Simply connected gradings of complex matrix algebra

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- A *grading* of an algebra A by a group G is a vector space decomposition
$$A = \bigoplus_{g \in G} A_g \qquad (1)$$
such that $A_g A_h \subseteq A_{gh}$.
- The *support* of a grading $A = \bigoplus_{g \in G} A_g$ is the set
$$\Lambda_G(A) = \Lambda_G = \{g \in G | \dim(A_g) \geq 1\}.$$

Examples

- The natural $\mathbb{Z}$-grading of a polynomial ring.
- The natural G-grading of the group algebra $\mathbb{C}G$.
- More generally, let G be a finite group, the *twisted group algebra* $\mathbb{C}^f G$ is an associative algebra with basis $\{u_g\}_{g\in G}$, where
$$u_{g_1}u_{g_2} = f(g_1, g_2)u_{g_1 g_2},$$
for an $f : G \times G \longrightarrow \mathbb{C}^*$.
- For associativity f is a two-cocycle, $f \in Z^2(G, \mathbb{C}^*)$. The natural G-grading of $\mathbb{C}^f G$ has the property that any component is one dimensional.
- This is a specific case of fine grading.

fine gradings

A grading $A = \bigoplus_{g \in G} A_g$ is called *fine* if $\dim(A_g) \leq 1$ for any $g \in G$.

Example: Let $G = C_n \times C_n = \langle \sigma \rangle \times \langle \tau \rangle$ and let $A = M_n(\mathbb{C})$. Define

$$M_n(\mathbb{C})_\sigma = \text{span}_{\mathbb{C}}(B_\sigma) \quad , \quad M_n(\mathbb{C})_\tau = \text{span}_{\mathbb{C}}(B_\tau),$$

where

$$B_\sigma = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad B_\tau = \begin{pmatrix} \eta_n & 0 & 0 & \cdots & 0 \\ 0 & \eta_n^2 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \eta_n^n \end{pmatrix}.$$

Here η_n is an *n-th* primitive root of unity.

elementary gradings

- Next, any grading of a vector space

$$V = \bigoplus_{g \in G} V_g \tag{2}$$

  induces a unique grading of $\mathrm{End}_{\mathbb{C}}(V)$ with property that $\phi_{g_1}(v_{g_2}) \in V_{g_1 g_2}$ for any $\phi_{g_1} \in (\mathrm{End}_{\mathbb{C}}(V))_{g_1}$ and $v_{g_2} \in V_{g_2}$, in the following way

$$(\mathrm{End}(V))_g := \{\phi \in \mathrm{End}(V) | \phi(V_h) \subseteq V_{gh}, \forall h \in G\}.$$

  This grading is called *elementary* (good).
- By choosing a basis of V which respects the grading (2) the elementary G-grading of $\mathrm{End}_{\mathbb{C}}(V) = M_n(\mathbb{C})$ is given by an n-tuple, $(g_1, g_2, \ldots, g_n) \in G^n$ where any elementary matrix $E_{ij}(\subseteq \mathrm{Hom}(V_{g_i}, V_{g_j}))$ is in the $g_i^{-1} g_j$ component.

induction

Let $A = \bigoplus_{g \in G} A_g$ be any G-graded algebra, $B = M_n(\mathbb{C}) = \bigoplus_{g \in G} B_g$ is a matrix algebra with an elementary grading given by the n-tuple $(g_1, g_2, \ldots, g_n) \in G^n$, that is $E_{ij} \in B_{g_i^{-1} g_j}$. Then $R = A \bigotimes B = \bigoplus_{g \in G} R_g$ has the following induced G-grading

$$R_g = \mathrm{span}_{\mathbb{C}}\{a \otimes E_{ij} | a \in A_h, g_i^{-1} h g_j = g\}. \qquad (3)$$

BSZ decomposition

Clearly, in order to understand group gradings of $M_n(\mathbb{C})$ it is *necessary* to understand fine and elementary gradings of $M_n(\mathbb{C})$. The following theorem shows that it is also *sufficient*.

Theorem-Bahturin Sehgal and Zaicev

Let $M_n(\mathbb{C}) = \bigoplus M_n(\mathbb{C})_g$ *be a* G*-grading. Then, this grading is induced from a fine grading of* $M_r(\mathbb{C})$ *for some* r *which divides* n.

