Groups that cohabit with rings

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Noncommutative rings and their applications, IV

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Isoclinism

Isoclinism for groups is an equivalence relation introduced by P. Hall (1940). Equivalence classes are called **families**.

We developed a type of isoclinism in a universal algebra context and applied it to rings (B., 2014). With MacHale and Ní Shé, we applied it to commuting probability for rings.

Here we introduce a new type of isoclinism for so-called triples that allows us to relate groups to rings, and create families that contain both groups and rings!

Commuting Probability

The **commuting probability** of a finite algebraic system S having a multiplication operation denoted by juxtaposition is

$$\Pr(S) := \frac{|\{(x,y) \in S \times S \ : \ xy = yx\}|}{|S|^2}$$

$\Pr(\cdot)$-values for groups include, in particular:

- $1/n$, $n \in \mathbb{N}$;
- $2/n$, $n \in \mathbb{N}$, $n \equiv 5 \pmod 8$ or $n \equiv 7 \pmod 8$.

[B-McHale] By contrast, for rings, k/n **is not a Pr(·)-value** if

- $k, n \in \mathbb{N}$, n is square-free, $k < n$, and either
  - n is even, or
  - n has at most 69 prime factors.

Comparing groups and rings: coincidence or not?

- For groups [Rusin, 1979], the possible commuting probability values not less than $11/32$ are:

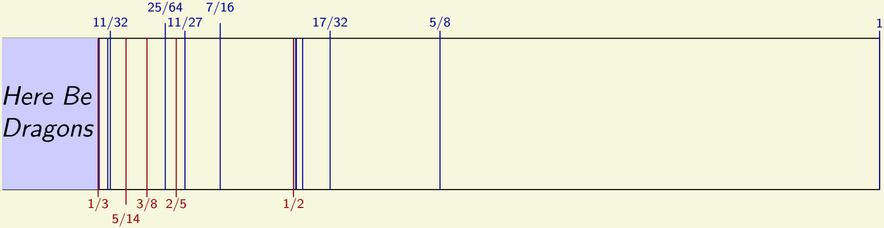
  1,

  $(2^{2k}+1)/2^{2k+1}$, $k \in \mathbb{N}$,

  $1/2$, **$7/16$**, $11/27$, $2/5$, $25/64$, $3/8$, $5/14$, **$11/32$**.

  Blue font values can be obtained with nilpotent groups (of class at most 2).

- For rings [B-MacHale-Ní Shé], we get exactly the blue values above.

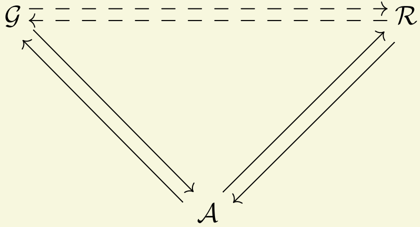


The mystery deepens!

In [B-MacHale, 2014], we found all isoclinism families of groups G such that $\mathrm{Pr}(G) \geq 1/3$.
For nilpotent groups of class 2, the number of families matches that for rings (at least for probability $\geq 11/32$, where the situation for rings was understood).

