Modules which satisfy ACC on annihilators

Mauricio Medina-Bárcenas
(Joint work with José Ríos and Jaime Castro)

National Autonomous University of Mexico

Noncommutative rings and their applications IV
Lens 2015

Definition (L. Bican, P. Jambor, T. Kepka, P. Nemec. 1980)

Let M and L be R-modules and $K \leq M$. The product of K with L in M is defined as

$$K_M L = \sum \{f(K) | f \in Hom_R(M, L)\}$$

In general, this product is not associative. If M is projective in $\sigma[M]$ then it is. In fact, if M is projective in $\sigma[M]$ then the complete lattice of submodules of M is a quantale with this product.

Prime submodule and Prime module

Definition

Let $M \in R-Mod$. A fully invariant submodule $N \leq M$ is a *prime submodule* in M if for any fully invariant submodules $K, L \leq M$ such that $K_M L \leq N$, then $K \leq N$ or $L \leq N$.

Definition

We say M is a *prime module* if 0 is a prime submodule.

Semiprime submodule and Semiprime module

Definition

A fully invariant submodule $N \leq M$ is a *semiprime submodule* in M if for any fully invariant submodule $K \leq M$ such that $K_M K \leq N$, then $K \leq N$.

Definition

We say M is a *semiprime module* if 0 is a semiprime submodule.

Proposition

Let $0 \neq M$ be an R-module and projective in $\sigma[M]$. If P is prime in M then there exists a minimal prime $P' \subseteq P$.

Proposition

Let $S := End_R(M)$ and assume M generates all its submodules. If N is a fully invariant submodule of M such that $Hom_R(M, N)$ is a prime (semiprime) ideal of S, then N is prime (semiprime) in M.

From now on, M will be an R-module projective in $\sigma[M]$.

Definition

A module M is retractable if $Hom_R(M, N) \neq 0$ for all $0 \neq N \leq M$.

Proposition

Let M be retractable. Then, $S := End_R(M)$ is a semiprime ring if and only if M is semiprime.

The Annihilator

Definition

Let $N \in \sigma[M]$. The annihilator of N in M is defined as

$$Ann_M(N) = \bigcap \{Ker(f) | f \in Hom_R(M, N)\}$$

$Ann_M(N)$ is the largest submodule of M such that $Ann_M(N)_M N = 0$. Moreover $Ann_M(N)$ is a fully invariant submodule.

Proposition

Let N be a proper fully invariant submodule of M. The following conditions are equivalent:

1. *N is semiprime in M.*
2. *If $m \in M$ is such that $Rm_M Rm \leq N$, then $m \in N$.*
3. *N is an intersection of prime submodules.*

By (3) of this proposition we see that if M is a semiprime module then M has prime submodules. So M has minimal primes submodules. This give us another description of $Ann_M(N)$ when $N \leq M$.

Lemma

Let M be a semiprime module and $N \leq M$. Let X be the set of all minimal prime submodules of M which do not contain N. Then $Ann_M(N) = \bigcap\{P | P \in X\}$.

Proposition

Let M be semiprime and $N \leq M$. If $N = Ann_M(U)$ with $U \leq M$ an uniform submodule, then N is a minimal prime in M.

ACC on annihilators

Definition

A left annihilator in M is a submodule

$$\mathcal{A}_X = \bigcap \{Ker(f) | f \in X\}$$

for some $X \subseteq End_R(M)$.

In particular if $N \leq M$ then $Ann_M(N)$ is a left annihilator in M.

We say that M satisfies ACC on annihilators if M satisfies ascending chain condition on left annihilators.

Definition

Let M be an R-module and N a submodule of M. We define the *powers* of N as:

1. $N^0 = 0$
2. $N^1 = N$
3. $N^m = N_M N^{m-1}$

Let $N \in \sigma[M]$. Denote by $\mathcal{Z}(N)$ the M-singular submodule of N.

Proposition

If M satisfies ACC on annihilators then $\mathcal{Z}(M)$ is nilpotent.

