Rings of Morita Contexts which are Maximal Orders

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- \( V \) is an \( R-S \) bimodule,
- \( W \) is an \( S-R \) bimodule.
- \( \theta : V \otimes_S W \to R \) is an \( R-R \) bilinear map,
Background

A Morita context is a set $M = (R, V, W, S)$ and two maps $\theta$ and $\psi$, where

- $V$ is an $R$-$S$ bimodule,
- $W$ is an $S$-$R$ bimodule.
- $\theta : V \otimes_S W \to R$ is an $R$-$R$ bilinear map,
- $\psi : W \otimes_R V \to S$ is an $S$-$S$ bilinear map.
Furthermore, the maps $\theta$ and $\psi$ satisfy the associativity conditions that are required to make

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a ring. $T$ is called the ring of the Morita context.
Notation

For any $v \in V$ and $w \in W$, 

$\theta(v \otimes w)$ is denoted by $vw$, 

$\psi(w \otimes v)$ by $wv$. 

$\text{Im}(\theta)$ by $VW$, 

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- \( \text{Im}(\theta) \) by \( VW \),
- \( \text{Im}(\psi) \) by \( WV \).
Let $R$ be a prime Goldie ring and $Q(R)$ be its simple Artinian quotient ring.

**Definition:** Let $I$ be an $R$-$R$ bisubmodule of $Q(R)$. $I$ is called fractional $R$-ideal if it satisfies

1. $I$ contains a regular element.
2. There exist regular elements $c_1, c_2 \in R$ such that $c_1 I \subseteq R$ and $I c_2 \subseteq R$. 
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A commutative ring $R$ is a Dedekind domain $\iff F(R) = \{I|I \text{ is a fractional } R\text{-ideal}\}$ is a group under multiplication.
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Background
Let $R$ be a right order in $Q(R)$. $R$ is a maximal right order in $Q(R)$ if there exists a right order $S$ in $Q(R)$ and a regular element $c \in R$ such that either $cS \subseteq R$ or $Sc \subseteq R$ implies that $S = R$. 

\[\iff\]
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**Fact:** $R$ is a maximal order $\iff O_l(I) = R = O_r(I)$ for every fractional $R$-ideal $I$. 
Definition

Let $I$ be a fractional $R$-ideal. $(R : I)_l = \{ q \in Q(R) | qI \subseteq R \}$ and $(R : I)_r = \{ q \in Q(R) | Iq \subseteq R \}$.

If $I$ is reflexive, then $I$ is called a $v$-ideal or reflexive ideal.
Definition

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$I_v = (R : (R : I)_l)_r$ and $vI = (R : (R : I)_r)_l$.
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$I_v = (R : (R : I)_l)_r$ and $vI = (R : (R : I)_r)_l$.

If $I_v = vI$, then $I$ is called a $v$-ideal or reflexive ideal.
If $R$ is a maximal order, then $D(R) = \{v\text{-ideals}\}$ is a group under the multiplication $\circ$, where $I \circ J : = (IJ)_v$. 
Theorem [Marubayashi, Zhang and Yang, 1998]

\[ T = \begin{pmatrix} R & V \\ W & S \end{pmatrix} \]

is a prime Goldie ring \iff

1. \( R \) and \( S \) are prime Goldie rings,
2. \( vW = 0 \Rightarrow v = 0 \) and \( Vw = 0 \Rightarrow w = 0 \),
3. \( VsW = 0 \Rightarrow s = 0 \).
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Then

\[ Q(T) = \begin{pmatrix} Q(R) & Q(V) \\ Q(W) & Q(S) \end{pmatrix} \]

is the quotient ring of \( T \), where

\[ Q(V) = VQ(S) = Q(R)V \text{ and } Q(W) = WQ(R) = Q(S)W. \]
Definition

Let $V_1$ be an $R$-$S$ submodule of $Q(V)$. $V_1$ is called a fractional $R$-$S$ module if and only if:

1. $V_1 Q(S) = Q(V) = Q(R) V_1$
2. There exist regular elements $c \in R$ and $d \in S$ such that $cV_1 \subseteq V$ and $V_1 d \subseteq V$. 
Definition

Let $V_1$ be a fractional $R$-$S$-module. Then

$O_l(V_1) = \{ q \in Q(R) | qV_1 \subseteq V_1 \}$ and

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$R_VS$ is a maximal module in $Q(V)$ if $O_l(V_1) = R$ and $O_r(V_1) = S$

for every fractional $R$-$S$-module $V_1$ of $Q(V)$. 
Definition

Let $V_1$ be a fractional $R$-$S$ submodule of $Q(V)$. 
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$V_{1v} := (S : (S : V_1)_l)_r$ and $vV_1 := (R : (R : V_1)_r)_l$.
Let $V_1$ be a fractional $R$-$S$ submodule of $Q(V)$. 

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$V_{1v} := (S : (S : V_1)_l)_r$ and $v V_1 := (R : (R : V_1)_r)_l$.

If $V_{1v} = v V_1$, then $V_1$ is called a $v$-$(R, S)$-module.
Theorem [Marubayashi, Zhang and Yang, 1998]

TFAE:

1. $T$ is a maximal order in $Q(T)$.
2. (i) $R$ and $S$ are maximal orders in $Q(R)$ and $Q(S)$, respectively;
   (ii) $(R : W)_l = V = (S : W)_r$ and $(R : V)_r = W = (S : V)_l$. 

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1. $T$ is a maximal order in $Q(T)$.

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   (ii) $(R : W)_l = V = (S : W)_r$ and $(R : V)_r = W = (S : V)_l$.

