Noncommutative Rings and Their Applications

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# $\tau$-Projective and Strongly $\tau$-Projective Modules

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According to Nakayama a ring $R$ is quasi-Frobenius ($QF$-ring) if $R$ is left (or right) artinian and if $\{e_1, e_2, \cdots, e_n\}$ is a basic set of primitive idempotents of $R$, then there exists a (Nakayama) permutation $\sigma$ of $\{1, 2, \cdots, n\}$ such that $soc(Re_k) \cong Re_{\sigma k}/Je_{\sigma k}$ and $soc(e_{\sigma k}R) \cong e_kR/e_kJ$, where $J = J(R)$ is the Jacobson radical of $R$. This remarkable description by Nakayama reduces the perfect duality in $QF$-rings to a duality between the Jacobson radical and the socle of the indecomposable projective components of the basic subring of $R$. This result was the primary motivation behind the introduction of the concept of *soc-injectivity* and the dual concept *rad-projectivity*, as follows:

**Definition 1** *Let $M$ and $N$ be right $R$-modules.*

$M$ is called socle-$N$-injective (soc-$N$-injective) if any $R$-homomorphism $f : Soc(N) \to M$ extends to $N$. Equivalently, for any semisimple submodule $K$ of $N$, any $R$-homomorphism $f : K \to M$ extends to $N$. $M$ is called soc-injective, if $M$ is soc-$R$-injective. A right $R$-module $M$ is called strongly soc-injective, if $M$ is soc-$N$-injective for all right $R$-modules $N$.

$$\begin{array}{ccccc} 0 & \longrightarrow & Soc(N) & \xrightarrow{f} & N \\ & & \downarrow f & \exists g \swarrow & \\ & & M & & \end{array}$$

$$\updownarrow$$

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{f} & N \\ & & \downarrow f & \exists g \swarrow & \\ & & M & & \end{array}$$

**Definition 2** *Let $M, N$ be right $R$-modules.*

$M$ is called radical $N$-projective (rad-$N$-projective) if, for any epimorphism $\sigma : N \longrightarrow K$ where $K$ is a homomorphic image of $N/rad(N)$ and any homomorphism $f : M \longrightarrow K$, there exists a homomorphism $g : M \longrightarrow N$ such that $f = \sigma \circ g$.

$$\begin{array}{ccccc} & & M & & \\ & \exists g \swarrow & \downarrow f & & \\ N & \overset{\sigma}{\longrightarrow} & K & \longrightarrow 0 \end{array}$$

$M$ is called rad-projective (resp., rad-quasi-projective) if $M$ is rad-$R_R$-projective (resp., rad-$M$-projective). The module $M$ is called strongly rad-projective if $M$ is rad-$N$-projective for every $R$-module $N$.

**Remark 3** *This notion is distinct from that of Clark, Lomp, Vanaja and Wisbauer in their book "Lifting Modules."*

In this talk we generalize and extend the notion of rad-pojectivity by introducing the notions of $\tau$-projective and strongly $\tau$-projective modules relative to any preradical $\tau$. When $\tau(M) = rad(M)$ we recover all the work that was carried out in on rad-projectivity, and obtain new and interesting results in the cases where $\tau(M) = soc(M)$, $\tau(M) = Z(M)$ and $\tau(M) = \delta(M)$, where $soc(M)$, $Z(M)$ and $\delta(M)$ denotes to the socle, the singular sub-module and the $\delta$-submodule of $M$, respectively.

A preradical $\tau$ of $Mod$-$R$ assigns to each $M \in Mod$-$R$ a submodule $\tau(M)$ in such a way that for each $R$-homomorphism $f : M \to N$ we have $f(\tau(M)) \subseteq \tau(N)$. Thus a preradical is a subfunctor of the identity functor of $Mod$-$R$. Every preradical $\tau$ commutes with direct sums and gives rise to a pretorsion class $T_\tau =: \{M \in Mod\text{-}R : \tau(M) = M\}$ which is closed under direct sums and factor modules. Clearly $\tau(R)M \subseteq \tau(M)$ for every $M \in Mod$-$R$. We sometimes call $\tau(M)$ the $\tau$-submodule of $M$. A preradical is said to be a radical if $\tau(M/\tau(M)) = 0$. Examples of preradicals include:

1. $rad(M) =: \cap \{N : N$ is a maximal submodule of $M\}$

   $= \sum \{L : L$ is a small submodule of $M\}$.

