On Internally Cancellable Rings

Meltem ALTUN

Hacettepe University, Ankara, Turkey

Joint work with A. Ç. ÖZCAN

Noncommutative Rings and their Applications
June 2015
Outline

Background

Unit Regular Elements and Internal Cancellation

Internal Cancellation with SSP

Special Clean Elements
Let $R$ be an associative unital ring.
$U(R)$ denotes the group of units of a ring $R$.
$\text{Reg}(R)$ is the set of all regular elements of a ring $R$.
$\text{Idem}(R)$ is the set of all idempotent elements of a ring $R$. 
Let $R$ be an associative unital ring.  
$U(R)$ denotes the group of units of a ring $R$. 
$\text{Reg}(R)$ is the set of all regular elements of a ring $R$. 
$\text{Idem}(R)$ is the set of all idempotent elements of a ring $R$. 

**Definition**

An element $a$ in a ring $R$ is called *regular* if there exists an element $x \in R$ such that $a = axa$. The ring $R$ is called *regular* if every element in $R$ is regular.
Let $R$ be an associative unital ring. 
$\text{U}(R)$ denotes the group of units of a ring $R$. 
$\text{Reg}(R)$ is the set of all regular elements of a ring $R$. 
$\text{Idem}(R)$ is the set of all idempotent elements of a ring $R$. 

**Definition**
An element $a$ in a ring $R$ is called *regular* if there exists an element $x \in R$ such that $a = axa$. The ring $R$ is called *regular* if every element in $R$ is regular.

**Definition**
An element $a$ in a ring $R$ is called *unit-regular* if there exists a unit element $u \in R$ such that $a = aua$. The ring $R$ is called *unit-regular* if every element in $R$ is unit-regular.
Definition [Bass, 1964]

A ring $R$ has stable range one ($sr(R) = 1$) if, for each $a, b \in R$ with $Ra + Rb = R$ there exists $x \in R$ such that $a + xb \in U(R)$. 
Definition [Bass, 1964]
A ring $R$ has stable range one ($sr(R) = 1$) if, for each $a, b \in R$ with $Ra + Rb = R$ there exists $x \in R$ such that $a + xb \in U(R)$.

[Vasershtein, 1971]
Stable range one condition is left-right symmetric.
Background

Definition [Bass, 1964]
A ring $R$ has stable range one ($\text{sr}(R) = 1$) if, for each $a, b \in R$ with $Ra + Rb = R$ there exists $x \in R$ such that $a + xb \in U(R)$.

[Vasershtein, 1971]
Stable range one condition is left-right symmetric.

Unit regular rings have stable range one.
Definition [Bass, 1964]

A ring $R$ has *stable range one* ($sr(R) = 1$) if, for each $a, b \in R$ with $Ra + Rb = R$ there exists $x \in R$ such that $a + xb \in U(R)$.

[Vasershtein, 1971]

Stable range one condition is left-right symmetric.


Unit regular rings have stable range one.

Definition [Khurana-Lam, 2005]

An element $a \in R$ is said to have *stable range one* ($sr(a) = 1$) if, for any $b \in R$ with $Ra + Rb = R$ there exists $x \in R$ such that $a + xb \in U(R)$.
**Definition**

A right $R$-module $M$ is said to be *internally cancellable* (IC, for short) if $M = M_1 \oplus M_2 = N_1 \oplus N_2$ (in the category of $R$-modules) and $M_1 \cong N_1$ together imply that $M_2 \cong N_2$. 

If $R$ is internally cancellable then $R$ is said to be a (right) IC ring.

Theorem [Ehrlich, 1976]

A right $R$-module $M$ is IC if and only if every regular element in the endomorphism ring $\text{End}_R(M)$ of $M$ is unit-regular.

The IC-property of rings is right-left symmetric.
Definition
A right $R$-module $M$ is said to be *internally cancellable* (IC, for short) if $M = M_1 \oplus M_2 = N_1 \oplus N_2$ (in the category of $R$-modules) and $M_1 \cong N_1$ together imply that $M_2 \cong N_2$. If $R_R$ is internally cancellable then $R$ is said to be a (right) IC ring.
Definition
A right $R$-module $M$ is said to be *internally cancellable* (IC, for short) if $M = M_1 \oplus M_2 = N_1 \oplus N_2$ (in the category of $R$-modules) and $M_1 \cong N_1$ together imply that $M_2 \cong N_2$. If $R_R$ is internally cancellable then $R$ is said to be a (right) IC ring.

