Free group algebras generated by symmetric elements inside division rings with involution

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Notation

- Rings are associative with 1.
- Morphisms, subrings and embeddings preserve 1.
- A division ring is a nonzero ring such that every nonzero element is invertible.
- Free groups $H = \langle X \mid \quad \rangle$, free algebras $k\langle X \rangle$, and free group algebras $k[H]$ are supposed to be noncommutative, i.e. $|X| \geq 2$.
- If D is a division ring and $R \hookrightarrow D$, we denote by $D(R)$ the division subring of D generated by R.

Some conjectures

Let D be a division ring with center Z.

(A) If D is finitely generated (as a division ring) over Z and $[D:Z]=\infty$, then D contains a free algebra. **(Makar-Limanov)**

(GA) If D is finitely generated (as a division ring) over Z, and $[D:Z]=\infty$, then D contains a free group algebra.

Involutions

Let $\mathbb{k}$ be a field. Given a $\mathbb{k}$-algebra A, a $\mathbb{k}$-involution on A is a $\mathbb{k}$-linear map $^\star\colon A \to A$ satisfying

$$(ab)^\star = b^\star a^\star, \quad \forall a, b \in A, \qquad \text{and} \qquad (a^\star)^\star = a, \quad \forall a \in A.$$

An element $a \in A$ is said to be symmetric if $a^\star = a$.

Example

If G is a group and $\mathbb{k}[G]$ denotes the group algebra of G over $\mathbb{k}$

$$\begin{array}{ccc} \mathbb{k}[G] & \longrightarrow & \mathbb{k}[G] \\ \sum_{x\in G} a_x x & \longmapsto & \sum_{x\in G} a_x x^{-1}, \end{array}$$

is a $\mathbb{k}$-involution $\mathbb{k}[G]$, called the canonical involution of $\mathbb{k}[G]$.

Question

Let D be a division ring with an involution $\star\colon D \to D$.
If D is finitely generated over Z, and $[D : Z] = \infty$, does D contain a free (group) algebra generated by symmetric elements?

Malcev-Neumann series ring

Definition

- $(G,<)$ is an ordered group if G is a group and $<$ is a total order such that for all $x,y,z\in G$

$$x<y\Rightarrow xz<yz \qquad x<y\Rightarrow zx<zy$$

- $(G,<)$ ordered group. K a division ring, KG a group ring.

$$KG\hookrightarrow K((G,<))=\left\{f=\sum_{x\in G}a_x x \mid a_x\in K,\ \mathrm{supp}\, f \text{ is well ordered}\right\}$$

  $K((G,<))$ is a division ring, Malcev-Neumann series ring.
- $K(G)$ is the division ring generated by KG inside $K((G,<))$.

Example

$$k\mathbb{Z}=k[t,t^{-1}]\hookrightarrow k((t))=\left\{\sum_{i\geq n}a_i t^i \mid a_i\in k,\ n\in\mathbb{Z}\right\}.$$

$k(\mathbb{Z})=k(t)$.

Main result

Theorem (Gonçalves-Ferreira-S.)

Let G be an orderable group, let $\Bbbk$ be a field and let $\Bbbk G$ be the group algebra.

Denote by $\Bbbk(G)$ be the division ring generated by $\Bbbk G$ inside $\Bbbk((G,<))$.

Then the following are equivalent:

- *$\Bbbk(G)$ contains a free group $\Bbbk$-algebra freely generated by symmetric elements with respect to the canonical involution.*
- *$\Bbbk(G)$ is not a locally P.I. $\Bbbk$-algebra.*
- *G is not abelian.*

Crossed products

Let R be a ring and G a group. A crossed product is a ring:

- As a set $RG = \{\sum_{x \in G} r_x \bar{x} \mid r_x \in R \text{ almost all } r_x = 0\}$.
- Addition: $\sum_{x \in G} r_x \bar{x} + \sum_{x \in G} s_x \bar{x} = \sum_{x \in G} (r_x + s_x) \bar{x}$.
- Multiplication:
  - Exist maps $\tau \colon G \times G \to R^{\times}$ and $\sigma \colon G \to \mathrm{Aut}(R)$ such that

$$\bar{x}\bar{y} = \tau(x, y)\overline{xy} \qquad \bar{x}r = r^{\sigma(x)}\bar{x}.$$

Examples

- If $G = \mathbb{Z}$, then $RG = R[t, t^{-1}; \alpha]$.
- $\tau(y, z) = 1,\ \sigma(y) = 1_R\ \ \forall\, y, z \in G \implies RG$ is the group ring.

