On classical rings of quotients of duo rings

Michał Ziembowski

Warsaw University of Technology

Lens
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Let R be a ring and S a multiplicative set in R (i.e. $S \cdot S \subseteq S$, $1 \in S$, and $0 \notin S$). Then a ring RS^{-1} is called a *right ring of quotients of R with respect to S* if there exists a ring homomorphism $\varphi : R \to RS^{-1}$ such that

(a) For any $s \in S$, $\varphi(s)$ is a unit of RS^{-1}.

(b) Every element of RS^{-1} has the form $\varphi(a)\varphi(s)^{-1}$ for some $a \in R$ and $s \in S$.

(c) $\ker \varphi = \{r \in R : rs = 0 \text{ for some } s \in S\}$.

It is well known that the ring R has a right ring of quotients with respect to S if and only if the following conditions are satisfied:

(1) For any $a \in R$ and $s \in S$, $aS \cap sR \neq \emptyset$.

(2) For $a \in R$, if $ta = 0$ for some $t \in S$, then $as = 0$ for some $s \in S$.

A multiplicative set S satisfying the above conditions (1) and (2) is called a *right denominator set*.

- If the set S consists of all regular elements of R (i.e. all elements $a \in R$ such that a is neither a left zero-divisor nor a right zero-divisor of R), then the right ring of quotients RS^{-1} is called the *classical right ring of quotients of* R and is denoted by $Q^r_{cl}(R)$.

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- In the same way we can consider left sided version of above and get a left ring of quotients $S^{-1}R$ of R with respect to a left denominator set S and the classical left ring of quotients of R which is denoted by $Q^l_{cl}(R)$.

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- In this talk we want to show that there exists a duo ring R such that its classical right ring of quotients $Q^r_{cl}(R)$ is left duo and not right duo. Using mentioned construction we will built up a duo ring with classical right ring of quotients which is neither right nor left duo.

Proposition 1

Let R be a right (resp. left) duo ring and P an ideal of R such that $S = R \setminus P$ is a right (resp. left) denominator set in R. Then RS^{-1} (resp. $S^{-1}R$) is right (resp. left) duo if and only if for any $a \in R$ we have $Sa \subseteq aS$ or $as = 0$ (resp. $aS \subseteq Sa$ or $sa = 0$) for some $s \in S$.

- Let G be the free abelian group generated by the set $\{x_i : i \in \mathbb{Z}\}$ and let φ be an endomorphism of G defined by

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- Given an element $g \in G$, we can write $g = x_{\ell_1}^{k_1} x_{\ell_2}^{k_2} \cdots x_{\ell_n}^{k_n}$ where $\ell_1 < \ell_2 < \cdots < \ell_n$ and $k_i \in \mathbb{Z} - \{0\}$. We call this the *canonical representation* for g. We call k_n the *final exponent* and we call x_{ℓ_n} the *final component*. (The canonical representation of $g = 1$ is somewhat special, being an empty product of such terms, and we write $g = 1$.)

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- For any $g_1, g_2 \in G$ we write $g_1 \prec g_2$ if $g_1 \neq g_2$ and $g_1^{-1} g_2$ has a (strictly) positive final exponent.
- It is easy to see that $(G, \preceq)$ is a totally ordered group and for any $g_1, g_2 \in G$, $g_1 \prec g_2$ implies $\varphi(g_1) \prec \varphi(g_2)$.

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- We define a multiplication and order relation on T in the following way. For $(m_1, g_1), (m_2, g_2) \in T$ we define

$$(m_1, g_1)(m_2, g_2) = (m_1 + m_2, \varphi^{m_2}(g_1)g_2),$$

and

$$(m_1, g_1) \leq (m_2, g_2) \iff \text{either } m_1 < m_2 \text{ or } m_1 = m_2 \text{ and } g_1 \preceq g_2.$$

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- It is easy to verify that $(T, \leq)$ is a positively strictly totally ordered monoid with $(0, 1)$ as a unity (an ordered monoid $(T, \cdot, \leq)$ is *positively ordered* if $s \geq 1$ for any $s \in T$).

- Let D be a division ring. Then we consider the set $D[[T]]$ of formal power series of the form

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- With pointwise addition and multiplication as defined above, $D[[T]]$ becomes a ring.

- A ring (resp. monoid) R is said to be a *right chain ring* (resp. *monoid*) if the right ideals of R are totally ordered by set inclusion, i.e., if $aR \subseteq bR$ or $bR \subseteq aR$ for any $a, b \in R$. Left chain rings (resp. monoids) are defined similarly. If R is left and right chain, then we say that R is a *chain ring* (resp. *monoid*).

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- **Fact 1.** The monoid T is chain.
- **Fact 2.** The ring $D[[T]]$ is chain and duo.

- For an element $f \in D[[T]]$ by $\pi(f)$ we denote the minimal element of $supp(f)$.

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- It is clear that the set

$$I = \{0\} \cup \{f \in R \setminus \{0\} : \pi(f) > (1, x_1^i x_2^j x_3) \text{ for any } i, j \in \mathbb{Z}\}$$

is a proper ideal of R. Now we consider the ring

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- Since $D[[T]]$ is a duo chain ring, so is R. So the set P of those elements of R that are right or left zero-divisors is an ideal of R, and $S = R \setminus P$ is a right and left denominator set in R. Hence we can consider a right and left ring of quotients RS^{-1} and $S^{-1}R$.

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- Note that since S coincides with the set of regular elements of R, RS^{-1} and $S^{-1}R$ are the classical right and left rings of quotients, respectively, and we have

$$RS^{-1} = Q^r_{cl}(R) = Q^l_{cl}(R) = S^{-1}R.$$

- **Fact 3.** For an element $f \in D[[T]]$ we have $\overline{f} \in S$ if and only if $\pi(f) \leq (0, x_1^k)$ for some non-negative integer k.

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- **Fact 4.** The ring $Q_{cl}^r(R)$ is not right duo.
- **Fact 5.** The ring $Q_{cl}^r(R)$ is left duo.

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- Note that the regular elements are $S \times S$, and $B \setminus S \times S$ is an ideal of B.
- We know that there exist $a \in R$ and $s \in S$ such that $sa \notin aS$.
- Thus for $(a, a) \in B$ and $(s, s) \in S \times S$ we have $(s, s)(a, a) \notin (a, a)(S \times S)$ and $(a, a)(s, s) \notin (S \times S)(a, a)$. So using Proposition 1 we deduce that $Q^r_{cl}(B)$ is neither right duo nor left duo.

THANK YOU FOR YOUR ATTENTION.