Some properties of $n$-Armendariz rings

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Abstract. In this presentation, for a positive integer $n$, we construct examples of rings which are $n$-Armendariz but which are not $(n+1)$-Armendariz. While ps-Armendariz rings are semi-
commutative as well as \( n \)-Armendariz for any \( n \), the reverse implications are not true in general. We give an example of a ring which is \( n \)-Armendariz for any \( n \), but which is not ps-Armendariz and we find conditions for which these classes of rings coincide. Further, we discuss a few more properties of \( n \)-Armendariz rings.

**Background.** Armendariz rings are interested objects of study during the last one and a half decade. Its origin is traced back to the year 1974 when E.P. Armendariz [2] proved that reduced rings satisfy this property.
A ring $R$ is *Armendariz* if given polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ with coefficients in $R$, the condition $f(x)g(x) = 0$ implies $a_i b_j = 0$ for every $i$ and for every $j$.

The term ‘*Armendariz rings*’ was coined by Rege and Chhawchharia in 1997.

The concept of Armendariz rings generates many new classes of related objects, for example, Kim, et al ([6]) generalised the concept Armendariz ring to power series which we call ps-Armendariz rings; Buhphang and Rege [3] studied the class of Armendariz modules; Lee and Wong in [7] introduced weak Armendariz rings as those rings such that whenever the product of
two linear polynomials is zero, then the products of their coefficients are zero.

In [8], the following observation was recorded regarding the $n$-Armendariz property and the integrally closed property:

Let $R$ be a subring of a ring $A$. Then $R$ is integrally closed in $A$ iff $A/R$ is an $n$-Armendariz $R$-module for every positive integer $n$.

However, the following question still remains unsettled:

If $R$ is integrally closed, does it imply that $A/R$ is an Armendariz $R$-module?

Equivalently,
Does there exist a subring $R$ of a ring $A$ such that $A/R$ is an $n$-Armendariz $R$-module, for all $n$, but it is not an Armendariz $R$-module?

It is also not known whether polynomial rings of $n$-Armendariz rings are $n$-Armendariz.

**Definition.** ([8]) For a fixed positive integer $n$, a left $R$-module $M$ is $n$-Armendariz if whenever polynomials $f(x) = a_0 + a_1 x$ in $R[x]$ and $g(x) = b_0 + b_1 x + \cdots + b_n x^n$ in $M[x]$ satisfy $f(x)g(x) = 0$, we have $a_i b_j = 0$, for all $i$ and for all $j$. A ring is $n$-Armendariz if it is $n$-Armendariz as a module over itself.
We confine our attention in this presentation to $n$-Armendariz rings.

It can be noted that following Lee-Wong’s definition, a ring $R$ is weak Armendariz iff it is 1-Armendariz.

**Basic properties.**

- If a ring $R$ is $n$-Armendariz for a positive integer $n$, then it is $m$-Armendariz for all positive integers $m \geq n$.
- Reduced rings, more generally, Armendariz rings are $n$-Armendariz for every $n$.
- Direct products and subrings of $n$-Armendariz rings are $n$-Armendariz.
Examples.

The ring $\mathbb{Z}_3[x, y]/(x^3, x^2 y^2, y^3)$ (due to [7]) is not 2-Armendariz, as

$$(x + yt)(x^2 + 2xyt + y^2 t^2) = (x + yt)^3 = 0$$

But $xy^2 \neq 0$. However, it is 1-Armendariz.

Using the same idea as above, we get that the ring $\mathbb{Z}_5[x, y]/(x^5, x^4 y^2, x^3 y^3, x^2 y^4, y^3)$ is not 4-Armendariz but it is 3-Armendariz.

The ring $\mathbb{Z}_8(+)^n \mathbb{Z}_8$ is not weak Armendariz and therefore it is not $n$-Armendariz for any $n$. 
The ring $M_r(K)$ of all $r \times r$ matrices over a field $K$ is not $n$-Armendariz for any $n$. So $n$-Armendariz is not a Morita invariant property.

**More properties.**

- With notations as in [1], if $D$ is a commutative domain and $M$ is a $D$-module, then for any $n > 0$, the ring $D(+)^{n}M$ is $n$-Armendariz $\iff M$ is $n$-Armendariz over $D$.
- $R$ is $n$-Armendariz $\iff$ for any idempotent element $e$ of $R$, the left ideals $Re$ and $R(1-e)$ are $n$-Armendariz.
Let \( n > 0 \) and suppose that \( R \) is a ring having a classical right ring of quotients \( Q(R) \). Then \( R \) is \( n \)-Armendariz \( \iff \) \( Q(R) \) is \( n \)-Armendariz. (Follows from [4]).

**Definitions.**

- A ring \( R \) is **linear-ps-Armendariz** if whenever a linear polynomial \( f(x) = a_0 + a_1x \) and a power series \( g(x) = \sum b_i x^i \) satisfy \( f(x)g(x) = 0 \), we have \( a_i b_j = 0 \), for all \( i \) and for all \( j \).
- \( R \) is **semi-commutative** if whenever \( a, b \in R \) satisfy \( ab = 0 \), we have \( arb = 0 \), for all \( r \in R \).
**Remarks.**

- Linear-ps-Armendariz rings are $n$-Armendariz for each positive integer $n$. $\mathbb{Z}(+)\mathbb{Q}/\mathbb{Z}$ is $n$-Armendariz for each $n$ but it is not linear-ps-Armendariz.
- Left (right) duo rings are semi-commutative.

**Proposition.**

If $R$ is linear-ps-Armendariz, then $R$ is semi-commutative.

**Proof:** If $ab = 0$, then for any $c \in R$,

$$ (a - acx)(b + cbx + c^2bx^2 + \ldots) = 0 $$

which implies $acb = 0$.

**Corollary** [6, Lemma 2.3]
If $R$ is ps-Armendariz, then $R$ is semi-commutative.

We recall that a ring is *abelian* if every idempotent element is central. Armendariz rings as well as semi-commutative rings are abelian. But we have a more general result:

**Proposition.** $n$-Armendariz rings are abelian.

**Proof:** By [7, Lemma 3.4] and using the fact that every $n$-Armendariz ring is 1-Armendariz.

The following figure illustrates the relations between the classes of rings discussed:
A ring $R$ is *von Neumann regular* if for all $a \in R$, $\exists b \in R$, such that $a = aba$. [1] and [4] proved that for a von Neumann regular ring the conditions Armendariz and semi-commutative are equivalent. Indeed, we have,

**Theorem.** If $R$ is von Neumann regular then the following are equivalent:
1. $R$ is ps-Armendariz
2. $R$ is linear ps-Armendariz
3. $R$ is semi-commutative
4. $R$ is Armendariz
5. $R$ is $n$-Armendariz, for all positive integer $n$
(6) $R$ is abelian.

If we replace von Neumann regular ring by a weaker class, viz., semiprime ring, then we obtain:

**Theorem.** If $R$ is semiprime ring, then the following are equivalent:

1. $R$ is semi-commutative
2. $R$ is linear-ps-Armendariz
3. $R$ is ps-Armendariz
References.


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