Corollary

Let $S = End_R(M)$. If M satisfies ACC on annihilators then $\mathcal{Z}_r(S)$ is nilpotent. Here $\mathcal{Z}_r(S)$ denotes the right singular ideal of S.

Corollary

Let $S = End_R(M)$. Suppose that M is a continuous module. If M satisfies ACC on annihilators then $J(S)$ is nilpotent; where $J(S)$ is the Jacobson radical of S.

Now we investigate the conditions semiprime and ACC on annihilators together.

Theorem

Let M be semiprime. Suppose that M satisfies ACC on annihilators, then:

1. *M has finitely many minimal prime submodules.*
2. *If $P_1, \ldots, P_n$ are the minimal prime submodules then $0 = P_1 \cap \ldots \cap P_n$.*
3. *If $P \leq M$ is prime in M then P is a minimal prime if and only if $P = Ann_M(N)$ for some $N \leq M$.*

Proof

Definition

Let $M \in R - Mod$ and $N \leq M$. N is an *annihilator submodule* if $N = Ann_M(K)$ for some $0 \neq K \leq M$.

By "prime annihilator" we mean an annihilator submodule which is a prime submodule.

Proof

To prove (1), *using that M satisfies ACC on annihilators it is seen that every annihilator submodule contains a finite product of prime annihilators. We have that $Ann_M(M) = 0$. Then there exists finitely many prime annihilators $P_1, ..., P_l$ such that $P_{1M} ..._M P_l = 0$. Hence the minimal primes in M have to be some of $\{P_1, .., P_l\}$. The proof of* (2) *and* (3) *are consequences of previous results.*

Let τ_g be the hereditary torsion theory in $\sigma[M]$ generated by all M-singular modules. If M is non M-singular then $\tau_g = \chi(M)$ where $\chi(M)$ denotes the hereditary torsion theory in $\sigma[M]$ cogenerated by M.

$Spec^{Min}(M)$ denotes the set of minimal primes in M

If τ is an hereditary torsion theory in $\sigma[M]$
$\mathcal{E}_\tau(M)$ is a complete set of representatives of isomorphism classes of indecomposable τ-torsionfree injective modules in $\sigma[M]$.

Theorem

Let M be semiprime. If M satisfies ACC on annihilators and any nonzero submodule of M contains an uniform submodule, then there is a bijective correspondence between $\mathcal{E}_{\tau_g}(M)$ and $Spec^{Min}(M)$.

To give the proof of this theorem we need next definition

Definition

Let $K \in \sigma[M]$. A proper fully invariant submodule N of M is associated to K if there exists a nonzero submodule $L \leq K$ such that $Ann_M(L') = N$ for all nonzero submodule L' of L. We can see that if $N \leq M$ is associated to $K \in \sigma[M]$ then N is prime in M.

Denote by $Ass_M(K)$ the set of prime submodules associated to $K \in \sigma[M]$. If K is an uniform module then $Ass_M(K)$ has at most one element.

Remark

Suppose that C is $\chi(M)$-cocritical, then there exist submodules $C' \leq C$ and $M' \leq M$ such that $C' \cong M'$. Since C is cocritical then it is uniform, so M' does. Hence $Ann_M(M') = P$ is a minimal prime and $Ass_M(M') = \{P\}$. Thus $Ass_M(C) = \{P\}$.

Proof

It could be shown that semiprime and ACC on annihilators implies that M is non M-singular, so $\chi(M) = \tau_g$. If $E \in \mathcal{E}_{\tau_g}(M)$ then E is uniform and hence it is τ_g-cocritical. Hence $Ass_M(E) = \{P\}$ with $P \in Spec^{Min}(M)$

Proof

Then, we define

$$\Psi : \mathcal{E}_{\tau_g}(M) \to Spec^{Min}(M)$$

as $\Psi(E) = P$
Now, in order to define Ψ^{-1}, *let* $P \in Spec^{Min}(M)$ *then* $P = Ann_M(N)$ *for some* $N \leq M$. *By hypothesis there exists* $U \leq N$ *uniform, then* $P = Ann_M(N) \subseteq Ann_M(U)$ *but* $Ann_M(U)$ *is a minimal prime, hence* $P = Ann_M(U)$. *Thus* $\Psi^{-1}(P) = \widehat{U}$.