Name	Stem order	$\Pr(\mathcal{F})$			$f(\mathcal{F})$		
$\mathcal{F}_1$	1	1	$=$	1.0000	1	$=$	1.0000
$\mathcal{F}_{2,k},\ k \in \mathbb{N}$	2^{2k+1}	$\dfrac{2^{2k}+1}{2^{2k+1}}$	$\leq$	0.6250	$\dfrac{2^{k}+1}{2^{k+1}}$	$\leq$	0.7500
$\mathcal{F}_3$	6	1/2	$=$	0.5000	2/3	$\approx$	0.6667
$\mathcal{F}_4$	16	7/16	$=$	0.4375	5/8	$=$	0.6250
$\mathcal{F}_5$	32	7/16			5/8		
$\mathcal{F}_{6,1}$	27	11/27	$\approx$	0.4074	5/9	$\approx$	0.5556
$\mathcal{F}_7$	10	2/5	$=$	0.4000	3/5	$=$	0.6000
$\mathcal{F}_8$	64	25/64	$\approx$	0.3906	9/16	$=$	0.5625
$\mathcal{F}_9$	24	3/8	$=$	0.3750	7/12	$\approx$	0.5833
$\mathcal{F}_{10}$	14	5/14	$\approx$	0.3571	4/7	$\approx$	0.5714
$\mathcal{F}_{11}$	32	11/32	$\approx$	0.3438	1/2	$=$	0.5000
$\mathcal{F}_{12}$	32	11/32			1/2		
$\mathcal{F}_{13}$	32	11/32			9/16	$=$	0.5625
$\mathcal{F}_{14}$	64	11/32			1/2	$=$	0.5000
$\mathcal{F}_{15}$	64	11/32			1/2		
$\mathcal{F}_{16}$	64	11/32			1/2		
$\mathcal{F}_{17}$	64	11/32			9/16	$=$	0.5625
$\mathcal{F}_{18}$	64	11/32			9/16		
$\mathcal{F}_{19}$	128	11/32			1/2	$=$	0.5000
$\mathcal{F}_{20}$	128	11/32			9/16	$=$	0.5625
$\mathcal{F}_{6,k},\ k > 1$	3^{2k+1}	$\dfrac{3^{2k}+2}{3^{2k+1}}$	$\leq$	0.3416	$\dfrac{3^{k}+2}{2^{k+1}}$	$\leq$	0.4074

Mystery solved: categorical equivalence



$\mathcal{G}$ is a category of **finite nilpotent groups of class (at most) 2**.
$\mathcal{R}$ is a category of **finite rings**.
$\mathcal{A}$ is a category of **finite alternating minimal triples** *(to be defined!)*.

The morphisms in $\mathcal{G}$ and $\mathcal{R}$ are homoclinisms, not homomorphisms.

Key fact *These categorical equivalences preserve the associated probabilities of objects (i.e. commuting probability in $\mathcal{G}$ and $\mathcal{R}$, and* **null probability** *in $\mathcal{A}$).*

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Alternating minimal triples: definitions

Consider a triple (A, B, k), where

- A, B are abelian groups and
- $k : A \times A \to B$ is bilinear, i.e.

  $\lambda_{x,k},\ \rho_{y,k} : A \to B$ are homomorphisms, $x, y \in A$, where

  $\lambda_{x,k}(y) = \rho_{y,k}(x) = k(x, y)$.

- k is **alternating** if $k(x, x) = 0$, $x \in A$. (This implies $k(x, y) = -k(y, x)$.)

- k is **non-degenerate**: $\lambda_{x,k}$ **and** $\rho_{x,k}$ are nontrivial for all $x \in A \setminus \{0\}$.
  k is **weakly non-degenerate**: $\lambda_{x,k}$ **or** $\rho_{x,k}$ are nontrivial for all $x \in A \setminus \{0\}$.

- k is **t-surjective** if the induced tensor map $\kappa : A^{\otimes 2} \to B$ is surjective.
  (Equivalently, $\{k(x, y) \mid x, y \in A\}$ generates B.)

- k is **minimal** if it is t-surjective and weakly non-degenerate.

Examples of triples: class 2 groups, and rings

Below, G is a class 2 group and R is a ring.

The **(group) commuting triple $\mathrm{CT}^{\mathcal{G}}(G)$** is (A, B, k) where

$$A = G/Z(G), \qquad B = G',$$

and $k : A \times A \to B$ is induced by

$$[\cdot, \cdot] : G \times G \to G', \qquad [x, y] = x^{-1}y^{-1}xy.$$

The **(ring) commuting triple $\mathrm{CT}^{\mathcal{R}}(R)$** is (A, B, k) where

$$A = R/Z(R), \qquad B = [R, R],$$

and $k : A \times A \to B$ is induced by

$$[\cdot, \cdot] : R \times R \to [R, R], \qquad [x, y] = xy - yx.$$

$\mathrm{CT}^{\mathcal{G}}(G)$ and $\mathrm{CT}^{\mathcal{R}}(R)$ are alternating minimal triples.

