Counting Additive $\mathbb{Z}_2\mathbb{Z}_4$ codes

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Outline

Outline

The Problem

TO COUNT ADDITIVE CODES

What we've done

- Free additive $\mathbb{Z}_2\mathbb{Z}_4$ codes
- Arbitrary additive $\mathbb{Z}_2\mathbb{Z}_4$ codes
- Decomposable codes

Outline

Translation-Invariant Propelinear Codes

- 1973, Delsarte [1].
- 1998, The only structures for the abelian group (2^n) are of the form $\mathbb{Z}_2^\alpha\mathbb{Z}_4^\beta$, with $\alpha + 2\beta = n$ [2].
- $\mathbb{Z}_2\mathbb{Z}_4$ codes are translation-invariant propelinear codes.

History of Additive Codes

- Borges, Fernández and collaborators, [3]-[6].
  - 2006, $\mathbb{Z}_2\mathbb{Z}_4-$Linear codes, Borges, Fernández, Pujol, Rifà, Villanueva [3]
  - 2010, Generator matrices and parity check matrices, Borges, Fernández, Pujol, Rifà, Villanueva [4]
  - 2011-..., Structure of MDS and self dual codes Bilal, Borges, Dougherty, Fernández [5] and Borges, Dougherty, Fernández [6].

History of Counting Problem

- Codes over finite fields: Gaussian coefficients.
- The number of subgroups of a given finite p-group:
  - 1948, Delsarte [7], Djubjuk [8],
  - 2000, Honold [9],
  - 2004, Calugreanu [10] ,
- 2013, Codes over finite chain rings and finite principal ideal rings: Dougherty and Saltürk [11].

Outline

[3]-[6]

- An additive $\mathbb{Z}_2\mathbb{Z}_4$ code C is a subgroup of $\mathbb{Z}_2^\alpha\mathbb{Z}_4^\beta$, it is isomorphic to an abelian structure $\mathbb{Z}_2^\gamma\mathbb{Z}_4^\delta$.
  - $|C| = 2^\gamma 4^\delta$.
  - The number of order two vectors is $2^{\gamma+\delta}$.

Example

Take C as a $\mathbb{Z}_2\mathbb{Z}_4$ code generated by

$$G = \left(\begin{array}{cccc|cc} 1 & 0 & 0 & 1 & 0 & 2 \\ \hline 1 & 0 & 1 & 1 & 2 & 1 \end{array}\right)$$

$$C = \Big\{(0000|00), (1011|21), (0000|02), (1011|23), \\ (1001|02), (0010|23), (1001|00), (0010|21)\Big\}$$

[3]-[6]

- For any vector $\mathbf{v} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, $\mathbf{v} = (\mathbf{v}_1|\mathbf{v}_2)$, $\mathbf{v}_1 = (x_1, \ldots, x_\alpha) \in \mathbb{Z}_2^\alpha$ and $\mathbf{v}_2 = (y_1, \ldots, y_\beta) \in \mathbb{Z}_4^\beta$.
- An extension of the usual Gray map Φ is defined as $\Phi : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_2^n$, where $n = \alpha + 2\beta$
$\Phi(\mathbf{v}_1|\mathbf{v}_2) = (\mathbf{v}_1|\phi(y_1), \ldots, \phi(y_\beta))$.

[3]-[6]

- X and Y denote the set of $\mathbb{Z}_2$ and $\mathbb{Z}_4$ coordinate positions, respectively.
  - $|X| = \alpha$ and $|Y| = \beta$.
  - X corresponds to the first α coordinates and Y corresponds to the last β coordinates.
- Define C_X and C_Y.

[3]-[6]

- C_b: The subcode of C which contains all order two codewords
  κ: The dimension of $(C_b)_X$.
    - C_b is a binary linear code.
- When $\alpha = 0$, then $\kappa = 0$.
- We say that a $\mathbb{Z}_2\mathbb{Z}_4$ code C is of type $(\alpha, \beta; \gamma, \delta; \kappa)$.

Example

From the previous example, take C as a $\mathbb{Z}_2\mathbb{Z}_4$ code generated by

$$G = \left(\begin{array}{cccc|cc} 1 & 0 & 0 & 1 & 0 & 2 \\ \hline 1 & 0 & 1 & 1 & 2 & 1 \end{array}\right)$$

- $\alpha = 4$, $\beta = 2$ since $|X| = 4$ and $|Y| = 2$.
- The order of C is $2^1 4^1$, hence $\gamma = 1$ and $\delta = 1$.
- The code C_b is generated by $(1001|02)$ and $(0000|21)$. $(C_b)_X$ is generated by (1001) and so $\kappa = 1$
- C is of type $(4, 2, 1, 1, 1)$.