Lemma

R a ring, G a group and RG a crossed product. Suppose N is a normal subgroup of G. Then

$$RG = (RN)\frac{G}{N}.$$

Locally indicable groups

Definition

A group G is locally indicable if for each nontrivial finitely generated subgroup H of G there exists $N \lhd H$ such that H/N is infinite cyclic.

Examples

- Torsion-free nilpotent groups
- (Locally) free groups
- **(Levi 43)** Orderable groups.
- **(Howie 82, Brodskii 84)** Torsion-free one-relator groups.
- Closed under extensions, cartesian products and free products.

G a locally indicable group.
Let kG be a crossed product.
Let H be a finitely generated subgroup of G.

There exists $N \lhd H$ such that H/N is infinite cyclic.

Let $t \in H$ such that Nt generates H/N.
Then each $h = dt^n$ for some unique $d \in N$ and $n \in \mathbb{Z}$. Thus

$$kH = kN\frac{H}{N} = \oplus kN\bar{t}^n.$$

The powers of $\bar{t}$ are kN-linearly independent.

Hughes-freeness

Definitions

k a division ring, G locally indicable group.
Suppose $kG \hookrightarrow D$ division ring of fractions. Then

1. $kG \hookrightarrow D$ is Hughes-free if
   - for each nontrivial finitely generated subgroup H of G, and
   - for each $N \lhd H$ such that H/N is infinite cyclic, and
   - for each t such that Nt generates H/N,

   then the powers of $\bar{t}$ are left linearly independent over $D(kN)$: i.e. for $d_0, \dots, d_n \in D(kN)$

$$d_0 + d_1\bar{t} + \cdots + d_n\bar{t}^n = 0 \Longrightarrow d_0 = d_1 = \cdots = d_n = 0$$

$$\begin{array}{ccc} kH = \oplus kN\bar{t}^n & \Longrightarrow & kH = kN[\bar{t}, \bar{t}^{-1}; \alpha] \\ \Downarrow & & \\ D(kN)[\bar{t}, \bar{t}^{-1}; \alpha] & \Longrightarrow & D(kH) \cong Q_{cl}(D(kN)[\bar{t}, \bar{t}^{-1}; \alpha]) \end{array}$$

Main example of Hughes-free embeddings

Definition

G an orderable group. Fix an order $<$ such that $(G,<)$ ordered group.
k a division ring, kG a crossed product group ring.
Consider

$$E = k((G,<)) = \left\{ f = \sum_{x\in G} r_x \bar{x} \mid r_x \in k,\ \text{supp}\, f \text{ is well ordered} \right\}$$

E is a division ring, the Mal'cev-Neumann series ring, and $kG \hookrightarrow E$.

Example

$$k\mathbb{Z} = k[t,t^{-1};\alpha] \hookrightarrow E = k((t;\alpha)) = \left\{ \sum_{i\geq n} a_i t^i \mid a_i \in k,\ n \in \mathbb{Z} \right\}.$$

Notation

Denote by $k(G)$ the division ring generated by kG inside $k((G,<))$.

Hughes' Theorems

Theorem (Hughes 70, Dicks-Herbera-S. 04)

k division ring, G locally indicable, kG crossed product.
Suppose kG has Hughes-free division rings of fractions D_1, D_2

$$\begin{array}{ccc} & \overset{\iota_1}{\nearrow} & D_1 \\ kG & & \Big\downarrow \cong \\ & \underset{\iota_2}{\searrow} & D_2 \end{array}$$

Then they are isomorphic division rings of fractions of kG.

Corollary

Suppose $kG \hookrightarrow D$ is Hughes-free. Let $\alpha \colon kG \to kG$ be an isomorphism of rings. Then

$$kG \xrightarrow{\alpha} kG \hookrightarrow D$$

is Hughes-free. Therefore

$$\begin{array}{ccc} kG & \xrightarrow{\alpha} & kG \\ \big\downarrow & & \big\downarrow \\ D & \overset{\alpha}{\dashrightarrow} & D \end{array}$$

Extension of involutions

Theorem (Gonçalves-Ferreira-S.)

Let k be a division ring, let G be a locally indicable group and let kG be a crossed product. Suppose that kG has a Hughes-free division ring of fractions D. Then any involution on kG extends to a unique involution on D.

