Mappings between $\mathbb{R}$-tors and other lattices.

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Noncommutative rings and their applications, IV  
Lens, France

8-11 June 2015
Notation:

- $\mathbb{R}$ will denote an associative ring with identity.
- $\mathbb{R}$-Mod will denote the category of unital left $\mathbb{R}$-modules.
- $S(M)$, the complete lattice of submodules of $M$.
- $S_{fi}(M)$, the lattice of fully invariant submodules of $M$.
- $\mathbb{R}$-simp a complete set of representatives of isomorphism classes of simple modules.
A \textit{preradical} $r$ for $R$-$\text{Mod}$ is a subfunctor of the identity functor for $R$-$\text{Mod}$, that is, for every module homomorphism $f : M \to N$ it happens that $f (r (M)) \subseteq r (N)$.

\[ \text{R-pr} : \]

The class of all preradicals in $R$-$\text{Mod}$.

For each $M \in R$-$\text{Mod}$ and $r, s \in \text{R-pr}$,

- **Order**: $r \preceq s$ if $r (M) \leq s (M)$ for each $M \in R$-$\text{Mod}$.
- **Meet**: $(r \wedge s) (M) = r (M) \cap s (M)$,
- **Join**: $(r \vee s) (M) = r (M) + s (M)$.
- **Product**: $(r \cdot s) (M) = r (s (M))$.
- **Coproduct**: $(r : s) (M)$ is such that $(r : s) (M) / r (M) = s (M / r (M))$. 
The lattice structure of $R\text{-pr}$

$R\text{-pr}$ with the partial ordering $\preceq$, is a complete, atomic, coatomic, modular, upper continuous and strongly pseudocomplemented big lattice.

The least element is the zero functor denoted by $0$ and the identity functor $1$ is the greatest element.

- The class of atoms of $R\text{-pr}$ is $\{\alpha^E_S \mid S \in R\text{-simp}\}$.
- The class of coatoms of $R\text{-pr}$ is $\{\omega^R_I \mid I \text{ is a maximal two sided ideal of } R\}$. 
For $r \in R\text{-pr}$, we say that

- $r$ is idempotent if $r \cdot r = r$.
- $r$ is a radical if $r : r = r$.
- $r$ is a left exact preradical if $r(N) = N \cap r(M)$ for every $N \leq M$ and $M \in R\text{-Mod}$.
- $r$ is a $t$–radical when $r(M) = r(R)M$ for every $M \in R\text{-Mod}$. 
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For each $r \in \mathbb{R}$-pr, we denote

- $\mathbb{T}_r = \{ M \mid r(M) = M \}$, the pretorsion class associated to $r$,
- $\mathbb{F}_r := \{ M \mid r(M) = 0 \}$, the pretorsion free class associated to $r$. 
Alpha and omega preradicals.

- For $N \in S_{fi}(M)$, there are two distinguished preradicals, $\alpha^M_N$ and $\omega^M_N$, assigning $M$ to $N$, which are defined as follows:

$$\alpha^M_N(L) := \sum \{ f(N) \mid f \in \text{Hom}_R(M, L) \}$$

$$\omega^M_N(L) := \cap \{ f^{-1}(N) \mid f \in \text{Hom}_R(L, M) \},$$

for each $L \in R\text{-Mod}$.

- If $N$ is a fully invariant submodule of $M$, we have that the class $\{ r \in R\text{-pr} \mid r(M) = N \}$ is precisely the interval $[\alpha^M_N, \omega^M_N]$. 
A *torsion theory* for $R$-Mod is an ordered pair $(\mathbb{T}, \mathbb{F})$ of classes of modules such that:

(i) $\text{Hom}(T, F) = 0$ for every $T \in \mathbb{T}$ and for every $F \in \mathbb{F}$.

(ii) If $\text{Hom}_R(C, F) = 0$ for all $F \in \mathbb{F}$, then $C \in \mathbb{T}$.