Outline

[3]-[6]

A $\mathbb{Z}_2\mathbb{Z}_4$ code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ is permutation equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$ code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with standard generator matrix of the form

$$G = \left(\begin{array}{cc|ccc} I_\kappa & T_b & 2T_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \hline \mathbf{0} & S_b & S_q & R & I_\delta \end{array} \right),$$

where I_k is the identity matrix; T_b, T_1, T_2, R, S_b are matrices over $\mathbb{Z}_2$ and S_q is a matrix over $\mathbb{Z}_4$.

[3]-[6]

The parameters of a $\mathbb{Z}_2\mathbb{Z}_4$ code has the following inequalities:

$$\begin{aligned} \alpha, \beta, \gamma, \delta, \kappa &\geq 0, \quad \alpha+\beta>0 \\ 0<\gamma+\delta &\leq \beta+\kappa, \quad \kappa \leq \min(\alpha, \gamma). \end{aligned}$$

Example

$$G_1 = \left(\begin{array}{cc|ccc} \mathbf{1} & \mathbf{0} & 2 & 0 & 0 \\ \mathbf{0} & \mathbf{1} & 2 & 0 & 0 \\ 0 & 0 & 2 & \mathbf{2} & 0 \\ \hline 0 & 0 & 3 & 1 & \mathbf{1} \end{array}\right) \quad G_2 = \left(\begin{array}{c|ccccc} \mathbf{1} & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{2} & \mathbf{0} & 0 & 0 \\ 0 & 2 & \mathbf{0} & \mathbf{2} & 0 & 0 \\ \hline 0 & 3 & 0 & 1 & \mathbf{1} & \mathbf{0} \\ 0 & 1 & 1 & 0 & \mathbf{0} & \mathbf{1} \end{array}\right)$$

The code generated by G_1 is of type $(2,3;3,1;2)$ and the code generated by G_2 is of type $(1,5;3,2;1)$.

Outline

[3]-[6]

The inner product of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_2^\alpha \mathbb{Z}_4^\beta$ is defined as follows

$$[\mathbf{u}, \mathbf{v}] = 2(\sum_{i=1}^{\alpha} u_i v_i) + \sum_{j=\alpha+1}^{\alpha+\beta} u_j v_j \in \mathbb{Z}_4$$

The additive dual code of C is

$$C^\perp = \{\mathbf{v} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \mid [\mathbf{u}, \mathbf{v}] = 0 \text{ for all } \mathbf{u} \in C\}.$$

[3]-[6]

If C is a $\mathbb{Z}_2\mathbb{Z}_4$ code with type $(\alpha, \beta; \gamma, \delta; \kappa)$, then $C^\perp$ is of type $(\alpha, \beta; \bar{\gamma}, \bar{\delta}; \bar{\kappa})$, where

$$\begin{aligned} \bar{\gamma} &= \alpha + \gamma - 2\kappa, \\ \bar{\delta} &= \beta - \gamma - \delta + \kappa, \\ \bar{\kappa} &= \alpha - \kappa. \end{aligned}$$

Example

Let C_3 be a $\mathbb{Z}_2\mathbb{Z}_4$ code generated by

$$G_3 = \left(\begin{array}{c|c} 1 & 0 \end{array} \right)$$

Hence C_3 is of type $(1, 1, 1, 0, 1)$.

Then the dual code of C_3 is the code $C_3^{\perp}$ generated by

$$\left(\begin{array}{c|c} 0 & 1 \end{array} \right)$$

$C_3^{\perp}$ is of type $(1, 1, 0, 1, 0)$.

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Outline

Definition ([12])

Let $q \neq 1$, k and n be positive numbers. q-ary Gaussian coefficients, $\begin{bmatrix} n \\ k \end{bmatrix}_q$, are defined as follows:

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\ldots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\ldots(q-1)}, \qquad k = 1, 2, \ldots$$

Theorem ([12])

The number of $[n,k]$-codes over $\mathbb{F}_q$ is given by the following Gaussian coefficient: $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

Example

The number of ternary linear codes of length 3 and dimension 1 is 13:

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}_3 = \frac{(3^3-1)(3^{3-1}-1)}{(3^2-1)(2^{2-1}-1)} = 13.$$

These linear codes are given by the following generator matrices:

$$\begin{bmatrix} 1 & X & Y \end{bmatrix}, \begin{bmatrix} 0 & 1 & X \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

where $X, Y \in \mathbb{Z}_3$.