Theorem [Ehrlich, 1976]
A right $R$-module $M$ is IC if and only if every regular element in the endomorphism ring $\text{End}_R(M)$ of $M$ is unit-regular.
Definition
A right $R$-module $M$ is said to be *internally cancellable* (IC, for short) if $M = M_1 \oplus M_2 = N_1 \oplus N_2$ (in the category of $R$-modules) and $M_1 \cong N_1$ together imply that $M_2 \cong N_2$. If $R_R$ is internally cancellable then $R$ is said to be a (right) IC ring.

Theorem [Ehrlich, 1976]
A right $R$-module $M$ is IC if and only if every regular element in the endomorphism ring $End_R(M)$ of $M$ is unit-regular.

The IC-property of rings is right-left symmetric.
Examples of IC rings

(1) Any unit-regular ring is IC.
Examples of IC rings

(1) Any unit-regular ring is IC.
(2) Any Abelian ring is IC.
Examples of IC rings

(1) Any unit-regular ring is IC.
(2) Any Abelian ring is IC.
(3) Any right artinian ring is IC.
Examples of IC rings

(1) Any unit-regular ring is IC.
(2) Any Abelian ring is IC.
(3) Any right artinian ring is IC.

Theorem
The following statements are equivalent for a ring $R$.

(1) $R$ is an IC ring;
Examples of IC rings

(1) Any unit-regular ring is IC.
(2) Any Abelian ring is IC.
(3) Any right artinian ring is IC.

Theorem
The following statements are equivalent for a ring $R$.

(1) $R$ is an IC ring;
(2) Given idempotents $e, f \in R$, if $eR \cong fR$, then $(1 - e)R \cong (1 - f)R$;
Background

Examples of IC rings

(1) Any unit-regular ring is IC.
(2) Any Abelian ring is IC.
(3) Any right artinian ring is IC.

Theorem

The following statements are equivalent for a ring $R$.

(1) $R$ is an IC ring;
(2) Given idempotents $e, f \in R$, if $eR \cong fR$, then $(1 - e)R \cong (1 - f)R$;
(3) Given idempotents $e, f \in R$, if $eR \cong fR$, then $ueu^{-1} = f$ for some $u \in U(R)$;
Examples of IC rings

(1) Any unit-regular ring is IC.
(2) Any Abelian ring is IC.
(3) Any right artinian ring is IC.

Theorem
The following statements are equivalent for a ring $R$.

(1) $R$ is an IC ring;
(2) Given idempotents $e, f \in R$, if $eR \cong fR$, then $(1 - e)R \cong (1 - f)R$;
(3) Given idempotents $e, f \in R$, if $eR \cong fR$, then $ueu^{-1} = f$ for some $u \in U(R)$;
(4) The left analogues of (2) and (3).
Consider the following statement:

\[(\diamondsuit) : Ra + Rb = R \text{ implies that } a + xb \in \bigcup(R) \text{ for some } x \in R,\]

where the elements \(a, b \in R\) are to be quantified.
Consider the following statement:

\[(\diamond) : Ra + Rb = R \text{ implies that } a + xb \in \mathbb{U}(R) \text{ for some } x \in R,\]

where the elements \(a, b \in R\) are to be quantified.

**Lemma [Song-Chu-Zhu, 2003, Khurana-Lam, 2005]**

The following are equivalent for a ring \(R\).

\[\text{Lemma [Song-Chu-Zhu, 2003, Khurana-Lam, 2005]}\]

\[\text{The following are equivalent for a ring } R.\]
Consider the following statement:

\[(\Diamond) : Ra + Rb = R \implies a + xb \in \mathbb{U}(R) \text{ for some } x \in R,\]

where the elements \(a, b \in R\) are to be quantified.

**Lemma [Song-Chu-Zhu, 2003, Khurana-Lam, 2005]**

The following are equivalent for a ring \(R\).