Triple morphisms

Let $T_i = (A_i, B_i, k_i)$ be minimal triples, $k = 1, 2$.

A **morphism** $\mu : T_1 \to T_2$ consists of two group homomorphisms, $\phi : A_1 \to A_2$ and $\psi : B_1 \to B_2$, such that

$$\psi(k_1(x, y)) = k_2(\phi(x), \phi(y)), \qquad x, y \in A_1,$$

i.e. the following diagram commutes.

$$\begin{array}{ccc} A_1 \times A_1 & \xrightarrow{\phi \times \phi} & A_2 \times A_2 \\ \downarrow k_1 & & \downarrow k_2 \\ B_1 & \xrightarrow{\psi} & B_2 \end{array}$$

$\mu : T_1 \to T_2$ is an **isomorphism** if ϕ, ψ are group isomorphisms.

Target and null probabilities

The **target probability** of a finite minimal triple $T := (A, B, k)$ with respect to $u \in B$ is

$$\mathrm{Pr}_u(T) = \frac{|\{(x,y) \in A \times A \ : \ k(x,y) = u\}|}{|A|^2},$$

In particular, the **null probability of T** is $\mathrm{Pr}_0(T)$.

Below, $T_i = (A_i, B_i, k_i)$ is a finite minimal triple, $i = 1, 2$.

Lemma

If $\mu \in \mathrm{iso}(T_1, T_2)$ *and* $u \in A_1$*, then* $\mathrm{Pr}_u(T_1) = \mathrm{Pr}_{\psi_\mu(u)}(T_2)$*.*
In particular, $\mathrm{Pr}_0(T_1) = \mathrm{Pr}_0(T_2)$*.*

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Group homoclinism

A **homoclinism from a group G to a group H** is a pair (ϕ, ψ) of homomorphisms $\phi : G/Z(G) \to H/Z(H)$ and $\psi : G' \to H'$ such that $\psi([u, v]) = [u', v']$ whenever $\phi(uZ(G)) = u'Z(H)$ and $\phi(vZ(G)) = v'Z(H)$.

$$\begin{array}{ccc} (G/Z(G))^{\times 2} & \xrightarrow{\ \phi \times \phi\ } & (H/Z(H))^{\times 2} \\ \downarrow{\scriptstyle k_G} & & \downarrow{\scriptstyle k_H} \\ G' & \xrightarrow{\ \psi\ } & H' \end{array}$$

(ϕ, ψ) is an **isoclinism** if ϕ, ψ are isomorphisms [P. Hall, 1940].

- A **family** $\mathcal{F} := [G]$ is an equivalence class under isoclinism.
- $G/Z(G)$ and G' are family invariants.
- A family $\mathcal{F}$ contains at least one **stem group**, i.e. $G \in \mathcal{F}$ with $Z(G) \leq G'$.
- If $|G/Z(G)| < \infty$, then

  $$H \in [G] \text{ is a stem group} \iff |H| < \infty \text{ has minimal order in } [G].$$

Ring homoclinism

A **homoclinism from a ring R to a ring S** is a pair (ϕ, ψ) of **additive group homomorphisms** $\phi : R/Z(R) \to S/Z(S)$ and $\psi : [R, R] \to [S, S]$ such that $\psi([u, v]) = [u', v']$ whenever $\phi(uZ(R)) = u'Z(S)$ and $\phi(vZ(R)) = v'Z(S)$.

$$\begin{array}{ccc} (R/Z(R))^{\times 2} & \xrightarrow{\phi^{\times 2}} & (S/Z(S))^{\times 2} \\ \Big\downarrow k_R & & \Big\downarrow k_S \\ [R, R] & \xrightarrow{\psi} & [S, S] \end{array}$$

(ϕ, ψ) is an **isoclinism** if ϕ, ψ are isomorphisms.