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Outline

Definition

- A ring R is a local ring if it has a unique maximal ideal m. This maximal ideal contains all non-units of the ring.
- A principal ideal ring is a ring such that every ideal is generated by a single element.
- A principal ideal ring where the ideals are linearly ordered is called a chain ring.

Theorem

Every code over a finite chain ring has a generator matrix that is permutation equivalent to a matrix of the following form

$$\begin{pmatrix} I_{k_0} & A_{0,1} & A_{0,2} & A_{0,3} & & & A_{0,e} \\ & \gamma I_{k_1} & \gamma A_{1,2} & \gamma A_{1,3} & & & \gamma A_{1,e} \\ & & \gamma^2 I_{k_2} & \gamma^2 A_{2,3} & & & \gamma^2 A_{2,e} \\ & & \ddots & \ddots & & & \\ & & & \ddots & & \ddots & \\ & & & & & \gamma^{e-1} I_{k_{e-1}} & \gamma^{e-1} A_{e-1,e} \end{pmatrix}$$

where $A_{i,j}$ are matrices with elements in a finite chain ring and e is the nilpotency index of γ.

A code with this generator matrix is said to be of type $(k_0, k_1, \cdots, k_{e-1})$.

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A code with this generator matrix is said to be of type $(k_0, k_1, \cdots, k_{e-1})$.

Theorem ([11], Dougherty and Saltürk)

Let R be a chain ring with maximal ideal $<\gamma>$, where γ has nilpotency e. Then number of distinct codes of type $(k_0, k_1, \cdots, k_{e-1})$ is

$$\frac{q^{\sum_{j=0}^{e-2} nk_j(e-(j+1))} \prod_{a=0}^{e-1} \prod_{i=0}^{k_a-1} (q^n - q^{\sum_{b=0}^{a-1} k_b} q^i)}{q^{\sum_{j=0}^{e-2}(e-(j+1))k_j^2 + \sum_{a=0}^{e-2}\{(e-(a+1))k_{a+1}\sum_{t=0}^{a} k_t\} + \sum_{r=0}^{e-2}(\sum_{l=r+1}^{e-1}(e-l)k_r k_l)} \prod_{i=0}^{e-1}(q^{k_i}-1)(q^{k_i}-q)\ldots(q^{k_i}-q^{k_i-1})}.$$

Theorem ([14], Wan)

Take $\mathbb{Z}_4$ as a finite chain ring. A linear code over $\mathbb{Z}_4$ is permutation equivalent to a linear code with the following generator matrix

$$\begin{pmatrix} I_{k_0} & A_{11} & A_{12} \\ 0 & 2I_{k_1} & 2A_{22} \end{pmatrix}.$$

where A_{ij} are matrices over $\mathbb{Z}_4$.

Corollary ([11], Dougherty and Saltürk)

The number of distinct linear codes of type (k_0, k_1) over $\mathbb{Z}_4$ is

$$\frac{2^{nk_0}\prod_{i=0}^{k_0-1}(2^n-2^i)\prod_{j=0}^{k_1-1}(2^n-2^{k_0+j})}{2^{k_0^2+2k_0k_1}\prod_{t=0}^{k_0-1}(2^{k_0}-2^t)\prod_{l=0}^{k_1-1}(2^{k_0}-2^t)}.$$

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Example

The number of distinct linear codes of type $(1,3)$ and length 4 over $\mathbb{Z}_4$ is 15 since

$$\frac{2^4(2^4-1)(2^4-2)(2^4-2^2)(2^4-2^3)}{2^7(2^1-1)(2^3-1)(2^3-2)(2^4-2^2)} = 15.$$

Example (cont.)

The generator matrices of those codes are as follows

$$\begin{bmatrix} 1 & X & Y & Z \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & X & Y \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & X \\ 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $X, Y, Z \in \mathbb{Z}_2$.

Outline

Basic Definitions

From the standard form, a free $\mathbb{Z}_2\mathbb{Z}_4$ code has a generator matrix of the following form

$$\left(\begin{array}{cc|ccc} 0 & S_b & S_q & R & I_\delta \end{array}\right)$$

where S_b is a binary matrix, S_q and R are quaternary matrices and I_δ is the identity matrix.