1. \(R\) is IC;
Consider the following statement:

$$(\diamondsuit) : Ra + Rb = R \text{ implies that } a + xb \in U(R) \text{ for some } x \in R,$$

where the elements $a, b \in R$ are to be quantified.

**Lemma [Song-Chu-Zhu, 2003, Khurana-Lam, 2005]**

The following are equivalent for a ring $R$.

1. $R$ is IC;
2. For each $a \in \text{Reg}(R)$ and $b \in R$, $(\diamondsuit)$ holds;
Consider the following statement:

$$ (\diamondsuit) : Ra + Rb = R \text{ implies that } a + xb \in U(R) \text{ for some } x \in R, $$

where the elements $a, b \in R$ are to be quantified.

**Lemma [Song-Chu-Zhu, 2003, Khurana-Lam, 2005]**

The following are equivalent for a ring $R$.

1. $R$ is IC;
2. For each $a \in \text{Reg}(R)$ and $b \in R$, $(\diamondsuit)$ holds;
3. For each $a, b \in \text{Reg}(R)$, $(\diamondsuit)$ holds;
Consider the following statement:

\[(\diamondsuit) : Ra + Rb = R \implies a + xb \in \mathbb{U}(R) \text{ for some } x \in R,\]

where the elements \(a, b \in R\) are to be quantified.

**Lemma [Song-Chu-Zhu, 2003, Khurana-Lam, 2005]**

The following are equivalent for a ring \(R\).

1. \(R\) is IC;
2. For each \(a \in \text{Reg}(R)\) and \(b \in R\), \((\diamondsuit)\) holds;
3. For each \(a, b \in \text{Reg}(R)\), \((\diamondsuit)\) holds;
4. For each \(a \in \text{Reg}(R)\) and \(b \in \text{Idem}(R)\), \((\diamondsuit)\) holds;
Consider the following statement:

\[(\diamondsuit) : Ra + Rb = R \text{ implies that } a + xb \in U(R) \text{ for some } x \in R,\]

where the elements \(a, b \in R\) are to be quantified.

**Lemma [Song-Chu-Zhu, 2003, Khurana-Lam, 2005]**

The following are equivalent for a ring \(R\).

1. \(R\) is IC;
2. For each \(a \in \text{Reg}(R)\) and \(b \in R\), \((\diamondsuit)\) holds;
3. For each \(a, b \in \text{Reg}(R)\), \((\diamondsuit)\) holds;
4. For each \(a \in \text{Reg}(R)\) and \(b \in \text{Idem}(R)\), \((\diamondsuit)\) holds;
5. For each \(a \in R\) and \(b \in \text{Idem}(R)\), \((\diamondsuit)\) holds;
Consider the following statement:

\[(\diamond) : Ra + Rb = R \text{ implies that } a + xb \in U(R) \text{ for some } x \in R,\]

where the elements \(a, b \in R\) are to be quantified.

**Lemma [Song-Chu-Zhu, 2003, Khurana-Lam, 2005]**

The following are equivalent for a ring \(R\).

1. \(R\) is IC;
2. For each \(a \in \text{Reg}(R)\) and \(b \in R\), \((\diamond)\) holds;
3. For each \(a, b \in \text{Reg}(R)\), \((\diamond)\) holds;
4. For each \(a \in \text{Reg}(R)\) and \(b \in \text{Idem}(R)\), \((\diamond)\) holds;
5. For each \(a \in R\) and \(b \in \text{Idem}(R)\), \((\diamond)\) holds;
6. For each \(a \in R\) and \(b \in \text{Reg}(R)\), \((\diamond)\) holds.
$R$ is unit-regular $\implies sr(R) = 1 \implies rsr(R) = 1 \iff R$ is IC
Consider the following condition:
Consider the following condition:

\((*)\): \(Ra + Rb = R\) implies that \(a + xb \in \mathbb{U}(R)\) and \(aR \cap xR = 0\) for some \(x \in R\),

where the elements \(a, b \in R\) are to be quantified.
Consider the following condition:

\[(\ast) : \quad Ra + Rb = R \implies a + xb \in \mathbb{U}(R) \text{ and } aR \cap xR = 0 \text{ for some } x \in R,\]

where the elements \(a, b \in R\) are to be quantified.