2. $soc(M) =: \sum \{S : S$ is a simple submodule of $M\}$

   $= \cap \{N : N$ is an essential submodule of $M\}$.

3. $Z(M) =: \{x \in M : r_R(x) \subseteq^{ess} R_R\}$.

4. $\delta(M) =: \sum \{L : L$ is a $\delta$-small submodule of $M\}$

   $= \cap\{N \subset M : M/N$ is a simple singular $R$-module$\}$.

Where according to Y. Zhou, a submodule $N$ of a right $R$-module $M$ is called $\delta$-small in $M$, and denoted by $N \subseteq^{\delta} M$, if $M \neq N + X$ for any proper submodule $X$ of $M$ with $M/X$ singular.

Clearly if $M$ is a right $R$-module, then $rad(M) \subseteq \delta(M)$ and if $M$ is projective, then $soc(M) \subseteq \delta(M)$.

**Definition 4** *A right R-module $M$ is called $\tau$-$N$-projective if, for every diagram:*

$$\begin{array}{ccccc} & & M & & \\ & \exists\lambda \swarrow & \downarrow f & & \\ N & \xrightarrow{g} & L & \longrightarrow & 0 \end{array}$$

*with $L$ an image of $N/\tau(N)$, equivalently $\tau(N) \hookrightarrow$* ker $g$*, there exists a homomorphism $\lambda : M \longrightarrow N$ such that $g\lambda = f$. The module $M$ is called $\tau$-projective (resp., $\tau$-quasi-projective) if $M$ is $\tau$-$R_R$-projective (resp., $\tau$-$M$-projective), and is called strongly $\tau$-projective if it is $\tau$-$N$-projective for every $R$-module $N$.*

If $\tau$ is the trivial preradical, i.e. $\tau(M) = 0$ for every right $R$-module $M$, then the notion of $\tau$-$N$-projectivity is the usual notion of $N$-projectivity.

**Example 5**

1. *If $M$ is strongly $\tau$-projective and either $\tau(R) = 0$ or $\tau(M) = 0$, then $M$ is projective. In fact, since $M$ is a homomorphic image of a free module, there is an exact sequence $R^{(\Lambda)} \xrightarrow{\eta} M \to 0$ for some set $\Lambda$. If $\tau(R) = 0$, then $\tau(R^{(\Lambda)}) = (\tau(R))^{(\Lambda)} = 0$ and so $\eta(\tau(R^{(\Lambda)})) = 0$; and if $\tau(M) = 0$, then $\eta(\tau(R^{(\Lambda)})) \subseteq \tau(M) = 0$. In both cases $\tau(R^{(\Lambda)}) \subseteq \ker \eta$ and by the assumption the map $\eta$ splits. Therefore $M$ is isomorphic to a direct summand of $R^{(\Lambda)}$, and so $M$ is projective.*

2. *Since $soc(\mathbb{Z}_{\mathbb{Z}}) = 0$ and no non-trivial maps from $\mathbb{Q}_{\mathbb{Z}}$ into $\mathbb{Z}_n$, any diagram:*

$$\begin{array}{ccccc} & & \mathbb{Q}_{\mathbb{Z}} & & \\ & & \downarrow f & & \\ \mathbb{Z}_{\mathbb{Z}} & \xrightarrow{\eta} & \mathbb{Z}_n & \to 0 & \end{array}$$

*can be completed, and so $\mathbb{Q}_{\mathbb{Z}}$ is soc-projective, i.e. soc-$\mathbb{Z}$-projective. Since $\mathbb{Q}_{\mathbb{Z}}$ is not projective, we infer from* (2) *that $\mathbb{Q}_{\mathbb{Z}}$ is not strongly soc-projective.*