- A **family** $\mathcal{F} := [R]$ is an equivalence class under isoclinism.
- $R/Z(R)$ and $[R, R]$ are family invariants.

Canonical form rings

If $S/Z(S)$ is a direct sum of cyclic groups, then $[S]$ contains at least one **canonical form ring** $R = \text{Can}(S)$ such that

- $(R, +)$ is the internal direct sum of subgroups A_1 and A_2.
- $xy \in A_2$ for all $x, y \in R$, and $xy = 0$ if either x or y lies in A_2.
- $[R, R] = Z(R) = A_2$.

$R = \text{Can}(S)$ **might not have minimal order in $\mathcal{F} := [S]$**, but...

$$|S| < \infty \implies |R| = |S/Z(S)| \cdot |[S, S]| < \infty.$$

Functors $F_1^{\mathcal{G}} : \mathcal{G} \to \mathcal{A}$ and $F_1^{\mathcal{R}} : \mathcal{R} \to \mathcal{A}$

- $\mathcal{G}$:= category of finite class 2 groups, with homoclinisms as morphisms.
- $\mathcal{R}$:= category of finite rings, with homoclinisms as morphisms.
- $\mathcal{A}$:= category of finite alternating minimal triples.

- The map $G \mapsto \mathrm{CT}^{\mathcal{G}}(G)$ induces a fully faithful functor $F_1^{\mathcal{G}} : \mathcal{G} \to \mathcal{A}$.
- The map $R \mapsto \mathrm{CT}^{\mathcal{R}}(R)$ induces a fully faithful functor $F_1^{\mathcal{R}} : \mathcal{R} \to \mathcal{A}$.

- $\mathrm{Pr}_0(F_1^{\mathcal{G}}(G)) = \mathrm{Pr}(G)$ for all finite groups G.
- $\mathrm{Pr}_0(F_1^{\mathcal{R}}(R)) = \mathrm{Pr}(R)$ for all finite rings R.

A ring R is **isoclinic to** a (class 2) group G if $F_1^{\mathcal{R}}(R) = F_1^{\mathcal{G}}(G)$.

Commuting probability is invariant under such isoclinisms.

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Functor $F_2^{\mathcal{R}} : \mathcal{A} \to \mathcal{R}$

Suppose $T = (A, B, k) \in \mathrm{Ob}(\mathcal{A})$.

Let $(R, +) := A \oplus B$.
Select a basis $\{a_i\}_{i \in I}$ of A, with a_i of order $1 < m_i \leq \infty$, $i \in I$.
Associate some linear order $<$ to I. Let

$$a_i * a_j = \begin{cases} 0 \oplus k(a_i, a_j), & i < j, \\ 0, & \text{otherwise}, \end{cases}$$

and $x * b = b * x := 0$ for all $x \in R$, $b \in B$.

There exists a unique binary operation $*$ on R that is distributive over addition and satisfies the above equations.

- The above construction induces a fully faithful functor $F_2^{\mathcal{R}} : \mathcal{A} \to \mathcal{R}$.
- $\mathrm{Pr}(F_2^{\mathcal{R}}(T)) = \mathrm{Pr}_0(T)$ for all $T \in \mathrm{Ob}(\mathcal{A})$.

Functor $F_2^{\mathcal{G}} : \mathcal{A} \to \mathcal{G}$

Suppose $T = (A, B, k) \in \mathrm{Ob}(\mathcal{A})$.