3. (i) $V$ is an $R$-$S$ maximal module in $Q(V)$ and $W$ is an $S$-$R$ maximal module in $Q(W)$;
   (ii) $(VW)_v = R =_v (VW)$ and $(WV)_v = S =_v (WV)$;
   (iii) $V_v = V =_v V$ and $W_v = W =_v W$. 
Suppose that $T$ is a maximal order in $Q(T)$. Then there exists a 1-1 correspondence between $D(V)$ and $D(R)$ given by:

$$V_1 \rightarrow (V_1 W)_v \text{ and } I \rightarrow (IV)_v$$
Theorem

Suppose that \( T \) is a maximal order in \( Q(T) \). Then there exists a group isomorphism between \( D(R) \) and \( D(T) \) given by

\[
I \leftrightarrow \begin{pmatrix} I & (IV)_v \\ (WI)_v & (WIV)_v \end{pmatrix}
\]
Applications

Asano order: A prime Goldie ring in which each non-zero ideal is invertible.
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Lemma: Suppose that $R$ and $S$ are Asano orders in $Q(R)$ and $Q(S)$, respectively.
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Lemma: Suppose that $R$ and $S$ are Asano orders in $Q(R)$ and $Q(S)$, respectively. Then

1. For each fractional $(R, S)$–module $V'$ in $Q(V)$ we have $(V')^{-1}V' = S$ and $V'(V')^{-1} = R$. 
Applications

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Lemma: Suppose that $R$ and $S$ are Asano orders in $Q(R)$ and $Q(S)$, respectively. Then

1. For each fractional $(R, S)$–module $V'$ in $Q(V)$ we have $(V')^{-1}V' = S$ and $V'(V')^{-1} = R$.

2. For each fractional $(S, R)$–module in $Q(W)$ we have $(W')^{-1}W' = R$ and $W'(W')^{-1} = S$. 
Definition

$V$ is an $(R, S)$–Asano module in $Q(V)$ if for each integral $(R, S)$–module $V'$, $(V')^{-1}V' = S$ and $V'(V')^{-1} = R$. Similarly we can define an $(S, R)$–Asano module $W$ in $Q(W)$. It follows from Lemma that if $R$ and $S$ are Asano orders in $Q(R)$ and $Q(S)$, respectively, then $V$ is an $(R, S)$–Asano module in $Q(V)$ and $W$ is an $(S, R)$–Asano module in $Q(W)$. Moreover, it is easy to see that if $V$ is an $(R, S)$–Asano module in $Q(V)$, then it is an $(R, S)$–maximal module in $Q(V)$. An analogous result can be given for $W$. 
Definition

$V$ is an $(R, S)$–Asano module in $Q(V)$ if for each integral $(R, S)$–module $V'$, $(V')^{-1}V' = S$ and $V'(V')^{-1} = R$. Similarly we can define an $(S, R)$–Asano module $W$ in $Q(W)$. Moreover, it is easy to see that $V$ is an $(R, S)$–Asano module in $Q(V)$, then it is an $(R, S)$–maximal module in $Q(V)$.
Definition

\( V \) is an \((R, S)\)–Asano module in \( Q(V) \) if for each integral \((R, S)\)–module \( V' \), \((V')^{-1}V' = S\) and \( V'(V')^{-1} = R\).

Similarly we can define an \((S, R)\)–Asano module \( W \) in \( Q(W) \).

- It follows from Lemma that if \( R \) and \( S \) are Asano orders in \( Q(R) \) and \( Q(S) \), respectively, then \( V \) is an \((R, S)\)–Asano module in \( Q(V) \) and \( W \) is an \((S, R)\)–Asano module in \( Q(W) \).
- Moreover, it is easy to see that \( V \) is an \((R, S)\)–Asano module in \( Q(V) \), then it is an \((R, S)\)–maximal module in \( Q(V) \).
- An analogous result can be given for \( W \).
Suppose that $VW = R$ and $WV = S$. 
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$I \mapsto IV = V'$ and $V' \mapsto V'W$,

where $I$ is a fractional $R$–ideal and $V'$ is a fractional $(R, S)$–module in $Q(V)$. 
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3. (i) $V$ is an $(R, S)$–Asano module in $Q(V)$ and $W$ is an 
   $(S, R)$–Asano module in $Q(W)$, and 
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Dedekind order: A prime Goldie ring which is a maximal order and hereditary.
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Definition: $V$ is called an $(R, S)$–Dedekind module in $Q(V)$ if

- $V$ is an $(R, S)$–maximal module in $Q(V)$, and
- every left $R$–submodule of $V$ is projective and every right $S$–submodule of $V$ is projective.
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Definition: $V$ is called an $(R, S)$–Dedekind module in $Q(V)$ if

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Similarly we can define an $(S, R)$–Dedekind module $W$ in $Q(W)$. 
Theorem

The following three conditions are equivalent:

1. $T$ is a Dedekind order in $Q(T)$. 

2. (i) $R$ is a Dedekind order in $Q(R)$ and $S$ is a Dedekind order in $Q(S)$, respectively, and (ii) $VW = R$ and $WV = S$.

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3. (i) $V$ is an $(R, S)$–Dedekind module in $Q(V)$ and $W$ is an $(S, R)$–Dedekind module in $Q(W)$, and
   (ii) $VW = R$ and $WV = S$. 
Definition [Akalan, 2008]

A prime Goldie ring $R$ is called a Generalized Dedekind prime ($G$-Dedekind, for short) ring if

- $R$ is a maximal order and
- Every $
u$-ideal is invertible.
Conjecture

$T$ is a $G$-Dedekind prime ring $\iff$

1. $R$ and $S$ are $G$-Dedekind prime rings,
2. $(R : W)_l = V = (S : W)_r$ and $(R : V)_r = W = (S : V)_l$. 
Acknowledgement

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