We deal with the nine combinations for the quantifiers “for all”, “for all regular elements” and “for all idempotents elements” for each \(a, b\) and we obtain new characterizations of unit regular rings and IC rings by these combinations.
Consider the following condition:

\((\ast): \quad Ra + Rb = R \implies a + xb \in U(R) \text{ and } aR \cap xR = 0 \text{ for some } x \in R,\)

where the elements \(a, b \in R\) are to be quantified.

We deal with the nine combinations for the quantifiers “for all”, “for all regular elements” and “for all idempotents elements” for each \(a, b\) and we obtain new characterizations of unit regular rings and IC rings by these combinations.

It is observed that \((\ast)\) and \((\lozenge)\) have different behavior.
Theorem [Khurana-Lam, 2005]
If $a$ is a unit regular element in a ring $R$, then $sr(a) = 1$
Theorem [Khurana-Lam, 2005]
If $a$ is a unit regular element in a ring $R$, then $sr(a) = 1$

**Theorem 1.1**
For any element $a$ in a ring $R$, the following are equivalent:
Theorem [Khurana-Lam, 2005]
If $a$ is a unit regular element in a ring $R$, then $sr(a) = 1$

Theorem 1.1
For any element $a$ in a ring $R$, the following are equivalent:

(1) $a$ is unit regular;
Theorem [Khurana-Lam, 2005]
If \( a \) is a unit regular element in a ring \( R \), then \( sr(a) = 1 \)

**Theorem 1.1**
For any element \( a \) in a ring \( R \), the following are equivalent:

1. \( a \) is unit regular;
2. Whenever \( Ra + Rb = R \), there exists \( x \in R \) such that \( a + xb \in U(R) \) and \( aR \cap xR = 0 \).
Corollary 1.2
The following are equivalent for a ring $R$:
Corollary 1.2
The following are equivalent for a ring $R$:

(1) $R$ is unit regular;
Corollary 1.2

The following are equivalent for a ring $R$:

1. $R$ is unit regular;
2. Whenever $Ra + Rb = R$, there exists $e \in \text{Idem}(R)$ such that $a + eb \in U(R)$ and $aR \cap eR = 0$;
Corollary 1.2
The following are equivalent for a ring $R$:

1. $R$ is unit regular;
2. Whenever $Ra + Rb = R$, there exists $e \in \text{Idem}(R)$ such that $a + eb \in U(R)$ and $aR \cap eR = 0$;
3. Whenever $Ra + Rb = R$, there exists $x \in R$ such that $a + xb \in U(R)$ and $aR \cap xR = 0$. 


Corollary 1.2
The following are equivalent for a ring $R$:

1. $R$ is unit regular;
2. Whenever $Ra + Rb = R$, there exists $e \in \text{Idem}(R)$ such that $a + eb \in \mathbb{U}(R)$ and $aR \cap eR = 0$;
3. Whenever $Ra + Rb = R$, there exists $x \in R$ such that $a + xb \in \mathbb{U}(R)$ and $aR \cap xR = 0$;
4. For each $a \in R$ and $b \in \text{Reg}(R)$, if $Ra + Rb = R$, then there exists $x \in R$ such that $a + xb \in \mathbb{U}(R)$ and $aR \cap xR = 0$;
Corollary 1.2

The following are equivalent for a ring $R$:

(1) $R$ is unit regular;

(2) Whenever $Ra + Rb = R$, there exists $e \in \text{Idem}(R)$ such that $a + eb \in \mathbb{U}(R)$ and $aR \cap eR = 0$;

(3) Whenever $Ra + Rb = R$, there exists $x \in R$ such that $a + xb \in \mathbb{U}(R)$ and $aR \cap xR = 0$;

(4) For each $a \in R$ and $b \in \text{Reg}(R)$, if $Ra + Rb = R$, then there exists $x \in R$ such that $a + xb \in \mathbb{U}(R)$ and $aR \cap xR = 0$;

(5) For each $a \in R$ and $b \in \text{Idem}(R)$, if $Ra + Rb = R$, then there exists $x \in R$ such that $a + xb \in \mathbb{U}(R)$ and $aR \cap xR = 0$. 
Next we consider the element $a$ to be regular in $(\ast)$ whenever $b \in R$, $b \in \text{Reg}(R)$ or $b \in \text{Idem}(R)$ and characterize IC rings.
Next we consider the element $a$ to be regular in $(\ast)$ whenever $b \in R$, $b \in \text{Reg}(R)$ or $b \in \text{Idem}(R)$ and characterize IC rings.

