Noetherian rings whose injective hulls of simple modules are locally Artinian.

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2013-07-03

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Pathological cases are Artinian rings and V-rings.

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1. commutative Noetherian rings (Matlis);
2. FBN rings (Jategaonkar); ($\diamond$) $\Rightarrow$ Jacobson Conjecture
3. $\Rightarrow$ Noetherian PI-algebras;
4. Noetherian semiprime rings of Krull dimension 1.
   $\Rightarrow$ $A_1(k)$ does satisfy ($\diamond$).
5. $k_q[x,y]$ or $A_1^q(k)$ do satisfy ($\diamond$) iff $q \in \sqrt{1}$ (Carvalho-Musson)

Rings with $(\diamond)$

k a field of characteristic 0. For a ring R let $A_1(R) = R[y][x; \frac{\partial}{\partial y}]$.

Example (Stafford)

$A_n(k) = A_1(A_{n-1}(k))$ does not satisfy $(\diamond)$ if $[k : \mathbb{Q}] \geq n - 1 \geq 0$.

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Theorem (Carvalho-L.-Pusat, 2010)

Let A be Noetherian algebra over $k = \overline{k}$ (+more assumptions). Then A satisfies $(\diamond)$ iff $A/A\mathfrak{m}$ does for all $\mathfrak{m} \in \mathrm{Max}(Z(A))$.

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Example (Heisenberg Lie algebras)

Let $\mathfrak{h}_n = \mathrm{span}(x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ be the $2n+1$-dimensional complex Heisenberg Lie algebra with relation $[x_i, y_i] = z$ for all i. Then $U(\mathfrak{h}_n)$ satisfies $(\diamond)$ if and only if $n = 1$.

Musson's example

Example (Musson '82)

Let $\mathfrak{g} = \mathrm{span}(x, y)$ with $[x, y] = y$. Then $U(\mathfrak{g}) = k[y][x; y\frac{\partial}{\partial y}]$ does not satisfy $(\diamond)$.

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Question (1)

For which finite dimensional nilpotent $\mathfrak{g}$ does $U(\mathfrak{g})$ satisfy $(\diamond)$?

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Question (1)

For which finite dimensional nilpotent $\mathfrak{g}$ does $U(\mathfrak{g})$ satisfy $(\diamond)$?

Question (2)

For which derivation δ does $k[y][x; \delta]$ satisfy $(\diamond)$?

Question 1

Theorem (Hatipoğlu-L., 2012)

The following are equivalent for a finite dimensional complex nilpotent Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

(a) *Injective hulls of simple* $U(\mathfrak{g})$*-modules are loc. Artinian.*

(b) $\operatorname{ind}(\mathfrak{g}_0) = \inf_{f \in \mathfrak{g}_0^*} \dim(\mathfrak{g}_0^f) \geq \dim(\mathfrak{g}_0) - 2$.

(c) $\mathfrak{g}_0$ *has an abelian ideal of codimension* 1 *or* $g_0 = \mathfrak{h} \times \mathfrak{a}$ *where* $\mathfrak{a}$ *is abelian and* $\mathfrak{h}$ *is one of the following:*

   (i) $\mathfrak{h} = \operatorname{span}(e_1, e_2, e_3, e_4, e_5)$ *with*

$$[e_1, e_2] = e_3, \ [e_1, e_3] = e_4, \ [e_2, e_3] = e_5.$$

   (ii) $\mathfrak{h} = \operatorname{span}(e_1, e_2, e_3, e_4, e_5, e_6)$ *with*

$$[e_1, e_2] = e_6, \ [e_1, e_3] = e_4, \ [e_2, e_3] = e_5.$$

Primitive factors of superalgebras

Theorem (Hatipoğlu-L., 2012)

Let A be a Noetherian associative superalgebra such that

1. *every primitive ideal is maximal and*
2. *every graded maximal ideal is generated by a normalizing sequence of generators.*

Then

(a) *injective hulls of a left simple A-module are loc. Artinian;*

(b) *injective hulls of a left simple A/P-module are loc. Artinian for all primitive ideals P of A.*

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Theorem (Hatipoğlu-L., 2012)

Any ideal of a finite dimensional nilpotent Lie superalgebra has a supercentralizing sequence of generators.

Primitive factors of nilpotent Lie superalgebras

Theorem (Bell-Musson 1990, Herscovich 2010)

Let $\mathfrak{g}$ be a finite dimensional nilpotent complex Lie superalgebra.