3. *Since $\delta(\mathbb{Z}_{\mathbb{Z}}) = 0$, it follows from (2) above that every strongly $\delta$-projective $\mathbb{Z}$-module is projective. In particular, if $M = \mathbb{Q}/\mathbb{Z}$, then $M$ as a $\mathbb{Z}$-module is a $\delta$-$\mathbb{Q}_{\mathbb{Z}}$-projective module with $M = \delta(M)$, which is not strongly $\delta$-projective. Note also that $M$ is not $\mathbb{Q}_{\mathbb{Z}}$-projective. For, if $\mathbb{Q}/\mathbb{Z}$ were $\mathbb{Q}_{\mathbb{Z}}$-projective, then the following diagram:*

$$\begin{array}{ccccc} & & \mathbb{Q}/\mathbb{Z} & & \\ & & \downarrow id & & \\ \mathbb{Q} & \xrightarrow{\eta} & \mathbb{Q}/\mathbb{Z} & \to 0 & \end{array}$$

*can be completed, and $\mathbb{Z}_{\mathbb{Z}}$ would be a summand of $\mathbb{Q}_{\mathbb{Z}}$; a contradiction.*

4. *The $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ is an example of a $\delta$-quasi-projective module which is not quasi-projective.*

5. *If* $R = \mathbb{Z}_{(2)}$ *is the localization of* $\mathbb{Z}$ *at the prime ideal generated by* 2*, then the field of fractions of* $R$ *is the field of rational numbers* $\mathbb{Q}$*. Since* $R$ *is a local ring and* $\mathbb{Q}_R$*, as a right* $R$*-module, has no maximal submodules,* $\mathbb{Q}_R$ *is strongly rad-projective which is not projective (since projective modules have maximal submodules). We should note that, in general, if* $R = \mathbb{Z}_{(p)}$ *is the localization of* $\mathbb{Z}$ *at any prime element* $p \in \mathbb{Z}$*, then* $\mathbb{Q}_R$ *is strongly rad-projective which is not projective.*

6. *In general, if $R$ is a discrete valuation ring, i.e. a principal ideal domain with exactly one non-zero maximal ideal, and $K$ is its quotient field (field of fractions), then $K_R$ as a right $R$-module has no maximal submodules. For, if $M$ is the unique maximal right ideal of $R$, write $M = xR$ for some $x \in R$. It can be shown that the $R$-submodules of $K$ are $0$, $K$ and $x^i R$, $i \in \mathbb{Z}$, from which we can easily infer that $rad(K_R) = K$. Since $R$ is a local ring, it follows from the above observation that, $K_R$ is strongly rad-projective which is not projective. Now, we have an abundance of examples of strongly rad-projective modules that are not projective. For example, if $k$ is a field and $R = k\,[[x]]$ is the formal power series, with one indeterminate variable $x$, and $K$ is its quotient field, then $K$ is strongly rad-projective that is not projective.*

**Proposition 6**

1. *If $M$ is $\tau$-$N$-projective and $K$ is a submodule of $N$, then $M$ is $\tau$-$N/K$-projective.*

2. *A direct sum $\oplus_{i \in I} M_i$ of modules is $\tau$-$N$-projective iff each $M_i$ is $\tau$-$N$-projective.*

3. *A direct summand of a $\tau$-$N$-projective module is $\tau$-$N$-projective.*

4. *If $A \overset{\theta}{\backsimeq} B$, then $M$ is $\tau$-$A$-projective iff $M$ is $\tau$-$B$-projective.*

5. *Let $M$ be a $\tau$-$M_i$-projective for all $i = 1, 2, ..., n$. Then $M$ is $\tau$-$\oplus_{i=1}^{n} M_i$-projective.*

6. $\oplus_{i=1}^{n} M_i$ *is* $\tau$*-quasi-projective iff each* $M_i$ *is* $\tau$*-*$M_j$*-projective for all* $i, j = 1, 2, 3, ..., n$*. In particular,* $M \oplus N$ *is* $\tau$*-quasi-projective iff both* $M$ *and* $N$ *are* $\tau$*-quasi-projective,* $M$ *is* $\tau$*-*$N$*-projective and* $N$ *is* $\tau$*-*$M$*-projective.*