Select bases $\{a_i\}_{i \in I}$ of A, and $\{b_i\}_{i \in J}$ of B, such that a_i is of order $1 < c_i \leq \infty$, and b_i is of order $1 < d_i \leq \infty$. We associate linear orders with both A and B, both denoted $<$.
Let G have power-commutator (pc) presentation of the form

$$\begin{array}{lll} \langle\, a_i', b_j', \text{ for } i \in I,\ j \in J \mid & & \\ \quad (a_i')^{c_i} = 1 \quad \text{and} \quad a_j' b_i' = b_i' a_j', & & \text{for } i \in I,\ j \in J, \\ \quad (b_i')^{d_i} = 1 \quad \text{and} \quad b_j' b_i' = b_i' b_j', & & \text{for } i, j \in J, \\ \quad a_j' a_i' = a_i' a_j' b_{i,j}', & & \text{for } i, j \in I,\ i < j \,\rangle \end{array}$$

where $b_{i,j}' = (b_{j_1}')^{l(i,j,1)} (b_{j_2}')^{l(i,j,2)} \cdots (b_{j_m}')^{l(i,j,m)}$, and these exponents are chosen so that $k(a_i, a_j) = \sum_{q=1}^{m} l(i,j,q) b_{j_q}$. *(using additive notation for B)*

(m and the indices $j_1, \ldots, j_m \in J$ depend on both i and j, but we suppress this dependence for simplicity.)

- The above construction induces a fully faithful functor $F_2^{\mathcal{G}} : \mathcal{A} \to \mathcal{G}$.
- $\mathrm{Pr}(F_2^{\mathcal{G}}(T)) = \mathrm{Pr}_0(T)$ for all $T \in \mathrm{Ob}(\mathcal{A})$.

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Main results

Theorem

- *$\mathcal{A}$, $\mathcal{R}$, and $\mathcal{G}$ are mutually equivalent categories.*
- *The equivalence preserves the associated probabilities.*
- *The sets of commuting probabilities of rings and of class 2 groups coincide.*

Theorem

Suppose $T = (A, B, k) \in \mathrm{Ob}(\mathcal{A})$.

- *$G := F_2^{\mathcal{G}}(T)$ is a stem group and $|G| = |A| \cdot |B|$.*
- *$R := F_2^{\mathcal{R}}(T)$ is a canonical form ring and $|R| = |A| \cdot |B|$.*
- *A stem group and a canonical form ring that are isoclinic have the same order.*

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Application 1: Burnside-Hirsch theorem

Theorem [Follows from Burnside (1911) and Hirsch (1950)**]**

Suppose $\Pr(G) = m/n$, *where* G *is a finite group and* $m, n \in \mathbb{N}$.

$|G|$ *odd* $\Longrightarrow n - m$ *is divisible by* 16.

$|G|$ *odd and not divisible by* 3 $\Longrightarrow n - m$ *is divisible by* 48.

Theorem

Suppose $\Pr(R) = m/n$, *where* R *is a finite ring and* $m, n \in \mathbb{N}$ *are coprime, where* n *has* k *distinct prime divisors.*

$|R|$ *odd* $\Longrightarrow n - m$ *is divisible by* $3^{k-1} \cdot 16^k$.

$|R|$ *odd and not divisible by* 3 $\Longrightarrow n - m$ *is divisible by* 48^k.

Application 2: Special and extraspecial groups

A **special p-triple** is an alternating minimal triple (A, B, k), where A is a finite elementary abelian p-group for some prime p.

A **symplectic p-triple** is a triple $T = (V, F, k)$, where V is a finite-dimensional vector space over a field F of order a power of p, and $k : V \times V \to F$ is a symplectic form.

Proposition

Suppose p is a prime. Under our categorical equivalences:

- *special p-triples correspond to (canonical form) finite $\mathbb{Z}_p$-algebras, and to finite special p-groups.*
- *symplectic p-triples $T = (V, \mathbb{F}_p, k) \in \mathrm{Ob}(\mathcal{A})$ correspond to finite $\mathbb{Z}_p$-algebras R with $|[R, R]| = p$, and to extraspecial p-groups.*

Open Problem

Is every commuting probability of a finite p-ring also the commuting probability of a finite $\mathbb{Z}_p$-algebra?