Prime ideals and radicals of polynomial rings and Z-graded rings

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1 Notation

- All rings are associative but not necessarily with 1.
- The usual extension of a ring A by adjoining 1 to A is denoted by A^*.
- To denote that I is an ideal of a ring A we write $I \lhd A$.
- Nil, J, G denote the nil, Jacobson and Brown-McCoy radical, respectively.
- $G(A) = \bigcap\{I \lhd A \mid A/I \text{ simple ring with } 1\}$. In particular a ring A is Brown-McCoy radical if it cannot be homomorphically mapped onto a (simple) ring with 1.

2 Some old results and problems

- 1956 Amitsur

Theorem 2.1 *For every ring A, $J(A[x]) = I[x]$ for some nil ideal I of A.*

Questions 2.2 *1. Is $J(A[x]) = Nil(A)[x]$?*

2. Is $J(A[x]) = Nil(A[x])$?

3. Is $Nil(A[x]) = Nil(A)[x]$?

- 1972 Krempa

Theorem 2.3 *The following conditions are equivalent*

a) For every ring A, $J(A[x]) = Nil(A)[x]$;

b) For every nil ring A the ring $M_2(A)$ of 2×2-matrices over A is nil;

c) Koethe problem has a positive solution.

Koethe Problem (1930): Is every left nil ideal of an arbitrary ring A contained in $Nil(A)$?.

- 1978 Krempa and Sierpińska

Theorem 2.4 *For every ring A, $J(A[x^{-1}, x]) = I[x, x^{-1}]$, where I is an ideal of A such that $J(A[x]) = I[x]$.*

- 1980 Bedi and Ram

Theorem 2.5 *1. If σ is an endomorphism of a ring A, then $J(A[x;\sigma] = (J(A) \cap I) + I[x;\sigma]x$, where $I = \{r \in A \mid rx \in J(A[x;\sigma])\}$.*

2. If σ is an automorphism of A, then $J(A[x,x^{-1};\sigma]) = K[x,x^{-1};\sigma]$, where K is a σ-invariant ideal of A such that for every $r \in K$, $rx \in J(A[x,x^{-1};\sigma])$.

It is not known whether $I = K$.

- $\sim$ 1980 Bergman.

Theorem 2.6 *If R is a Z-graded ring, then $J(R)$ is homogeneous.*

Questions 2.7 (1993 E.P.). *Let A be a nil ring.*

1. Is $A[x]$ Brown-McCoy radical?

2. Let X be a set of cardinality ≥ 2.

a) Is the polynomial ring $A[X]$ in commuting indeteminates from X Brown-McCoy radical?

b) Is the polynomial ring $A\langle X\rangle$ it non-commuting indeterminates from X Brown-McCoy radical?

3. Suppose that R is a Z-graded ring. Is $Nil(R)$ homogeneous?

3 Approximations of positive or negative solutions of Koethe problem

• 2000 Smoktunowicz constructed a counterexample to Amitsur's Problem 3.

• 2001 Smoktunowicz and E.P. constructed a counterexample to Amitsur's Problem 2.

• 1998 Smoktunowicz and E.P.

Theorem 3.1 *For a given ring A, $A[x]$ is Brown-McCoy radical if and only if A cannot be homomorphically mapped onto a prime ring A' such that for every $0 \neq I \lhd A'$, $Z(A') \cap I \neq 0$.*

This in particular shows that if A is a nil ring, then $A[x]$ is Brown-McCoy radical.

Let $\mathcal{P}$ be the class of prime rings A with large center, i.e., for every $0 \neq I \lhd A$, $I \cap Z(A) \neq 0$.

Define for an arbitrary ring A, $S(R) = \bigcap\{I \lhd R \mid R/I \in \mathcal{P}\}$.

Corollary 3.2 *$G(A[x]) = S(A)[x]$ for every ring A.*

• 2001 Beidar, Fong and E.P.

Theorem 3.3 *If A is a nil ring, then $A[x]$ is Behrens radical, i.e., $A[x]$ cannot be homomorphically mapped onto a ring containing non-trivial idempotents.*

- 2006 Smoktunowicz

Theorem 3.4 *If A is a nil ring and P is a primitive ideal of $A[x]$ (so under the assumption that Koethe problem has a negative solution), then $P = I[x]$ for an ideal I of A.*

- 2008 Smoktunowicz

Example 3.5 *There exists a positively graded ring, which is graded-nil and Jacobson semisimple.*

- 2008 Smoktunowicz

Theorem 3.6 *If R is positively graded, graded-nil and I is a primitive ideal of R, then I is homogeneous.*

4 Polynomial rings in several indeterminates

• 2002 Smoktunowicz, 2003 Ferrero and Wisbauer.