7. *If* $M$ *is a* $\tau$*-projective right* $R$*-module and* $N$ *is a finitely generated right* $R$*-module, then* $M$ *is* $\tau$*-*$N$*-projective.*

8. *If* $M$ *is finitely generated and* $\tau$*-*$M_i$*-projective for all* $i \in I$*, then* $M$ *is* $\tau$*-*$\oplus_{i \in I} M_i$*-projective.*

9. *If* $N$ *is a generator, then every finitely generated* $\tau$*-*$N$*-projective module is strongly* $\tau$*-projective.*

10. *If* $M$ *is a finitely generated* $\tau$*-projective right* $R$*-module, then* $M$ *is strongly* $\tau$*-projective.*

11. *If* $\tau(R) = 0$*, then every finitely generated* $\tau$*-projective right* $R$*-module is projective.*

12. *If $A$, $B$ and $N$ are right $R$-modules with $A \overset{\theta}{\backsimeq} B$, then $A$ is $\tau$-$N$-projective iff $B$ is $\tau$-$N$-projective.*

**Remark 7** *Note that if the right $R$-module $M$ is $N$-projective, then it is $K$-projective for every submodule $K$ of $N$. This is not true for $\tau$-$N$-projective modules. In fact, if $M = \mathbb{Z}_n$, $N = \mathbb{Q}_{\mathbb{Z}}$ and $K = \mathbb{Z}_{\mathbb{Z}}$, then $M_{\mathbb{Z}}$ is rad-$N$-projective but not rad-$K$-projective.*

**Corollary 8** *The following statements are true:*

1. *For every family $\{M_i\}_{i\in I}$ of right $R$-modules, $\oplus_{i\in I} M_i$ is (strongly) $\tau$-projective iff $M_i$ is (strongly) $\tau$-projective for every $i \in I$.*

2. *A direct summand of a (strongly) $\tau$-projective module is again (strongly) $\tau$-projective.*

3. *If $M_R$ is a finitely generated $R$-projective module (i.e. projective relative to $R_R$), then $M$ is projective.*

Let me take you back to soc-injectivity and the following theorem:

**Theorem 9** *For a right R-module M, the following conditions are equivalent :*

1. $M$ *is strongly soc-injective.*

$$\begin{array}{ccccc} 0 & \longrightarrow & K \subseteq Soc(N) & \xrightarrow{f} & N \\ & & \downarrow f & \exists g \swarrow & \\ & & M & & \end{array}$$

2. $M$ *is soc-*$E(M)$*-injective.*

3. $M = E \oplus T$, *where* $E$ *is injective and* $T$ *has zero socle. Moreover, if* $M$ *has non-zero socle then* $E$ *has essential socle.*

The exact dualization of the above theorem is the following:

**Theorem 10** *The following are equivalent:*

1. *Every right $R$-module is $\tau$-$N$-projective.*
2. *Every homomorphic image of $N$ is $\tau$-$N$-projective.*
3. $N = \tau(N) \oplus A$ *with $A$ semisimple.*
4. $N = \tau(N) + soc(N)$.

**Proposition 11** *The following conditions are equivalent for a* **finitely generated** *right $R$-module $N$:*

1. Every right $R$-module is $rad$-$N$-projective.
2. Every right $R$-module is $\delta$-$N$-projective.
3. Every homomorphic image of $N$ is $rad$-$N$-projective.
4. Every homomorphic image of $N$ is $\delta$-$N$-projective.
5. $N$ is semisimple.

**Proposition 12** *The following conditions are equivalent for a right $R$-module $N$:*

1. *Every right $R$-module is $soc$-$N$-projective.*

2. *Every homomorphic image of $N$ is $soc$-$N$-projective.*

3. *$N$ is semisimple.*

Recall that a ring $R$ is right hereditary if every submodule of a projective right $R$-module is projective; equivalently if every factor module of an injective right $R$-module is injective.