Example

The codes C_4 and C_5 generated by the following matrices, respectively, are free

$$G_4 = \left(\begin{array}{c|cc} 1 & 1 & 1 \end{array}\right) \quad \text{and} \quad G_5 = \left(\begin{array}{c|ccc} 1 & 3 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array}\right).$$

Example (cont.)

$$C_4 = \{(0|00), (1|11), (0|22), (1|33)\}$$

and

$$C_5 = \Big\{(0|000), (1|310), (0|220), (1|130), (0|301), (1|211), (0|121), (1|031), (0|202), (1|112), (0|022), (1|332), (0|103), (1|013), (0|323), (1|233)\Big\}$$

Lemma ([13], Dougherty and Saltürk)

Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ *generate a free code if and only if* $(\mathbf{v}_1)_Y, (\mathbf{v}_2)_Y, \ldots, (\mathbf{v}_k)_Y$ *generate a quaternary free code.*

A free code generated by s vectors has type $(\alpha, \beta, 0, s, \kappa)$.

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Example

The codes C_4 and C_5 generated by the following matrices are of types $(1, 2; 0, 1; 0)$ and $(1, 3; 0, 2; 0)$ respectively:

$$G_4 = \left(\begin{array}{c|cc} 1 & 1 & 1 \end{array} \right) \quad \text{and} \quad G_5 = \left(\begin{array}{c|ccc} 1 & 3 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array} \right)$$

The Number

Theorem ([13], Dougherty and Saltürk)

The number of free $\mathbb{Z}_2\mathbb{Z}_4$ codes generated by s vectors in $\mathbb{Z}_2^{\alpha}\mathbb{Z}_4^{\beta}$ is

$$2^{s(\beta+\alpha-s)} \begin{bmatrix} \beta \\ s \end{bmatrix}_2 .$$

The Number

$$2^{s(\beta+\alpha-s)}\begin{bmatrix}\beta\\ s\end{bmatrix}_2 = \frac{(4^\beta-2^\beta)(4^\beta-2^\beta 2)(4^\beta-2^\beta 2^2)\dots(4^\beta-2^\beta 2^{s-1})2^{s\alpha}}{(4^s-2^s)(4^s-2^s 2)(4^s-2^s 2^2)\dots(4^s-2^s 2^{s-1})}$$

Example

The number of free $\mathbb{Z}_2\mathbb{Z}_4$ codes with $\alpha = \beta = 1$ generated by 1 vector is 2 since $2^{(1+1-1)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_2 = 2$

These codes are generated by the following generator matrices:

$$\left(\begin{array}{c|c} 1 & 1 \end{array} \right) \quad \text{and} \quad \left(\begin{array}{c|c} 0 & 1 \end{array} \right)$$

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Example

The number of free $\mathbb{Z}_2\mathbb{Z}_4$ codes with $\alpha = 1$ and $\beta = 2$ generated by 1 vector is 12 since $2^{(2+1-1)} \begin{bmatrix} 2 \\ 1 \end{bmatrix}_2 = 12$.

These codes are generated by the following generator matrices:

$$\left(\begin{array}{c|cc} 1 & 0 & 1 \end{array}\right), \quad \left(\begin{array}{c|cc} 1 & 1 & 1 \end{array}\right), \quad \left(\begin{array}{c|cc} 1 & 2 & 1 \end{array}\right), \quad \left(\begin{array}{c|cc} 1 & 3 & 1 \end{array}\right),$$

$$\left(\begin{array}{c|cc} 1 & 1 & 0 \end{array}\right), \quad \left(\begin{array}{c|cc} 1 & 1 & 2 \end{array}\right), \quad \left(\begin{array}{c|cc} 0 & 0 & 1 \end{array}\right), \quad \left(\begin{array}{c|cc} 0 & 1 & 1 \end{array}\right),$$

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Recurrence relations, [13]

Define $\left\{ \begin{array}{c} \alpha, \beta \\ s \end{array} \right\}$ to be the number of $\mathbb{Z}_2\mathbb{Z}_4$ codes of type $(\alpha, \beta; 0, s; \kappa)$.

