Jespers and Ruiz, 2006)

Let φ be an involution of a non-abelian group G and let R be a commutative ring os characteristic different from 2. Then, the following are equivalent:

1. RG^+ is commutative.
2. G is an SLC group with cannonical involution.

Theorem (Broche, Jespers, P.M. and Ruiz 2009)

Let R be a commutative ring. Suppose G is a non-abelian group and φ is an involution on G. Then, $(RG)^-$ is commutative if and only if one of the following conditions holds:

1. $K = \langle g \in G |\ g \notin G^+ \rangle$ is abelian (and thus $G = K \cup Kx$, where $x \in G^+$, and $\varphi(k) = xkx^{-1}$ for all $k \in K$) and $R_2^2 = \{0\}$.
2. $R_2 = \{0\}$ and G contains an abelian subgroup of index 2 that is contained in G^+.
3. $char(R) = 4$, $|G'| = 2$, $G/G' = (G/G')^+$, $g^2 \in G^+$ for all $g \in G$, and G^+ is commutative in case $R_2^2 \neq \{0\}$.
4. $char(R) = 3$, $|G'| = 3$, $G/G' = (G/G')^+$ and $g^3 \in G^+$ for all $g \in G$.

Theorem (Giambruno, PM and Sehgal 2013)

Let F be a field, $char(F) \neq 2$, and let G be a group with no 2-elements. Let * be an involution on FG induced by an involution of G and suppose that no dihedral group is involved in G. Then the Lie algebra FG^- is nilpotent if and only if either FG is Lie nilpotent or $char(F) = p > 2$ and the following conditions hold.

1. The set P of p-elements in G is a subgroup,
2. * is trivial on G/P ,
3. there exist normal *-invariant subgroups A and B, $B \subset A$ such that B is a finite central p-subgroup of G, A/B is central in G/B and both G/A and $\{a \in A \mid aa^* \in B\}$ are finite. 2.

SLC groups

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A **loop** is a set L together with a (closed) binary operation $(a, b) \mapsto ab$ for which there is a two-sided identity element 1 and such that the right and left translation maps

$$R_x : a \mapsto ax \quad \text{and} \quad L_x : a \mapsto xa$$

are bijections for all $x \in L$. This requirement implies that, for any $a, b \in L$, the equations $ax = b$ and $ya = b$ have unique solutions.

The **loop algebra** of L over an associative and commutative ring with unity R was introduced in 1944 by R.H. Bruck as a means to obtain a family of examples of nonassociative algebras.
It is defined in a way similar to that of a group algebra; i.e., as the free R-module with basis L, with a multiplication induced distributively from the operation in L.

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Definition

A ring R is **alternative** if

$$x(xy) = (xx)y \text{ and } (xy)y = x(yy) \text{ for all } x, y \in R.$$

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Theorem

Let L be a loop. Then L is a loop with an alternative loop ring if and only if it has the following properties:

(i) If three elements associate in some order then they associate in all orders and

(ii) If $g, h, k \in L$ do not associate, then $gh.k = g.kh = h.gk$.

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(ii) If $g, h, k \in L$ do not associate, then $gh.k = g.kh = h.gk$.

It follows that if RL is alternative over one ring R as in the definition, then it is also alternative over *all* such rings.

Definition

A group G, with center $\mathcal{Z}(G)$, is called an **LC group** (or, that it has **limited commutativity**) if it is not commutative and for any pair of elements $x, y \in G$ we have that $xy = yx$ if and only if either $x \in \mathcal{Z}(G)$ or $y \in \mathcal{Z}(G)$ or $xy \in \mathcal{Z}(G)$.

Theorem

A loop L is RA if and only if it is not commutative and, for any two elements a and b of L which do not commute, the subloop of L generated by its centre together with a and b is a group G such that

(i) for any $u \notin G$, $L = G \cup Gu$ is the disjoint union of G and the coset Gu;

(ii) G is an LC group.

(iii) G has a unique nonidentity commutator s, which is necessarily central and of order 2;

(iv) the map

$$g \mapsto g^* = \begin{cases} g & \text{if } g \text{ is central} \\ sg & \text{otherwise} \end{cases}$$

is an involution of G (i.e., an antiautomorphism of order 2);

(v) multiplication in L is defined by

$$\begin{aligned} g(hu) &= (hg)u \\ (gu)h &= gh^*u \\ (gu)(hu) &= g_0h^*g \end{aligned}$$

where $g, h \in G$ and $g_0 = u^2$ is a central element of G.