In the soc-injective case, we had the following result:

**Theorem 13** *The following conditions are equivalent:*

1. *Every quotient of a soc-injective right R-module is soc-injective.*

2. *Every quotient of an injective right R-module is soc-injective.*

3. *Every semisimple submodule of a projective module is projective.*

4. *soc($R_R$) is projective.*

In the $\tau$-projective case, we have:

**Theorem 14** *For a right $R$-module $M$, the following statements are equivalent:*

1. *Every submodule of a $\tau$-$E(M)$-projective right $R$-module is*

   *$\tau$-$E(M)$-projective.*

2. *Every submodule of a projective right $R$-module is*

   *$\tau$-$E(M)$-projective.*

3. *Every right ideal of $R$ is $\tau$-$E(M)$-projective.*

4. *Every factor module of $E(M)/\tau(E(M))$ is injective.*

# 1 $\tau$-Projective Covers and The Dual Baer Criterion

A result of Eckmann and Schopf asserts that every right $R$-module $M$ can be embedded in an injective envelope (hull) of $M$. In dualizing this result, Bass has shown that, every (finitely generated) right $R$-module has a projective cover if and only if $R$ is a right (semi) perfect ring. On the other hand, a result of Baer, known by the *Baer Criterion*, asserts that a right $R$-module $M$ is injective if and only if it is injective relative to $R_R$. In general, the dual to the Baer Criterion is not true, as there are examples of $R$-projective modules that are not projective. For example $\mathbb{Q}_{\mathbb{Z}}$ is $\mathbb{Z}$-projective but not projective. Where a right $R$-module $M$ is $R$-projective, if it is projective relative to the right $R$-module $R_R$.

**Definition 15** *Let R be a ring and* $\Omega$ *be a class of right R-modules which is closed under isomorphisms. An R-homomorphism* $\phi : P \longrightarrow M$ *is called an* $\Omega$*-cover of the right R-module M, if* $P \in \Omega$ *and* $\phi$ *is an epimorphism with small kernel (i.e.,* $L + \ker(\phi) = P$ *implies that* $L = P$ *whenever L is a submodule of P). That is to say, if* $\Omega$ *is the class of (strongly)* $\tau$*-projective right R-modules, the R-homomorphism* $\phi : P \longrightarrow M$ *is called (strongly) rad-projective cover of M.*

**Theorem 16** *If $\tau = \delta$, $soc$ or $rad$, then the following statements are equivalent:*

1. *$R$ is semiperfect.*

2. *Every finitely generated right $R$-module has a strongly $\tau$-projective cover.*

3. *Every finitely generated right $R$-module has a $\tau$-projective cover.*

4. *Every finitely generated right $R$-module has a $\tau$-quasi-projective cover.*

5. *Every 2-generated right $R$-module has a $\tau$-quasi-projective cover.*

6. *Every simple right $R$-module has a $\tau$-projective cover.*

Since $R$-projective modules are $\tau$-projective, as an immediate consequence of the above theorem, the next corollary provides new characterizations of semiperfect rings.

**Corollary 17** *The following statements are equivalent:*

1. *R is semiperfect.*

2. *Every* $2$*-generated right* $R$*-module has a quasi-projective cover.*

3. *Every* $2$*-generated right* $R$*-module has a* $rad$*-quasi-projective cover.*

4. *Every* $2$*-generated right* $R$*-module has a* $soc$*-quasi-projective cover.*

5. *Every* $2$*-generated right* $R$*-module has a* $\delta$*-quasi-projective cover.*

6. *Every simple right $R$-module has an $R$-projective cover.*

7. *Every simple right $R$-module has a $rad$-projective cover.*

8. *Every simple right $R$-module has a $soc$-projective cover.*

9. *Every simple right $R$-module has a $\delta$-projective cover.*

With the help of an argument due to Ketkar and Vanaja (*R-projective modules over a semiperfect ring, Canad. Math. Bull. 24 (1981), 365-367.)*, we can establish the following theorem.