We have the following recurrence relations:

Theorem

$$\left\{ \begin{array}{c} \alpha, \beta \\ s \end{array} \right\} = 2^{\alpha+\beta-s} \left\{ \begin{array}{c} \alpha, \beta-1 \\ s-1 \end{array} \right\} + 2^{2s} \left\{ \begin{array}{c} \alpha, \beta-1 \\ s \end{array} \right\}.$$

Theorem

$$\left\{ \begin{array}{c} \alpha, \beta \\ s \end{array} \right\} = 2^{\alpha+2\beta-2s} \left\{ \begin{array}{c} \alpha, \beta-1 \\ s-1 \end{array} \right\} + 2^{s} \left\{ \begin{array}{c} \alpha, \beta-1 \\ s \end{array} \right\}.$$

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Recurrence relations, [13]

Define $\left\{ \begin{matrix} \alpha, \beta \\ s \end{matrix} \right\}$ to be the number of $\mathbb{Z}_2\mathbb{Z}_4$ codes of type $(\alpha, \beta; 0, s; \kappa)$.

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Outline

Theorem ([13], Dougherty and Saltürk)

The number of distinct $\mathbb{Z}_2\mathbb{Z}_4$ codes of type $(\alpha, \beta; \gamma, \delta; \kappa)$ is

$$N_{\alpha,\beta;\gamma,\delta;\kappa} = 2^{(\alpha+\beta-\gamma-\delta)\delta+(\beta-\delta-\gamma+\kappa)\kappa} \begin{bmatrix} \beta \\ \delta \end{bmatrix}_2 \begin{bmatrix} \alpha \\ \kappa \end{bmatrix}_2 \begin{bmatrix} \beta-\delta \\ \gamma-\kappa \end{bmatrix}_2 .$$

$$G = \left(\begin{array}{cc|ccc} I_\kappa & T_b & 2T_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \hline \mathbf{0} & S_b & S_q & R & I_\delta \end{array}\right)$$

Example

The number of $\mathbb{Z}_2\mathbb{Z}_4$ codes of type $(1,2;2,1;1)$ is 3 since

$$N_{1,2;2,1;1} = 2^0 \begin{bmatrix} 2 \\ 1 \end{bmatrix}_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}_2 = 3.$$

These codes are generated by the following matrices:

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \left(\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 1 & 1 \end{array}\right), \left(\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & 0 & 2 \\ \hline 0 & 1 & 0 \end{array}\right).$$

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Example

The number of $\mathbb{Z}_2\mathbb{Z}_4$ codes of type $(2,2;2,0;1)$ is 18 since

$$N_{2,2;2,0;1} = 2^1 \begin{bmatrix} 2 \\ 0 \end{bmatrix}_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}_2 = 18.$$

These codes are generated by the following matrices:

$$\left(\begin{array}{cc|cc} 1 & X & Y & 0 \\ 0 & 0 & Z & 2 \end{array}\right), \left(\begin{array}{cc|cc} 1 & X & 0 & T \\ 0 & 0 & 2 & 0 \end{array}\right),$$
$$\left(\begin{array}{cc|cc} 0 & 1 & Y & 0 \\ 0 & 0 & Z & 2 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & 0 & T \\ 0 & 0 & 2 & 0 \end{array}\right),$$

where $X \in \{0,1\}$ and $Y, Z, T \in \{0,2\}$. Thus we obtain the 18 codes.

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Recurrence Relations

Define $\left\{ \begin{array}{c} \alpha, \beta \\ \gamma, \delta, \kappa \end{array} \right\}$ to be the number of $\mathbb{Z}_2\mathbb{Z}_4$ codes of type $(\alpha, \beta; \gamma, \delta; \kappa)$.

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Then we have the following recurrence relations:

Recurrence Relations, [13]

Theorem

$$\left\{ \begin{array}{c} \alpha, \beta \\ \gamma, \delta, \kappa \end{array} \right\} = 2^{\beta-\gamma-\delta+\kappa} \left\{ \begin{array}{c} \alpha-1, \beta \\ \gamma-1, \delta, \kappa-1 \end{array} \right\} + 2^{\delta+\kappa} \left\{ \begin{array}{c} \alpha-1, \beta \\ \gamma, \delta, \kappa \end{array} \right\}.$$

Theorem

$$\left\{ \begin{array}{c} \alpha, \beta \\ \gamma, \delta, \kappa \end{array} \right\} = 2^{\alpha+\beta-\gamma-\delta} \left\{ \begin{array}{c} \alpha-1, \beta \\ \gamma-1, \delta, \kappa-1 \end{array} \right\} + 2^{\delta} \left\{ \begin{array}{c} \alpha-1, \beta \\ \gamma, \delta, \kappa \end{array} \right\}.$$

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Corollary ([13], Dougherty and Saltürk)

The number of $\mathbb{Z}_2\mathbb{Z}_4$ codes of type $(\alpha, \beta; \gamma, \delta; \kappa)$ is equal to the number of $\mathbb{Z}_2\mathbb{Z}_4$ codes of type $(\alpha, \beta; \alpha + \gamma - 2\kappa, \beta - \gamma - \delta + \kappa; \alpha - \kappa)$.