(iii) If $\text{Hom}_R(T, C) = 0$ for all $T \in \mathbb{T}$, then $C \in \mathbb{F}$. 
Notice that

- $\mathbb{T}$ is a torsion class (i.e. a class closed under taking quotients, direct sums and extensions).
- $\mathbb{F}$ is a torsion-free class (i.e. a class closed under taking submodules, direct products and extensions).
- $(\mathbb{T}, \mathbb{F})$ is a hereditary torsion theory if and only if $\mathbb{T}$ is closed under submodules. If in addition $\mathbb{F}$ is closed under quotients, $(\mathbb{T}, \mathbb{F})$ is a cohereditary and hereditary torsion theory.
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<th><strong>R – TORS</strong></th>
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<td>The big lattice of all torsion theories in $R$-$\text{Mod}$.</td>
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<td>All hereditary torsion theories which are cohereditary</td>
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For $\tau \in R$-tors, we shall write $T_\tau$ for the $\tau$-torsion class and $F_\tau$ for the $\tau$-torsion free class, so that $\tau = (T_\tau, F_\tau)$.

For a class $\mathcal{A}$ of modules:
- $\xi(\mathcal{A})$ is the hereditary torsion theory generated by $\mathcal{A}$.
- $\chi(\mathcal{A})$ is the hereditary torsion theory cogenerated by $\mathcal{A}$.

We shall write $\xi$ (respectively, $\chi$) for the least (resp., greatest) element of $R$-tors.

For $\tau \in R$-tors, let $\tau^\perp$ stand for the pseudocomplement in $R$-tors of $\tau$. 
Mappings between $R$-tors and $R$-pr.
Well Known Facts:

- There exist lattice isomorphisms

\[ \varphi : \mathbb{R}\text{-rid} \rightarrow \mathbb{R}\text{-TORS}, \]

\[ \zeta : \mathbb{R}\text{-ler} \rightarrow \mathbb{R}\text{-tors} \]

where both are given by \( r \mapsto (\mathbb{T}_r, \mathbb{F}_r) \).

- In both cases, the inverse is \( (\mathbb{T}, \mathbb{F}) \mapsto t_{(\mathbb{T}, \mathbb{F})} \), where

\[ t_{(\mathbb{T}, \mathbb{F})} = \bigvee \{ \alpha_T \mid T \in \mathbb{T} \}. \]

(Note that \( t_{(\mathbb{T}, \mathbb{F})} \), the so-called torsion part, coincides with \( \bigwedge \{ \omega^F_0 \mid F \in \mathbb{F} \} \).)
There exists a canonical isomorphism

\[ \eta : S_{fi}(R) \to R\text{-}\text{trad}, \]

\[ \eta(I) := \alpha^R_I \text{ (which is, observe, left multiplication by } I). \] Its inverse sends \( r \mapsto r(R). \)
Define a mapping

\[ t : \text{R-tors} \longrightarrow \text{R-pr}, \]
\[ \tau \mapsto t_\tau. \]

**Remark**

1. The mapping \( t \) is always injective, order-preserving and preserves infima. This also holds in the arbitrary case.

2. In general, \( t \) **does not preserve suprema**. Indeed, while it is true, for \( \tau, \sigma \in \text{R-tors} \), that \( t_\tau \lor t_\sigma \leq t_{\tau \lor \sigma} \), equality does not always hold.
**Example**

Let $R$ be the subring of $\mathbb{Z}_2^\mathbb{N}$ spanned by 1 and $\mathbb{Z}_2^{(\mathbb{N})}$, so that $R$ consists of sequences of zero and ones eventually constant. Denote as $Z$ the left exact preradical sending each module to its singular submodule. It can be proved that:

1. Every simple ideal is a direct summand of $R$, and therefore $\text{soc}_p(R) = \text{soc}(R) = \mathbb{Z}_2^{(\mathbb{N})}$.

2. $\mathbb{Z}_2^{(\mathbb{N})} \leq e R$.

3. $T_{\tau_G} = F_{\tau_{SP}}$, and $\tau_G \in R$-jans.

4. $\tau_{SP}^\perp = \tau_G$.

5. $\tau_{SP} \vee \tau_G = \tau_{SP} \vee \tau_{SP}^\perp = \xi(SP) \vee \chi(SP) = \chi$.

6. Thus, $t_{\tau_{SP} \vee \tau_G}(R) = R$, but $t_{\tau_{SP}}(R) + t_{\tau_G}(R) = \text{soc}_p(R) = \mathbb{Z}_2^{(\mathbb{N})} \neq R$. 
Lemma

Let \( \{\tau_i\}_{i \in I} \subseteq R\text{-qtors} \). Then \( t \bigvee_{i \in I} \tau_i = \bigvee_{i \in I} t \tau_i \), taking the supremum on the left in \( R\text{-tors} \) and the one on the right in \( R\text{-pr} \).