**Theorem 1.3**

The following are equivalent for a ring $R$. 
Next we consider the element $a$ to be regular in $(\ast)$ whenever $b \in R$, $b \in \text{Reg}(R)$ or $b \in \text{Idem}(R)$ and characterize IC rings.

**Theorem 1.3**
The following are equivalent for a ring $R$.

1. $R$ is IC;
Next we consider the element $a$ to be regular in $(\ast)$ whenever $b \in R$, $b \in \text{Reg}(R)$ or $b \in \text{Idem}(R)$ and characterize IC rings.

**Theorem 1.3**

The following are equivalent for a ring $R$.

1. $R$ is IC;
2. For each $a \in \text{Reg}(R)$ and $b \in R$, if $Ra + Rb = R$, then there exists $x \in R$ such that $a + xb \in U(R)$ and $aR \cap xR = 0$;
Next we consider the element $a$ to be regular in $(\ast)$ whenever $b \in R$, $b \in \text{Reg}(R)$ or $b \in \text{Idem}(R)$ and characterize IC rings.

**Theorem 1.3**
The following are equivalent for a ring $R$.

1. $R$ is IC;
2. For each $a \in \text{Reg}(R)$ and $b \in R$, if $Ra + Rb = R$, then there exists $x \in R$ such that $a + xb \in U(R)$ and $aR \cap xR = 0$;
3. For each $a \in \text{Reg}(R)$ and $b \in \text{Reg}(R)$, if $Ra + Rb = R$, then there exists $x \in R$ such that $a + xb \in U(R)$ and $aR \cap xR = 0$. 
Next we consider the element $a$ to be regular in $(\ast)$ whenever $b \in R$, $b \in \text{Reg}(R)$ or $b \in \text{Idem}(R)$ and characterize IC rings.

**Theorem 1.3**
The following are equivalent for a ring $R$.

1. $R$ is IC;
2. For each $a \in \text{Reg}(R)$ and $b \in R$, if $Ra + Rb = R$, then there exists $x \in R$ such that $a + xb \in U(R)$ and $aR \cap xR = 0$;
3. For each $a \in \text{Reg}(R)$ and $b \in \text{Reg}(R)$, if $Ra + Rb = R$, then there exists $x \in R$ such that $a + xb \in U(R)$ and $aR \cap xR = 0$;
4. For each $a \in \text{Reg}(R)$ and $b \in \text{Idem}(R)$, if $Ra + Rb = R$, then there exists $x \in R$ such that $a + xb \in U(R)$ and $aR \cap xR = 0$. 
Now we consider the elements $a$ and $b$ to be idempotent in ($\ast$) and see that this situation always holds.
Now we consider the elements $a$ and $b$ to be idempotent in ($\ast$) and see that this situation always holds.

**Theorem 1.4**
For any idempotents $e, f$ in a ring $R$, if $Re + Rf = R$, then there exists a unit regular element $x \in Rf$ such that $e + xf \in U(R)$ and $eR \cap xR = 0$. 

**Remark 1.5**
In other words, Theorem 1.4 says that ($\ast$) always holds for each $a \in \text{Idem}(R)$ and $b \in R$. This condition is also equivalent to the following conditions.

(1) ($\ast$) holds for each $a \in \text{Idem}(R)$ and $b \in R$;

(2) ($\ast$) holds for each $a \in \text{Idem}(R)$ and $b \in \text{Reg}(R)$. 


Now we consider the elements $a$ and $b$ to be idempotent in ($*$) and see that this situation always holds.

**Theorem 1.4**
For any idempotents $e, f$ in a ring $R$, if $Re + Rf = R$, then there exists a unit regular element $x \in Rf$ such that $e + xf \in U(R)$ and $eR \cap xR = 0$.