- Let $C_n \times C_n = \langle \tau \rangle \times \langle \sigma \rangle$. We can define a cohomology class by the relation $u_\tau u_\sigma = \eta_n u_\sigma u_\tau$. Here η_n is an *n-th* primitive root of unity.
- Then $\mathbb{C}^{[f]}(C_n \times C_n)$ is isomorphic to $M_n(\mathbb{C})$ by

$$u_\sigma \mapsto B_\sigma \quad u_\tau \mapsto B_\tau.$$

- We see that unlike group algebras, twisted group algebra $\mathbb{C}^f G$ can be simple, that is matrix algebra $M_n(\mathbb{C})$ s.t $n^2 = |G|$. In this case we say that G is *of central type* and f (or $[f]$) is nondegenerate.

By the theorem of BSZ in order to understand gradings of $M_n(\mathbb{C})$ it is sufficient to understand fine gradings of $M_n(\mathbb{C})$. It turns out that fine gradings of $M_n(\mathbb{C})$ are in 1-1 correspondence with simple complex twisted group algebras.

Theorem-Bahturin and Zaicev

Let $A = M_n(\mathbb{C}) = \bigoplus_{g\in G} A_g$ be a grading such that $dim(A_g) \leq 1$ for any $g \in G$. Then there exists a subgroup $H \leq G$ and a nondegenerate cocycle $f \in Z^2(H, \mathbb{C}^*)$ such that A is isomorphic (as graded algebras) to the twisted group algebra $\mathbb{C}^f H$.

isomorphism of gradings

- A main concern in the investigation of graded algebras is the notion of equivalence.

Definition

Two graded algebras

$$A = \bigoplus_{g \in G} A_g, \quad B = \bigoplus_{h \in H} B_h \tag{4}$$

are *graded-isomorphic if* $G = H$ and there exists an algebra isomorphism $\psi : A \to B$ such that $\psi(A_g) = B_g$ for any $g \in G$.

- $\mathbb{C}^{f_1}G$ and $\mathbb{C}^{f_2}G$ are graded-isomorphic if and only if $[f_1] = [f_2] \in H^2(G, \mathbb{C}^*)$.
- Aljadeff and Haile show that the BSZ decomposition is *unique* up to an isomorphism of graded algebras.

- Return to the $C_n \times C_n = \langle \tau \rangle \times \langle \sigma \rangle$ grading of $M_n(\mathbb{C})$. Notice that the primitive n-th root was not explicitly determined.
- It is not hard to show that the cocycles which correspond to different primitive roots of unity are not cohomologous. It is natural to define a coarser equivalence relation. Under such equivalence, grading which correspond to different primitive roots of unity in the above example belong to the same equivalence class.

equivalence of gradings

- Define graded-equivalence in the following way

Definition

Two gradings

$$A = \bigoplus_{g \in G} A_g, \quad B = \bigoplus_{h \in H} B_h$$

are equivalent if there exist an algebra isomorphism $\psi : A \to B$ and a group isomorphism $\phi : G \to H$ such that $\psi(A_g) = B_{\phi(g)}$ for any $g \in G$.

- In the sequel we will show a 1-1 correspondence between graded equivalence classes and equivalence classes of Galois covering in a more general linear categories.

Action of Aut(G)

- Let $A = \bigoplus_{g \in G} A_g$ be a G-grading, and let $\phi \in$Aut(G). Then $A = \bigoplus_{g \in G} B_g$ is a G-grading where $B_g = A_{\phi(g)}$.
- This is an action of Aut(G) on the G-graded isomorphism classes of A. The equivalence classes are the orbits under this action. In particular, for twisted group algebras, this action identifies with the well known action of Aut(G) on $H^2(G, \mathbb{C}^*)$.
- Clearly,if $[f] \in H^2(G, \mathbb{C}^*)$ is a nondegenerate cohomology class then $\phi([f])$ is again nondegenerate for any $\phi \in$Aut(G).

quotient grading

- One of our main goals is to compute the intrinsic fundamental group of $M_n(\mathbb{C})$. This is the inverse limit of all the connected gradings of $M_n(\mathbb{C})$ with respect to quotient maps.