Notation: $L \in \sigma[M]$. $\widehat{L}$ denote the M-injective hull of L

Theorem

Let M be semiprime. Suppose that M satisfies ACC on annihilators and any nonzero submodule of M contains an uniform submodule. If $P_1, ..., P_n$ are the minimal primes in M, then

$$\widehat{N_1} \oplus ... \oplus \widehat{N_n} = \widehat{M}$$

where $N_i = Ann_M(P_i)$.

Goldie Modules

As examples of modules which satisfy the last theorems we have the semiprime Goldie Modules

Definition

Let M be an R-module. M is a Goldie Module if M satisfies ACC on annihilators and has finite uniform dimension.

We can see that if we put $M = R$ then with this definition R is a left Goldie ring. So all semiprime left Goldie rings satisfy the last theorems. We have the following corollaries

Corollary

Let R be a semiprime left Goldie ring, then there is a bijective correspondence between indecomposable non-singular injective modules, up to isomorphism, and minimal prime ideals of R.

Corollary

Let R be a semiprime left Goldie ring. If $P_1, ..., P_n$ are the minimal primes ideals of R then

$$E(R) = E(N_1) \oplus ... \oplus E(N_n)$$

where $N_i = (0 : P_i)$

Using the decomposition of the M-injective hull of M given above, we have that

Proposition

Let M be semiprime. Suppose that M is a Goldie Module and $P_1, P_2, ..., P_n$ are the minimal prime in M submodules. If $N_i = Ann_M(P_i)$ for $1 \leq i \leq n$, then

$$P_i = M \cap \bigoplus_{j \neq i} \widehat{N_j}$$

for all $1 \leq i \leq n$.

In this context we have a generalization of Goldie's theorem

Theorem

Let $M \in R - Mod$ with finite uniform dimension. The following conditions are equivalent:

1. *M is semiprime and non M-singular*
2. *M is semiprime and satisfies ACC on annihilators*
3. *Let $N \leq M$, then $N \leq_e M$ if and only if there exists a monomorphism $f : M \to N$.*

We have some corollaries which give some examples of Goldie modules

Corollary

Let M be a semiprime R-module. Then, M has finite uniform dimension and enough monoforms if and only if M is a Goldie module.

Corollary

Let $M \in R-Mod$ with finite uniform dimension. If M is a semiprime module and has M-Gabriel dimension, then M is a Goldie module.

Corollary

Let M be semiprime with Krull dimension. Then M is a semiprime Goldie module.

Goldie Modules and their endomorphism rings

Theorem

Let $M \in R - Mod$, $S = End_R(M)$ and $T = End_R(\widehat{M})$. The following conditions are equivalent:

1. *M is a semiprime Goldie module.*
2. *T is semisimple right artinian and is the classical right quotient ring of S.*
3. *S is a semiprime right Goldie ring.*
4. *M is weakly compressible with finite uniform dimension, and for all $N \leq_e M$, $Hom_R(M/N, M) = 0$.*

Duo Modules

Proposition

Suppose that M is a semiprime and non M-singular duo module. The following conditions are equivalent:

1. *M has finite uniform dimension.*
2. *M has finitely many minimal prime submodules.*
3. *M satisfies ACC on annihilators.*
4. *M satisfies ACC on pseudocomplements.*

Theorem

If M is a semiprime duo module, then the following conditions are equivalent:

1. *M is a prime Goldie module.*
2. *$\widehat{M}$ is indecomposable and M is non M-singular.*
3. *M is uniform and non M-singular.*

Continuous modules and Goldie modules

Theorem

Suppose that M is continuous, retractable, non M-singular and satisfies ACC on annihilators. Then M is a semiprime Goldie module.

Corollary

Let R be a continuous and non singular ring. If R satisfies ACC on left annihilators then R is a semiprime left Goldie ring.

Thank you for your attention!