1. *For $f \in \mathfrak{g}_0^*$ there exists a graded primitive ideal $I(f)$ of $U(\mathfrak{g})$ such that*
$$U(\mathfrak{g})/I(f) \simeq \mathrm{Cliff}_q(\mathbb{C}) \otimes A_p(\mathbb{C}),$$
where $2p = \dim(\mathfrak{g}_0/\mathfrak{g}_0^f)$ and $q \geq 0$.
2. *For every graded primitive ideal P of $U(\mathfrak{g})$ there exists $f \in \mathfrak{g}_0^*$ such that $P = I(f)$.*

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2. *For every graded primitive ideal P of $U(\mathfrak{g})$ there exists $f \in \mathfrak{g}_0^*$ such that $P = I(f)$.*

We are left with classifying finite dimensional nilpotent Lie algebras $\mathfrak{g}$ with $\mathrm{ind}(\mathfrak{g}) = \dim(\mathfrak{g}) - 2$.

Question 2

Example (Musson '82)

$k[y][x; y\frac{\partial}{\partial y}]$ does not satisfy $(\diamond)$.

Question (2)

For which derivation δ does $k[y][x;\delta]$ satisfy $(\diamond)$?

($\diamond$) for Ore extensions

Theorem (Carvalho-Hatipoğlu-L., 2012)

Let R be a commutative Noetherian domain over k. Set $S = R[x;\delta]$ for some $\delta \in \mathrm{Der}_k(R)$. Suppose that

1. *R is not δ-simple;*
2. *R is δ-primitive (D.Jordan), i.e. $\exists$ a maximal ideal m of R that does not contain any non-zero δ-ideal.*
3. *every non-zero δ-ideal contains a non-zero Darboux element, i.e. an element a with Ra being an δ-ideal.*

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Then

$$0 \longrightarrow S/Sm \longrightarrow S/Sm(x-1) \longrightarrow S/S(x-1) \longrightarrow 0$$

is a non-Artinian essential extension of the simple module S/Sm, i.e S does not satisfies (◇).

$k[y][x;\alpha,\delta]$

Corollary

$k[y][x;\delta]$ satisfies $(\diamond)$ iff $\delta = \lambda\frac{\partial}{\partial y}$ for some $\lambda \in k$.

Corollary (Carvalho-Hatipoğlu-L., 2012)

The following are equivalent for an automorphism α and an α-derivation δ of $k[y]$:

1. *injective hulls of simple $k[y][x;\alpha,\delta]$-modules are locally Artinian;*
2. *$\alpha \neq id$ has finite order or $\alpha = id$ and δ is locally nilpotent.*
3. *$k[y][x;\alpha,\delta]$ is isomorphic to $A_1^q(k)$ or $k_q[x,y]$ for $q \in \sqrt{1}$.*

Locally nilpotent derivations

Theorem (Carvalho-Hatipoğlu-L., 2012)

Injective hulls of simple $R[x;\delta]$-modules are locally Artinian provided δ is locally nilpotent and R is an affine commutative k-algebra.

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Sketch: Set $\mathfrak{a} = \mathrm{span}(\{\delta^i(y_j) \mid i \geq 0, 1 \leq j \leq n\})$ and $\mathfrak{g} = kx \oplus \mathfrak{a} \subseteq R[x;\delta]$. Set

$$[x, \delta^i(y_j)] = \delta^{i+1}(y_j) \qquad \forall i, j$$

Then $\exists U(\mathfrak{g}) \twoheadrightarrow R[x;\delta]$ and as $\mathfrak{g}$ is nilpotent and $\mathfrak{a}$ is an abelian ideal of codimensiom 1, $U(g)$ and hence $R[x;\delta]$ satisfies $(\diamond)$.

Locally nilpotent derivations II

Lemma

Let R be a domain with locally nilpotent derivation δ and $y \in R$ with $\delta(y) = 1$. Set $S = R[x;\delta]$. For every $a \in R \setminus R^\delta$ consider

$$0 \longrightarrow S/S(x+a) \longrightarrow S/S(x+a)x \longrightarrow S/Sx \longrightarrow 0.$$

Then $S/S(x+a)$ embeds essentially in $S/S(x+a)x$ and $\mathrm{Kdim}({}_S S/Sx) = \mathrm{Kdim}({}_{R^\delta} R^\delta)$.

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Example (Stafford, cf. Coutinho, Bernstein-Lunts)

For $D = A_{n-1}(k)$ there exists $a \in D[y]$ such that $x + a$ generates a maximal left ideal in $A_n(k) = A_1(D) = D[y][x; \frac{\partial}{\partial y}]$.

$R = A_1(\mathbb{Q})[y]$ satisfies $(\diamond)$ while $A_2(\mathbb{Q}) = R[x; \frac{\partial}{\partial y}]$ doesn't.

Thank you