**Remark 1.5**
In other words, Theorem 1.4 says that ($*$) always holds for each $a \in \text{Idem}(R)$ and $b \in \text{Idem}(R)$. This condition is also equivalent the following conditions.
Now we consider the elements $a$ and $b$ to be idempotent in $(\ast)$ and see that this situation always holds.

**Theorem 1.4**
For any idempotents $e, f$ in a ring $R$, if $Re + Rf = R$, then there exists a unit regular element $x \in Rf$ such that $e + xf \in U(R)$ and $eR \cap xR = 0$.

**Remark 1.5**
In other words, Theorem 1.4 says that $(\ast)$ always holds for each $a \in \text{Idem}(R)$ and $b \in \text{Idem}(R)$. This condition is also equivalent the following conditions.

1. $(\ast)$ holds for each $a \in \text{Idem}(R)$ and $b \in R$;
Now we consider the elements $a$ and $b$ to be idempotent in $(*)$ and see that this situation always holds.

**Theorem 1.4**
For any idempotents $e, f$ in a ring $R$, if $Re + Rf = R$, then there exists a unit regular element $x \in Rf$ such that $e + xf \in U(R)$ and $eR \cap xR = 0$.

**Remark 1.5**
In other words, Theorem 1.4 says that $(*)$ always holds for each $a \in \text{Idem}(R)$ and $b \in \text{Idem}(R)$. This condition is also equivalent to the following conditions.

1. $(*)$ holds for each $a \in \text{Idem}(R)$ and $b \in R$;
2. $(*)$ holds for each $a \in \text{Idem}(R)$ and $b \in \text{Reg}(R)$.
In [Garg, Grover and Khurana, 2014], the authors ask the following question:
In [Garg, Grover and Khurana, 2014], the authors ask the following question:

- *If every regular element of $R$ has idempotent stable range one, then is $R$ perspective?*
In [Garg, Grover and Khurana, 2014], the authors ask the following question:

- If every regular element of $R$ has idempotent stable range one, then is $R$ perspective?

**Definition [Khurana-Lam, 2005]**

An element $a \in R$ is said to have *idempotent stable range one* if, for any $b \in R$ with $Ra + Rb = R$ there exists $e \in Idem(R)$ such that $a + eb \in U(R)$.
A ring $R$ is called *perspective* if any two isomorphic direct summands of $R$ have a common complement, i.e. if $eR \cong fR$ for any $e, f \in \text{Idem}(R)$, then there exists a direct summand $C$ of $R$ such that $R = eR \oplus C = fR \oplus C$. 

Abelian rings and rings with stable range one are perspective rings.

Definition
A ring $R$ is said to have the summand sum property (SSP, for short) if the sum of two direct summands of $R$ is again a direct summand.

Regular rings and abelian rings have the summand sum property.

A ring $R$ is called perspective if any two isomorphic direct summands of $R$ have a common complement, i.e. if $eR \cong fR$ for any $e, f \in \text{Idem}(R)$, then there exists a direct summand $C$ of $R$ such that $R = eR \oplus C = fR \oplus C$.

- Abelian rings and rings with stable range one are perspective rings.
A ring $R$ is called perspective if any two isomorphic direct summands of $R$ have a common complement, i.e. if $eR \cong fR$ for any $e, f \in \text{Idem}(R)$, then there exists a direct summand $C$ of $R$ such that $R = eR \oplus C = fR \oplus C$.

- Abelian rings and rings with stable range one are perspective rings.

Definition
A ring $R$ is said to have the summand sum property (SSP, for short) if the sum of two direct summands of $R$ is again a direct summand.
A ring $R$ is called perspective if any two isomorphic direct summands of $R$ have a common complement, i.e. if $eR \cong fR$ for any $e, f \in \text{Idem}(R)$, then there exists a direct summand $C$ of $R$ such that $R = eR \oplus C = fR \oplus C$.

- Abelian rings and rings with stable range one are perspective rings.

Definition
A ring $R$ is said to have the summand sum property (SSP, for short) if the sum of two direct summands of $R$ is again a direct summand.