Theorem 4.1 *If X is infinite then for any (not necessarily nil) ring A, $A[X]$ is Brown-McCoy radical if and only if $A\langle X\rangle$ is Brown-McCoy radical.*

• 2002 Smoktunowicz

Theorem 4.2 *If $A[x]$ is Jacobson radical, then $A[x,y]$ is Brown-McCoy radical.*

• 2003 Ferrero and Wisbauer.

Problem. Is $S(A[x]) = S(A)[x]$ for every ring A?

• 2006 Chebotar, Ke, Lee and E.P.

Theorem 4.3 *If A is a nil algebra over a field of a positive characteristic, then $S(A[x]) = A[x]$ and $A[x,y]$ is Brown-McCoy radical.*

It is not known whether if A is nil algebra over a field of characteristic 0, then $A[x,y]$ is Brown-McCoy radical.

• In the context of these questions Beidar asked (unpublished):

1. Does there exists a prime ring A with trivial center such that the Martindale central closure of A is a simple ring with 1?

2. Does there exist a prime nil ring A such that the central closure of A is a simple ring with 1?

• 2008 Chebotar constructed an example answering the first of these questions.

The second question is still open.

5 Some recent results on Z-graded rings

In what follows R is a Z-graded ring.

• Smoktunowicz (arxiv) $Nil(R)$ is homogeneous.

• If R is positively graded, then every homogeneous subring of $J(R)$ is Jacobson radical.

The second of these results was also proved by

• P. H. Lee and E.P. (JPAA, to appear).

Theorem 5.1 *Every Z-graded algebra of characteristic $p > 0$, which is graded-nil is S-radical.*

Theorem 5.2 *Every Z-graded ring, which is graded-nil, is Brown-McCoy radical.*

Corollary 5.3 *Every homogeneous subring of a Z-graded ring R is Brown-McCoy radical if and only if R is graded-nil.*

Let $\mathcal{P}_h$ be the class of prime Z-graded rings R such that $I \cap Z(R) \neq 0$ for every non-zero homogeneous ideal I of R.

Define for a Z-graded ring R, $S_h(R) = \bigcap\{I \lhd R, I - homogeneous \mid R/I \in \mathcal{P}_h\}$.

Theorem 5.4 *If R is positively graded, then $G(R) = S_h(R)$.*

6 Quasi duo skew polynomial rings and Z-graded rings

A ring A with 1 is called *left quasi-duo* if every maximal left ideal of A is two-sided. Right quasi-duo rings are defined similarly.

• 2005 Lam and Dugas.

Problem. Is every left quasi-duo ring right quasi-duo?

Remark. If there existed a left quasi-duo ring, which is not right quasi-duo, then there would exist a right primitive rings, which is left quasi-duo (so not left primitive).

• 1979 Irving. There exist right primitive skew polynomial rings, which are not left primitive.

• 2008 Leroy, Matczuk and E.P.

Theorem 6.1 *Let A be a domain with an automorphism σ. If $A[x;\sigma]$ is left quasi-duo, then A is commutative and $\sigma = id$.*

Theorem 6.2 *For a ring A with an endomorphism σ, $A[x;\sigma]$ is a right (left) quasi-duo ring if and only if*

(a) A is right (left) quasi-duo and $J(A[x;\sigma]) = (J(A) \cap N(A)) + N(A)[x;\sigma]x$, where $N(A) = \{a \in A \mid ax - nilpotent\}$

and

(b) $N(R)$ is a σ-stable ideal of A, the factor ring $A/N(A)$ is commutative and the endomorphism σ induces identity on $A/N(A)$.

Theorem 6.3 *The following are equivalent*

1. $A[x, x^{-1};\sigma]$ is right (left) quasi-duo;

2. $J(A[x, x^{-1};\sigma]) = N(A)[x, x^{-1};\sigma]$, $A/N(A)$ is commutative and the automorphism of $A/N(A)$ induced by σ is equal to $id_{A/N(A)}$;

3. $A[x, x^{-1};\sigma]/J(A[x, x^{-1};\sigma])$ is commutative.

• 2010 Leroy, Matczuk and E.P.

Theorem 6.4 *A Z-graded ring R is right (left) quasi-duo if and only if R_0 is right (left) quasi-duo and $R/A(R)$ is a commutative ring, where $A(R) = \{r \in R \mid R_n r \subseteq J(R)$ for every $0 \neq n \in Z\} = \{r \in R \mid rR_n \subseteq J(R)$ for every $0 \neq n \in Z\}$.*