Some properties of n-Armendariz rings

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Abstract. In this presentation, for a positive integer *n*, we construct examples of rings which are *n*-Armendariz but which are not *(n+1)*-Armendariz. While ps-Armendariz rings are semi-

commutative as well as *n*-Armendariz for any *n*, the reverse implications are not true in general. We give an example of a ring which is *n*-Armendariz for any *n,* but which is not ps-Armendariz and we find conditions for which these classes of rings coincide. Further, we discuss a few more properties of *n*-Armendariz rings.

Background. Armendariz rings are interested objects of study during the last one and a half decade. Its origin is traced back to the year 1974 when E.P. Armendariz [2] proved that reduced rings satisfy this property.

A ring R is *Armendariz* if given polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ with coefficients in R, the condition
$f(x)g(x) = 0$ implies $a_i b_j = 0$ for every i and for every j.

The term '*Armendariz rings*' was coined by Rege and Chhawchharia in 1997.

The concept of Armendariz rings generates many new classes of related objects, for example, Kim, et al ([6]) generalised the concept Armendariz ring to power series which we call ps-Armendariz rings; Buhphang and Rege [3] studied the class of Armendariz modules; Lee and Wong in [7] introduced weak Armendariz rings as those rings such that whenever the product of two linear polynomials is zero, then the products of their coefficients are zero.

In [8], the following observation was recorded regarding the n-Armendariz property and the integrally closed property:

- Let R be a subring of a ring A. Then R is integrally closed in A iff A/R is an n-Armendariz R-module for every positive integer n.

However, the following question still remains unsettled:

- If R is integrally closed, does it imply that A/R is an Armendariz R- module?

Equivalently,

Does there exist a subring R of a ring A such that A/R is an n-Armendariz R-module, for all n, but it is not an Armendariz R-module?

- It is also not known whether polynomial rings of n-Armendariz rings are n-Armendariz.

Definition. ([8]) For a fixed positive integer n, a left R-module M is n-*Armendariz* if whenever polynomials $f(x) = a_0 + a_1 x$ in $R[x]$and $g(x) = b_0 + b_1 x + \cdots b_n x^n$ in $M[x]$ satisfy$f(x)g(x) = 0$, we have $a_i b_j = 0$, for all i and for all j. A ring is n-*Armendariz* if it is n-Armendariz as a module over itself.

We confine our attention in this presentation to n-Armendariz rings.

It can be noted that following Lee-Wong's definition, a ring R is weak Armendariz iff it is 1-Armendariz.

Basic properties.

- If a ring R is n-Armendariz for a positive integer n, then it is m-Armendariz for all positive integers $m \geq n$.
- Reduced rings, more generally, Armendariz rings are n-Armendariz for every n.
- Direct products and subrings of n-Armendariz rings are n-Armendariz.

Examples.

- The ring $\mathbb{Z}_3[x,y]/(x^3, x^2y^2, y^3)$ (due to [7]) is not 2-Armendariz, as

$$(x+yt)(x^2+2xyt+y^2t^2) = (x+yt)^3 = 0$$

But $xy^2 \neq 0$. However, it is *1*-Armendariz.

- Using the same idea as above, we get that the ring $\mathbb{Z}_5[x,y]/(x^5, x^4y^2, x^3y^3, x^2y^4, y^3)$ is not 4-Armendariz but it is 3-Armendariz.
- The ring $\mathbb{Z}_8(+)\mathbb{Z}_8$ is not weak Armendariz and therefore it is not n-Armendariz for any n.

- The ring $M_r(K)$ of all $r \times r$ matrices over a field K is not n-Armendariz for any n. So n-Armendariz is not a Morita invariant property.

More properties.

- With notations as in [1], if D is a commutative domain and M is a D-module, then for any $n > 0$, the ring $D(+)M$ is n-Armendariz $\Leftrightarrow$ M is n-Armendariz over D.
- R is n-Armendariz $\Leftrightarrow$ for any idempotent element e of R, the left ideals Re and $R(1\text{-}e)$ are n-Armendariz.

- Let $n > 0$ and suppose that R is a ring having a classical right ring of quotients $Q(R)$. Then R is n-Armendariz $\Leftrightarrow$ $Q(R)$ is n-Armendariz. (Follows from [4]).

Definitions.

- A ring R is *linear-ps-Armendariz* if whenever a linear polynomial $f(x) = a_0 + a_1 x$ and a power series $g(x) = \sum b_i x^i$ satisfy $f(x)g(x) = 0$, we have $a_i b_j = 0$, for all i and for all j.
- R is *semi-commutative* if whenever $a, b \in R$ satisfy $ab = 0$, we have $arb = 0$, for all $r \in R$.

Remarks.

- Linear-ps-Armendariz rings are n-Armendariz for each positive integer n. $\mathbb{Z}(+)\mathbb{Q}/\mathbb{Z}$ is n-Armendariz for each n but it is not linear-ps-Armendariz.
- Left (right) duo rings are semi-commutative.

Proposition.

If R is linear-ps-Armendariz, then R is semi-commutative.

Proof: If $ab = 0$, then for any $c \in R$,

$$(a - acx)(b + cbx + c^2bx^2 + \ \dots) = 0$$

which implies $acb = 0$.