- Regular rings and abelian rings have the summand sum property.
Internal Cancellation with SSP

**Theorem 2.1**
If $R$ is an IC ring with SSP, then $R$ is perspective.
Theorem 2.1
If $R$ is an IC ring with SSP, then $R$ is perspective.

In Theorem 2.1, SSP is not superfluous.
Theorem 2.1
If \( R \) is an IC ring with SSP, then \( R \) is perspective.

In Theorem 2.1, SSP is not superfluous.

Example
Let \( R = \mathbb{M}_2(\mathbb{Z}) \).
Theorem 2.1
If $R$ is an IC ring with SSP, then $R$ is perspective.

In Theorem 2.1, SSP is not superfluous.

Example
Let $R = \mathbb{M}_2(\mathbb{Z})$.

$\triangleright$ $R$ is IC because $R \cong \text{End}_\mathbb{Z}(\mathbb{Z} \oplus \mathbb{Z})$ and $\mathbb{Z} \oplus \mathbb{Z}$ is an IC $\mathbb{Z}$-module.
Theorem 2.1
If \( R \) is an IC ring with SSP, then \( R \) is perspective.

In Theorem 2.1, SSP is not superfluous.

Example
Let \( R = \mathbb{M}_2(\mathbb{Z}) \).

- \( R \) is IC because \( R \cong \text{End}_\mathbb{Z}(\mathbb{Z} \oplus \mathbb{Z}) \) and \( \mathbb{Z} \oplus \mathbb{Z} \) is an IC \( \mathbb{Z} \)-module.
- But the element \( \begin{pmatrix} 12 & 5 \\ 0 & 0 \end{pmatrix} \) is unit regular but not clean. [Khurana and Lam, 2004]
Theorem 2.1
If $R$ is an IC ring with SSP, then $R$ is perspective.

In Theorem 2.1, SSP is not superfluous.

Example
Let $R = \mathbb{M}_2(\mathbb{Z})$.

- $R$ is IC because $R \cong \text{End}_\mathbb{Z}(\mathbb{Z} \oplus \mathbb{Z})$ and $\mathbb{Z} \oplus \mathbb{Z}$ is an IC $\mathbb{Z}$-module.
- But the element $\begin{pmatrix} 12 & 5 \\ 0 & 0 \end{pmatrix}$ is unit regular but not clean. [Khurana and Lam, 2004]
- Hence $R$ is not perspective by [Garg, Grover and Khurana, 2014]
Internal Cancellation with SSP

Theorem 2.1
If $R$ is an IC ring with SSP, then $R$ is perspective.

In Theorem 2.1, SSP is not superfluous.

Example
Let $R = \mathbb{M}_2(\mathbb{Z})$.

- $R$ is IC because $R \cong \text{End}_\mathbb{Z}(\mathbb{Z} \oplus \mathbb{Z})$ and $\mathbb{Z} \oplus \mathbb{Z}$ is an IC $\mathbb{Z}$-module.
- But the element $\begin{pmatrix} 12 & 5 \\ 0 & 0 \end{pmatrix}$ is unit regular but not clean. [Khurana and Lam, 2004]
- Hence $R$ is not perspective by [Garg, Grover and Khurana, 2014]
- On the other hand, $\mathbb{Z} \oplus \mathbb{Z}$ has not SSP as a $\mathbb{Z}$-module, hence $R$ has not SSP by [Goodearl, 1991]
By the previous theorem, we have a partial answer to the question.
By the previous theorem, we have a partial answer to the question.

**Theorem 2.2**
Let $R$ be a ring with SSP. Then the following are equivalent.
By the previous theorem, we have a partial answer to the question.

**Theorem 2.2**

Let $R$ be a ring with SSP. Then the following are equivalent.

1. $R$ is perspective;
By the previous theorem, we have a partial answer to the question.

**Theorem 2.2**

Let $R$ be a ring with SSP. Then the following are equivalent.