Definition

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Proposition

A group G, with center $\mathcal{Z}(G)$, is an SLC group if and only if $G/\mathcal{Z}(G) \cong C_2 \times C_2$.

Theorem (Leal - PM, 1993)

A group G is SLC if and only if G can be written in the form $G = D \times A$, where A is abelian and D is an indecomposable 2-group generated by its centre and two elements x and y which satisfy

(i) $\mathcal{Z}(D) = C_{2^{m_1}} \times C_{2^{m_2}} \times C_{2^{m_3}}$, where $C_{2^{m_i}}$ is cyclic of order 2^{m_i} for $i = 1, 2, 3$, $m_1 \geq 1$ and $m_2, m_3 \geq 0$;

(ii) $(x, y) \in C_{2^{m_1}}$;

(iii) $x^2 \in C_{2^{m_1}} \times C_{2^{m_2}}$ and $y^2 \in C_{2^{m_1}} \times C_{2^{m_2}} \times C_{2^{m_3}}$.

Theorem (Jespers, Leal and PM, 1995)

Let G be a finite group. Then $G/\mathcal{CZ}(G) \cong C_2 \times C_2$ if and only if G can be written in the form $G = D \times A$, where A is abelian and $D = \langle \mathcal{Z}(D), x, y \rangle$ is of one of the following five types of indecomposable 2-groups:

Type	$\mathcal{Z}(D)$	D
D_1	$\langle t_1 \rangle$	$\langle x, y, t_1 \mid (x,y) = t_1^{2^{m_1}-1}, x^2 = y^2 = t_1^{2^{m_1}} \rangle$
D_2	$\langle t_1 \rangle$	$\langle x, y, t_1 \mid (x,y) = t_1^{2^{m_1}-1}, x^2 = y^2 = t_1, t^{2^{m_1}} = 1 \rangle$
D_3	$\langle t_1 \rangle \times \langle t_2 \rangle$	$\langle x, y, t_1, t_2 \mid (x,y) = t_1^{2^{m_1}-1}, x^2 = t_1^{2^{m_1}} = t_2^{2^{m_2}} = 1, y^2 = t_2 \rangle$
D_4	$\langle t_1 \rangle \times \langle t_2 \rangle$	$\langle x, y, t_1, t_2 \mid (x,y) = t_1^{2^{m_1}-1}, x^2 = t_1, y^2 = t_2, t_1^{2^{m_1}} = t_2^{2^{m_2}} = 1 \rangle$
D_5	$\langle t_1 \rangle \times \langle t_2 \rangle \times \langle t_3 \rangle$	$\langle x, y, t_1, t_2, t_3 \mid (x,y) = t_1^{2^{m_1}-1}, x^2 = t_2, y^2 = t_3, t_1^{2^{m_1}} = t_2^{2^{m_2}} = t_3^{2^{m_3}} = 1 \rangle$

Jordan Nilpotency

Theorem (Goodaire and PM)

Let RG denote the group ring of a group G over a commutative coefficient ring R with 1. Then RG is Jordan nilpotent of index 3 if and only if

1. $car(R) = 4$ and G is abelian or,
2. $car(R) = 2$ and either G is abelian or G has a unique nonidentity commutator.

Theorem (Goodaire and PM)

Suppose the characteristic of R is different from 2 and $\alpha \mapsto \alpha^*$ is an involution on the group ring RG that extends linearly an involution on G. Then the Jordan ring $(RG)^+$ of symmetric elements is Jordan nilpotent of index 3 if and only if $car(R) = 4$ and G is abelian, or an SLC group with $*$ canonical, or a nonabelian group with the following properties:

(a) any $g \in G$ with $g^* = g$ is central;

(b) G has an abelian subgroup A of index 2;

(c) there exists $c \notin A$ with the property that for any $a \in A$, either $ac = ca$ or the commutator (a, c) is central of order 2;

(d) $a^* = (a, c)a$ for all $a \in A$.