Example

Since the number of $\mathbb{Z}_2\mathbb{Z}_4$ codes of type $(2, 2; 2, 0; 1)$ is 18 from the previous example, we consider codes of type $(2, 2; 2, 1; 1)$ where $\bar{\gamma} = 2 + 2 - 2 = 2$, $\bar{\delta} = 2 - 2 - 0 + 1 = 1$, $\bar{\kappa} = 2 - 1 = 1$.

The parameters above are the parameters of the dual codes.

Then we have the number of $\mathbb{Z}_2\mathbb{Z}_4$ codes of type $(2, 2; 2, 1; 1)$ from the formula which is the same as the number of $\mathbb{Z}_2\mathbb{Z}_4$ codes of type $(2, 2; 2, 0; 1)$.

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Decomposable codes

Lemma ([13], Dougherty and Saltürk)

A decomposable $\mathbb{Z}_2\mathbb{Z}_4$ code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ is the direct product of a binary code of dimension κ in an α dimensional space and a quaternary code in $\mathbb{Z}_4^\beta$ of quaternary type $(\delta, \gamma - \kappa)$.

Lemma ([13], Dougherty and Saltürk)

The number of decomposable codes of type $(\alpha, \beta; \gamma, \delta; \kappa)$ is

$$\left(\begin{bmatrix} \alpha \\ \kappa \end{bmatrix}_2\right)\left(\frac{2^{\beta k_0} \prod_{a=0}^{1} \prod_{i=0}^{k_a - 1} (2^\beta - 2^{\sum_{b=0}^{a-1} k_b} 2^i)}{2^{k_0^2 + 2k_0 k_1} \prod_{i=0}^{1} (2^{k_i} - 1)(2^{k_i} - 2) \dots (2^{k_i} - 2^{k_i - 1})}\right),$$

where $k_0 = \delta$, $k_1 = \gamma - \kappa$.

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where $k_0 = \delta$, $k_1 = \gamma - \kappa$.

Indecomposable codes

Theorem ([13], Dougherty and Saltürk)

The number of indecomposable codes of type $(\alpha, \beta; \gamma, \delta; \kappa)$ *is*

$$2^{(\alpha+\beta-2\gamma-\delta)\delta+\beta\kappa} \begin{bmatrix} \beta \\ \delta \end{bmatrix}_2 \begin{bmatrix} \alpha \\ \kappa \end{bmatrix}_2 \begin{bmatrix} \beta-\delta \\ \gamma-\kappa \end{bmatrix}_2$$

$$- \quad (\begin{bmatrix} \alpha \\ \kappa \end{bmatrix}_2)(\frac{2^{\beta k_0} \prod_{a=0}^{1} \prod_{i=0}^{k_a-1} (2^{\beta} - 2^{\sum_{b=0}^{a-1} k_b} 2^i)}{2^{k_0^2+2k_0k_1} \prod_{i=0}^{1} (2^{k_i}-1)(2^{k_i}-2)\dots(2^{k_i}-2^{k_i-1})}),$$

where $k_0 = \delta$, $k_1 = \gamma - \kappa$.

Example

The number of $\mathbb{Z}_2\mathbb{Z}_4$ codes of type $(2,2;2,0;1)$ is 18. The number of decomposable codes of type $(2,2;2,0;1)$ is 9. Because the number of binary codes of dimension $\kappa = 1$ is $\begin{bmatrix} \alpha \\ \kappa \end{bmatrix}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_2 = 3$, and the number of quaternary codes in $\mathbb{Z}_4^2$ of quaternary type $(\delta, \gamma - \kappa) = (0,1)$ is 3. Then the product, $3 \times 3 = 9$, gives the number of decomposable codes.

Example (cont. example)

These codes are generated by the following matrices:

$$\left(\begin{array}{cc|cc} 1 & X & 0 & 0 \\ 0 & 0 & Y & 2 \end{array}\right), \left(\begin{array}{cc|cc} 1 & X & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right),$$

$$\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & Y & 2 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$$

where $X \in \{0,1\}$ and $Y \in \{0,2\}$.
The remaining 9 codes are indecomposable ones.

Summary

- We count the number of additive codes of any types.

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Regards

Thank you for your attention... ☺