1. $R$ is perspective;
2. Every regular element of $R$ has idempotent stable range one.
Definition [Abrams-Rangaswamy, 2010]

An element $a$ in $R$ is called *special clean* if there exists a decomposition $a = e + u$ such that $aR \cap eR = 0$ where $e \in \text{Idem}(R)$, $u \in U(R)$. The ring $R$ is called *special clean* if every element of $R$ is special clean.
Definition [Abrams-Rangaswamy, 2010]
An element $a$ in $R$ is called special clean if there exists a decomposition $a = e + u$ such that $aR \cap eR = 0$ where $e \in \text{Idem}(R)$, $u \in U(R)$. The ring $R$ is called special clean if every element of $R$ is special clean.

Proposition 3.1
The following are equivalent for a ring $R$. 
Definition [Abrams-Rangaswamy, 2010]
An element $a$ in $R$ is called special clean if there exists a decomposition $a = e + u$ such that $aR \cap eR = 0$ where $e \in \text{Idem}(R)$, $u \in U(R)$. The ring $R$ is called special clean if every element of $R$ is special clean.

Proposition 3.1
The following are equivalent for a ring $R$.
(1) $R$ is IC;
Definition [Abrams-Rangaswamy, 2010]
An element $a$ in $R$ is called *special clean* if there exists a decomposition $a = e + u$ such that $aR \cap eR = 0$ where $e \in \text{Idem}(R)$, $u \in \text{U}(R)$. The ring $R$ is called *special clean* if every element of $R$ is special clean.

Proposition 3.1
The following are equivalent for a ring $R$.

(1) $R$ is IC;

(2) For every $a \in \text{Reg}(R)$, there exists $u \in \text{U}(R)$ such that $au$ is special clean.
Theorem [Camillo-Khurana, 2001]

$R$ is unit regular if and only if $R$ is a special clean ring.
Theorem [Camillo-Khurana, 2001]

$R$ is unit regular if and only if $R$ is a special clean ring.

- Any special clean element is unit regular.
Theorem [Camillo-Khurana, 2001]

\( R \) is unit regular if and only if \( R \) is a special clean ring.

- Any special clean element is unit regular.

- This gives the following fact for a ring \( R \):
  
  Every regular element is special clean \( \iff \) IC
Theorem [Camillo-Khurana, 2001]

$R$ is unit regular if and only if $R$ is a special clean ring.

- Any special clean element is unit regular.

- This gives the following fact for a ring $R$:
  Every regular element is special clean $\implies$ IC

Lemma 3.2

Any left non-zero divisor regular element over an abelian ring is a unit.
Theorem 3.3
Let $R$ be an abelian ring. Then for every $a \in \text{Reg}(R)$, there exists a unique decomposition $a = e + u$ such that $aR \cap eR = 0$ where $e \in \text{Idem}(R)$, $u \in U(R)$. 

Corollary 3.4 [Akalan-Vaš, 2013]
If $R$ is abelian, then $R$ is unit regular if and only if for every $a \in R$, there exists a unique decomposition $a = e + u$ such that $aR \cap eR = 0$ where $e \in \text{Idem}(R)$, $u \in U(R)$. 

Special Clean Elements
**Theorem 3.3**
Let $R$ be an abelian ring. Then for every $a \in \text{Reg}(R)$, there exists a unique decomposition $a = e + u$ such that $aR \cap eR = 0$ where $e \in \text{Idem}(R), \ u \in \text{U}(R)$.

**Corollary 3.4 [Akalan-Vaš, 2013]**
If $R$ is abelian, then $R$ is unit regular if and only if for every $a \in R$, there exists a unique decomposition $a = e + u$ such that $aR \cap eR = 0$ where $e \in \text{Idem}(R), \ u \in \text{U}(R)$. 

Theorem 3.3
Let $R$ be an abelian ring. Then for every $a \in \text{Reg}(R)$, there exists a unique decomposition $a = e + u$ such that $aR \cap eR = 0$ where $e \in \text{Idem}(R), u \in U(R)$.

Corollary 3.4 [Akalan-Vaš, 2013]
If $R$ is abelian, then $R$ is unit regular if and only if for every $a \in R$, there exists a unique decomposition $a = e + u$ such that $aR \cap eR = 0$ where $e \in \text{Idem}(R), u \in U(R)$. 
References


M. Henriksen, On a class of regular rings that are elementary divisor rings, *Arch. Math.* 24(1973), 133-141.


References