- $(kG)^{op} \cong k^{op}G^{op}$.
- $kG \hookrightarrow D$ is Hughes-free if and only if $(kG)^{op} \hookrightarrow D^{op}$ is Hughes-free.
- Let $\star \colon kG \to kG$ be an involution. Then $\star \colon kG \to kG^{op}$ is an isomorphism.
  Hence

$$\begin{array}{ccc} kG & \xrightarrow{\star} & (kG)^{op} \\ \downarrow & & \downarrow \\ D & \dashrightarrow^{\star} & D^{op} \end{array}$$

Proof of the main result

Theorem (Gonçalves-Ferreira-S.)

Let G be an orderable group, let $\mathbb{k}$ be a field and let $\mathbb{k}[G]$ be the group algebra.
Denote by $\mathbb{k}(G)$ the Malcev-Neumann division ring of fractions of $\mathbb{k}[G]$. Then the following are equivalent:

- *$\mathbb{k}(G)$ contains a free group $\mathbb{k}$-algebra of rank 2 freely generated by symmetric elements with respect to the canonical involution.*
- *$\mathbb{k}(G)$ is not a locally P.I. $\mathbb{k}$-algebra.*
- *G is not abelian.*

Free monoid case

Let $x, y \in G$ such that generate a free monoid.
Let $H = \langle x, y \rangle$ be the group they generate.
Let $N \lhd H$ such that $H/N = \langle Nt \rangle$ is infinite-cyclic.

- By Hughes-freeness $\mathbb{k}(H) \cong \mathbb{k}(N)(t;\alpha) \hookrightarrow \mathbb{k}(N)((t;\alpha))$
- We can suppose $x = ut^m$ and $y = vt^n$ with $n, m \geq 1$.
- $f = x^{-1} + x = (1 + x^2)x^{-1}, \quad g = y^{-1} + y = (1 + y^2)y^{-1}$.
- $f^{-1} = x(1 + x^2)^{-1}$ and $g^{-1} = y(1 + y^2)^{-1}$
- $$f^{-1} = ut^m + \sum_{i \geq m+1} \alpha_i t^i \quad \text{and} \quad g^{-1} = vt^n + \sum_{j \geq n+1} \beta_j t^j.$$
- f^{-1} and g^{-1} are symmetric and generate a free $\mathbb{k}$-algebra.
- $1 + f^{-1}$ and $1 + g^{-1}$ generate a free group $\mathbb{k}$-algebra. **(Lichtman)**

No free monoid case: Important particular case

Theorem (Gonçalves-Ferreira-S.)

Let $\mathbb{k}$ *be a field and consider the group*

$$G = \langle x, y : [[x, y], x] = [[x, y], y] = 1 \rangle.$$

Let D *denote the Ore field of fractions of the group algebra* $\mathbb{k}[G]$. *Then*

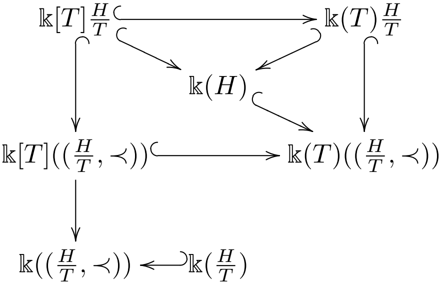
$$1 + y(1-y)^{-2} \quad \text{and} \quad 1 + y(1-y)^{-2}x(1-x)^{-2}y(1-y)^{-2}$$

are symmetric elements with respect to the canonical involution on D *and freely generate a free group* $\mathbb{k}$*-algebra in* D.

Lemma (S.)

Let G *be a noncommutative orderable group.*
If G *does not contain a free monoid, then there exist* $x, y \in G$ *such that the group* $H = \langle x, y \rangle$ *contains a normal subgroup* T *such that* H/T *is torsion-free nilpotent of class two.*

No free monoid case: General case

- Let $x, y \in G$ such that the group $H = \langle x, y \rangle$ contains a normal subgroup T such that H/T is torsion-free nilpotent of class two.
- 

- $\exists\ A_1, B_1, A_2, B_2 \in \Bbbk[\frac{H}{T}]$ such that $A_1B_1^{-1}, A_2B_2^{-1} \in \Bbbk(\frac{H}{T})$ are symmetric that generate a free group algebra.
- $\exists\ \hat{A}_1, \hat{B}_1, \hat{A}_2, \hat{B}_2 \in \Bbbk[T]\frac{H}{T} = \Bbbk[H]$ such that $\hat{A}_1\hat{B}_1^{-1}, \hat{A}_2\hat{B}_2^{-1} \in \Bbbk(H) = \Bbbk(T)(\frac{H}{T})$ are symmetric that generate a free group algebra.