Definition

Let $A = \bigoplus_{g \in G} A_g$ be a G-grading of A and let N be a normal subgroup of G. *The G/N-quotient* of this grading is $A = \bigoplus_{\bar{g} \in G/N} A_{\bar{g}}$ where $A_{\bar{g}} = \bigoplus_{h \in \bar{g}} A_h$.

- It is important to notice that quotients respect equivalence between graded algebras. Hence, the concept of quotient induces a natural order on the graded equivalence classes of an algebra.

connected gradings

- A G-grading of an algebra A is *connected* if the support of this grading generates the group G. It is natural to restrict only to connected gradings.
- Next, we want to find all the maximal connected graded classes of $M_n(\mathbb{C})$. Clearly, first we need to find maximal elementary classes and maximal fine classes with respect to quotients.

uniqueness of maximal elementary grading class

- For any r there is an elementary connected grading class of $M_r(\mathbb{C})$ by the free group F_{r-1}. This class is unique.
- Any elementary connected grading class of $M_r(\mathbb{C})$ is a quotient of the above F_{r-1}-grading class.
- This F_{r-1}-grading class is not a quotient of any other connected grading. That is, this F_{r-1}-grading class is the unique maximal connected elementary grading class of $M_r(\mathbb{C})$ (Cibils, Redondo and Solotar [2010]).
- The natural grading of any twisted group algebra is maximal.

maximal connected gradings of $M_n(\mathbb{C})$

- We can now determine the maximal gradings of $M_n(\mathbb{C})$.
- Given a decomposition $n = rq$, a G-grading which is induced from a simple twisted group algebra $\mathbb{C}^f H$, where $|H| = q^2$ by the above elementary F_{r-1}-grading of $M_r(\mathbb{C})$ is maximal.
- It turns out that these are all the maximal grading classes of $M_n(\mathbb{C})$.

maximal connected gradings of $M_n(\mathbb{C})$

Theorem A

There is a one-to-one correspondence between maximal connected gradings grading classes of $M_n(\mathbb{C})$, and the set of pairs $X_n = \{(G, \gamma)\}$, where G is a group of central type of order dividing n^2 and γ is an $Aut(G)$-orbit of a nondegenerate cohomology class in $H^2(G, \mathbb{C}^*)$.

Since for any $n \geq 2$, $|X_n| \geq 2$ there is no *universal covering* of $M_n(\mathbb{C})$ (i.e., a unique maximal connected grading class) as observed by CRS.

Main questions

- Theorem A gives rise to a number of questions.
- Let $n \in \mathbb{N}$, classify the groups of central type of order n^2. In particular,

Question 1

For which $n \in \mathbb{N}$, $C_n \times C_n$ is the only group of central type of order n^2?

- Furthermore, for each group of central type, classify the $\mathrm{Aut}(G)$-orbits of the nondegenerate cohomology classes. In particular,

Question 2

Is the action of $Aut(G)$ on the nondegenerate cohomology classes in $H^2(G, \mathbb{C}^*)$ transitive for any group G of central type?

Question 1

Classifying all the groups of central type is a very ambitious task. However, we give an *exact* answer to Question 1. First, we notice the following observations:

1. For a prime p, $C_p \times C_p$ is the only group of central type of order p^2.
2. If there is only one group of central type of order $(mk)^2$ then there is only one group of central type of order m^2.
3. For any $m \neq 1$ there exist at least two non-isomorphic abelian groups of central type of order m^4, namely

$$(C_m \times C_m) \times (C_m \times C_m), \quad C_{m^2} \times C_{m^2}.$$

Consequently, if there is only one group of central type of order n^2 then n is square free.

Theorem B

Let $p_1 < p_2 < \ldots < p_r$ be primes and let $n = \prod_{i=1}^{r} p_i^{k_i}$. Then there is only one group of central type of order n^2 if and only if n is square free ($k_i = 1$) and $p_j \not\equiv \pm 1 (mod\ p_i)$ for all $1 \leq i, j \leq r$.