Corollary [6, Lemma 2.3]

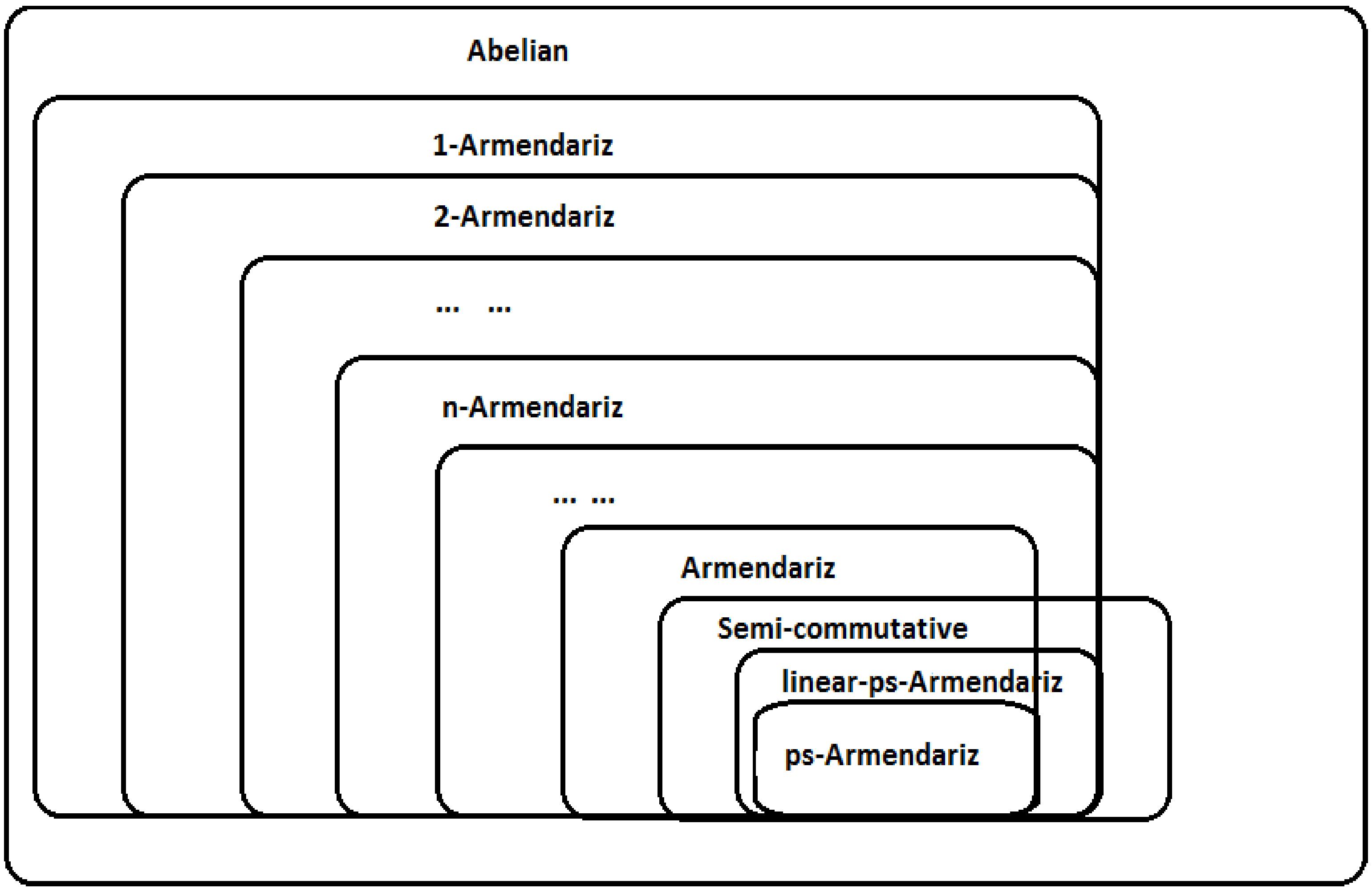
If R is ps-Armendariz, then R is semi-commutative.

We recall that a ring is *abelian* if every idempotent element is central. Armendariz rings as well as semi-commutative rings are abelian. But we have a more general result:

Proposition. n-Armendariz rings are abelian.

Proof: By [7, Lemma 3.4] and using the fact that every n-Armendariz ring is 1-Armendariz.

The following figure illustrates the relations between the classes of rings discussed:


Abelian
1-Armendariz
2-Armendariz
... ...
n-Armendariz
... ...
Armendariz
Semi-commutative
linear-ps-Armendariz
ps-Armendariz

A ring R is *von Neumann regular* if for all $a \in R$, $\exists\ b \in R$, such that $a = aba$. [1] and [4] proved that for a von Neumann regular ring the conditions Armendariz and semi-commutative are equivalent. Indeed, we have,

Theorem. If R is von Neumann regular then the following are equivalent:

(1) R is ps-Armendariz

(2) R is linear ps-Armendariz

(3) R is semi-commutative

(4) R is Armendariz

(5) R is n-Armendariz, for all positive integer n

(6) R is abelian.

If we replace von Neumann regular ring by a weaker class, viz., semiprime ring, then we obtain:

Theorem. If R is semiprime ring, then the following are equivalent:

(1) R is semi-commutative

(2) R is linear-ps-Armendariz

(3) R is ps-Armendariz

References.

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Poster presentation at NCRA IV,
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