On Symmetrized Weight Compositions

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First of All

• An alphabet $A$ is a finite left module over a finite ring $R$ with unity.

• A code of length $n$ is just a submodule of $A^n$. The Hamming weight counts the number of non-zero components in a tuple.
Two Notions of Equivalence

Consider two codes $C_1$ and $C_2$ of length $n$. We may think the two codes refer to the same thing in each of the following:

If $C_1 \cong C_2$ as (left) $R$-submodules of $A^n$ through an isomorphism that preserves Hamming weight (distance!),

or

if $C_1$ and $C_2$ are monomially equivalent.

Is this true?
In her PhD thesis, MacWilliams proved the Hamming weight EP (later this was called being MacWilliams!) for field alphabets.

- The alphabet $A$ has the Extension Property (EP) with respect to Hamming weight if every monomorphism preserving Hamming weight extends to a monomial transformation.
In [7], H. Ward and J. Wood reproved this via a character theoretic proof.

Key Word: 
*generating characters*

Now the question arises:
To what extent can this proof be generalized?! 
Can it work for arbitrary rings?
Nakayama’s Definitions
On Frobeniusean Algebras,
1939-1941
(1) A finite ring $R$ is Frobenius iff $R \hat{R}$ is cyclic.
(2) $\text{soc}(A)$ is cyclic if and only if $A$ can be embedded into $R \hat{R}$. 
Yes Frobenius is needed !!

1- (Wood [8] 1999): Every finite Frobenius ring has the extension property with respect to the Hamming weight.

Besides, Wood proved a partial converse (for commutative rings) in the same paper.

2- (Greferath, Nechaev, Wisbauer [3] 2004): More generally, if \( A \) is a Frobenius bi-module over the finite ring \( R \), then \( A \) has the extension property with respect to Hamming weight.

3- (Wood [10] 2009): Wood reproved this same result following the style appearing in his 1999’s paper.

One more thing was proved...
$\mathbb{R}A$ is MacWilliams if and only if

1. $A$ is pseudo-injective, and
2. $A$ can be embedded in the character group $\hat{\mathbb{R}}$ of $\mathbb{R}$ (or equivalently, $\text{soc}(A)$ is cyclic).

What Happens with Non-Cyclic Socles?
Theorem: Let $R = M_m(F_q)$ be the ring of all $m \times m$ matrices over a finite field $F_q$, and let $A = M_{m,k}(F_q)$ be the left $R$-module of all $m \times k$ matrices over $F_q$. If $k > m$, there exist linear codes $C_+, C_- \subset A^N$, $N = \prod_{i=1}^{k-1} (1 + q^i)$, such that they are isomorphic through a weight preserving map which does not extend to a monomial transformation.
that all this displayed so far concerns Hamming weight, so,

**Once again for swc?!**

- For any $G \subseteq \text{Aut}_R(A)$, define an **equivalence relation** $\sim$ on $A$: $a \sim b$ if $a = b\tau$ for some $\tau \in G$. Let $A/G$ denote the orbit space of this relation. The **G-symmetrized weight composition** is a function $\text{swc} : A^n \times A/G \to \mathbb{Q}$ defined by,

$$\text{swc}(x, a) = |\{ i : x_i \sim a \}|,$$

where $x = (x_1, \ldots, x_n) \in A^n$ and $a \in A/G$. Thus, $\text{swc}$ counts the number of components in each orbit.
Analogies Deduced

- In 2013, in [2], N. Elgarem, N. Megahed and J. Wood proved that the embeddability in $\hat{R}$ (cyclic socle) is sufficient for satisfying the extension property with respect to the G-symmetrized weight composition for any subgroup $G$ of $\text{Aut}_R(A)$,

but the necessity remained a question.

A seemingly doomed trial suggests bridging to Hamming weight ...
Define an equivalence relation $\approx$ on $A$:

$$a \approx b \text{ if } \text{Ann}_a = \text{Ann}_b.$$ 

The **Annihilator weight**, denoted $\text{aw}$, is then defined so that it counts the number of components in each orbit (i.e. having the same annihilator).

**Lemma**

In a **pseudo-injective** module, $\approx$ and $\sim_{\text{Aut}_R(A)}$ make the **same** partition.
Theorem
Let $R$ be a principal ideal ring, $\mathbb{R}_A$ a pseudo-injective module, and let $C$ be a submodule of $A^n$ for some $n$. Then a monomorphism $f : C \rightarrow A^n$ ($C \subseteq A^n$) preserves Hamming weight if and only if it preserves $\text{Aut}_R(A)$-swc.
Theorem

If $R\mathcal{A}$ is pseudo-injective, then $\mathcal{A}$ has the extension property with respect to $\text{Aut}_R(\mathcal{A})$-swc if and only if $\text{soc}(\mathcal{A})$ is cyclic.
Example:
If $L$ is any finite field, and $K \subseteq L$ is a subfield. The $K$-module $KL$ is pseudo-injective (by an extended basis argument). Thus the alphabet $KL$ has the extension property with respect to $\text{Aut}_K(L)$-swc if and only if $K = L$. 
References:


References:


References:


Thank You