**Theorem 18** *Let $R$ be a semiperfect ring with $\tau(R) \subseteq \delta(R)$. If $M_R$ is a $\tau$-projective module with small radical, then $M_R$ is projective.*

**Corollary 19** *Over a right perfect ring $R$ with $\tau(R) \subseteq \delta(R)$, every $\tau$-projective right $R$-module is projective.*

**Theorem 20** *If $\tau = \delta$, $soc$ or $rad$, then the following statements are equivalent:*

1. *$R$ is right perfect.*

2. *Every right $R$-module has a strongly $\tau$-projective cover.*

3. *Every right $R$-module has a $\tau$-projective cover.*

4. *Every semisimple right $R$-module has a strongly $\tau$-projective cover.*

5. *Every semisimple right $R$-module has a $\tau$-projective cover.*

**Corollary 21** *the following statements are equivalent:*

1. *$R$ is right perfect.*

2. **Every right $R$-module has an $R$-projective cover.**

3. *Every semisimple right $R$-module has an $R$-projective cover.*

4. *Every semisimple right $R$-module has a $rad$-projective cover.*

5. *Every semisimple right $R$-module has a $soc$-projective cover.*

6. *Every semisimple right $R$-module has a $\delta$-projective cover.*

It is well-known that a ring $R$ is right perfect if and only if every flat right $R$-module is projective. In the next theorem we show that if $R$ is a ring with $\tau(R) \subseteq \delta(R)$, then $R$ is right perfect if and only if every flat right $R$-module is $\tau$-quasi-projective. Our work depends on a remarkable result due to Bican, El-Bashir & Enochs which asserts that every $R$-module has a flat cover.

**Theorem 22** *If $R$ is a ring with $\tau(R) \subseteq \delta(R)$, then the following statements are equivalent:*

1. *$R$ is right perfect.*

2. *Every flat right $R$-module is strongly $\tau$-projective.*

3. *Every flat right $R$-module is $\tau$-quasi-projective.*

**Corollary 23** *the following statements are equivalent:*

1. $R$ *is right perfect.*

2. *Every flat right* $R$*-module is quasi-projective.*

3. *Every flat right* $R$*-module is rad-quasi-projective.*

4. *Every flat right* $R$*-module is soc-quasi-projective.*

5. *Every flat right* $R$*-module is* $\delta$*-quasi-projective.*

**Example 24** *Strongly* $\tau$*-projective right* $R$*-module need not be flat. For, if* $p_1$ *&* $p_2$ *are two distinct prime numbers and*

$$R =: \left\{ \frac{m}{n} : m, n \in \mathbb{Z},\ p_i \nmid n \right\},$$

*then* $R$ *is a commutative semilocal domain such that* $E(R/p_iR)$*, for* $i = 1, 2$*, is a strongly* $\delta$*-projective* $R$*-module which is not flat.*

$QF$-ring is right (and left) perfect ring, the next result is now an immediate consequence of the above results.

**Corollary 25** *If $R$ is a ring with $\tau(R) \subseteq \delta(R)$, then $R$ is quasi-Frobenius if and only if every $\tau$-projective right $R$-module is injective.*

**Remark 26** *It is also well-known that $R$ is QF iff every injective right $R$-module is projective. Such a result cannot be extended to strongly $\delta$-projective modules. In fact, if $p_i \in \mathbb{Z}, 1 \leq i \leq 2$, are two distinct prime numbers, and $R =: \left\{\frac{m}{n} : m, n \in \mathbb{Z} \text{ and } p_i \nmid n\right\}$, then R is a commutative, semilocal domain such that $M = radM = \delta(M)$, where $M$ is any injective $R$-module. Now, since $R$ is semilocal, it follows that every injective $R$-module is strongly $\delta$-projective. To see this, consider the following diagram:*

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow f & & \\ L & \xrightarrow{\eta} & K & \rightarrow 0 \end{array}$$

*with $K$ a homomorphic image of $L/\delta(L)$. Since $radL \subseteq \delta(L)$ and $R$ is semilocal, $rad(K) = 0$ and the only map from $M$ into $K$ is the trivial map. This means every such diagram can be completed and so $M$ is strongly $\delta$-projective. However $R$ is not a perfect ring, and hence not quasi-Frobenius.*

# Thank You