Question 2

- Again, classifying the Aut(G)-orbits of the nondegenerate cohomology classes seems beyond our reach for now. As for the transitivity question, Aljadeff, Haile and Natapov introduced a list of groups of central type with the property that the action of $Aut(G)$ on the nondegenerate cohomology classes in $H^2(G, \mathbb{C}^*)$ is transitive. This list contains all the abelian groups of central type.
- However, groups of central type with a non-transitive action on the nondegenerate cohomology classes do exist. We give two examples of such groups which are essentially different.

Question 2, examples

- There exists a group G of central type of order 2^8 s.t the action of $\mathrm{Aut}(G)$ on the nondegenerate cohomology classes is not transitive.
  The proof is by introducing two nondegenerate cohomology classes in $H^2(G, \mathbb{C}^*)$ of distinct orders.
- There exist groups of central type with a non-transitive action on the nondegenerate cohomology classes such that all their Sylow subgroups are abelian. The proof is by introducing two nondegenerate cohomology classes $[f_1], [f_2] \in H^2(G, \mathbb{C}^*)$ such that $[f_1]$ is trivial on the center of G and $[f_2]$ is non-trivial on the center of G.

maximal connected gradings of $M_n(\mathbb{C})$

Let n be of the form of Theorem B and let $M_n(\mathbb{C}) = \bigoplus_{g\in G} M_n(\mathbb{C})_g$ be a G-grading. Then, by the BSZ decomposition there is a subgroup $H \leq G$ of central type of order r^2 which determines a fine gradings of $M_r(\mathbb{C})$ where r divide n. By Theorem B this subgroup is $C_r \times C_r$. Now, by Aljadeff, Haile and Natapov the action of $\mathrm{Aut}(C_r \times C_r)$ on the nondegenerate cohomology classes is transitive. We proved the following theorem,

Theorem C

Let n be a natural number of the form of Theorem B. Let X be a maximal connected G-gradings of $M_n(\mathbb{C})$. Then there exists a decomposition $n = n_1 \cdot n_2$ such that

$$G = F_{n_1-1} * (C_{n_2} \times C_{n_2}).$$

- In a series of papers CRS, show a correspondence between Galois coverings of a K-category B and the connected gradings of B.
- For any linear K-category, the K-space of morphisms from any object to itself is a K-algebra.
- Conversely, any K-algebra A can be considered as a one object K-category, where A is the algebra of morphisms from this object to itself.
- We can now justify the definition of equivalence of gradings.

Proposition

Two Galois coverings of a K-algebra A are equivalent if and only if, the corresponding gradings are equivalent.

Therefore there is a 1-1 correspondence between the maximal elements in the category of Galois coverings of A, namely the *simply-connected Galois coverings* and the maximal connected gradings of A with respect to quotient gradings.

the intrinsic fundamental group

- Let B be a K-category. CRS define its *intrinsic fundamental group*, $\pi_1(\mathrm{B})$ using Galois coverings.
- Let A be a $\mathbb{C}$-algebra, and consider A as a one object $\mathbb{C}$-category. By the connection between Galois coverings and connected gradings, $\pi_1(A)$ is the inverse limit of all the connected gradings of A with respect to quotient maps.

intrinsic fundamental group of $M_p(\mathbb{C})$

- Let p be a prime number. By Theorem C there are exactly two maximal connected grading classes of $M_p(\mathbb{C})$, the F_{p-1} elementary grading class and the $C_p \times C_p$ fine grading class.
- By showing that these grading classes have a unique maximal common quotient grading class graded by C_p, CRS show that

$$\pi_1(M_p(\mathbb{C})) = F_{p-1} \times C_p.$$

intrinsic fundamental group of $M_n(\mathbb{C})$

- It is clear that in order to compute $\pi_1(M_n(\mathbb{C}))$ we need first to compute all the maximal connected gradings of $M_n(\mathbb{C})$ then to compute the common quotient between each two maximal connected gradings.
- By computing the common quotient between each pair of maximal connected gradings provided by Theorem C we are able to prove the following theorem

Theorem D

Let n be a non-prime number of the form of Theorem B, then

$$\pi_1(M_n(\mathbb{C})) = F_{n-1} * C_n.$$

The end

Thank you very much!