Thus, over any left perfect ring, \( R\text{-qtors} \) is a complete sublattice both of \( R\text{-tors} \) and (via a canonical embedding) of \( R\text{-pr} \).
**Theorem**

Let $R$ be a left perfect ring. Then $R$-qtors is a complete sublattice of $R$-tors and $t|_{R$-qtors} : R$-qtors $\rightarrow$ R-pr is a complete lattice embedding.
Mappings between $R$-tors and $S_f(R)$
Define a mapping, *evaluation* (at the ring),

\[ e : \text{R-tors} \longrightarrow S_{fi}(R), \]

\[ \tau \mapsto t_{\tau}(R). \]

**Remark**

Notice that \( e \) preserves orderings and arbitrary infima. In general, \( e \) does not preserve even binary suprema.
A module $M \in \mathbb{R} \text{-Mod}$ is called a *Kasch module* if and only if every $S \in \mathbb{R} \text{-simp}$ is embeddable in $M$.

$\mathbb{R}$ is a left Kasch ring if $E(\mathbb{R})$ is an injective cogenerator for $\mathbb{R} \text{-Mod}$.

**Proposition**

*If the mapping $e$ is injective, then $\mathbb{R}$ is a left Kasch ring.*
Recall that a ring \( R \) is said to be *left fully idempotent* (or *left weakly regular*) when every left ideal is idempotent.

**Proposition**

*If \( R \) is left fully idempotent and \( e \) is injective, then \( R \) is a left Bronowitz-Teply ring* (i.e. \( R\text{-qtors} = R\text{-tors} \)).

**Lemma**

*If \( R\text{-trad} = R\text{-ler} \), then \( R \) is a semisimple ring.*

**Proposition**

*If \( R \) is regular (in the sense of von Neumann) and \( e \) is an isomorphism, then \( R \) is semisimple.*
For an arbitrary ring $R$, abbreviate as $\mathcal{E}(R)$ the statement “$e : R\text{-tors} \rightarrow S_{fi}(R)$ is a lattice isomorphism”.

**Theorem**

Let $R$ be a ring such that $\mathcal{E}(R)$ and $RSP$ is finite. Then $R = R_s \times R_e$ for some (possibly trivial) semisimple ring $R_s$ and some ring $R_e$ such that $\mathcal{E}(R_e)$ and $soc_p(R_e) = 0$. 
Let us now consider the inverse of \( e \), when \( e \) is a lattice isomorphism. Set

\[
\alpha, \beta : S_{fi}(R) \rightarrow \text{R-tors}
\]

\[
\alpha(I) := \xi(I) = \bigwedge \{ \tau \in \text{R-tors} \mid I \in \mathbb{T}_\tau \}
\]

and

\[
\beta(I) := \chi(R/I) = \bigvee \{ \tau \in \text{R-tors} \mid R/I \in \mathbb{F}_\tau \},
\]

for each \( I \in S_{fi}(R) \).

**Theorem**

*If any one of \( \alpha, \beta \) or \( e \) is a lattice isomorphism, then all three are isomorphisms, \( \alpha = \beta \) and its inverse is \( e \).*
Mappings between $S_{fi}(M)$ and R-ler
Now, we continue the previous study for any $M \in R\text{-Mod}$ and $S_{fi}(M)$.

- For $L, K \in S_{fi}(M)$, consider the product

$$K_M L := \alpha^M_K(L)$$

- We say that $M$ is \textit{fully idempotent} if and only if $N_M N = N$ for every $N \in S_{fi}(M)$.

- Notice that $R$ is fully idempotent if and only if every two-sided ideal is idempotent.
Set,

\[ \lambda_M : S_{fi}(M) \to R-pr \]

\[ \lambda_M(N) = \alpha^M_N \]

and

\[ e_M : R-tors \to S_{fi}(M) \]

\[ e_M(\tau) := t_\tau(M) \]

It is clear that \( e_R = e \).

Taking the isomorphism \( \zeta : R-ler \to R-tors \), notice that \( e_M \circ \zeta : r \mapsto r(M) \).
Proposition

Let $M \in R\text{-Mod}$ be such that the assignation $\lambda_M : S_{fi}(M) \to R\text{-ler}$ given by $\lambda_M(N) = \alpha^M_N$ is well-defined and a lattice isomorphism. Then, the following conditions hold.

(a) $S_{fi}(M)$ is an atomic frame.
(b) $M$ is fully idempotent.
(c) $e_M \circ \zeta$ is the inverse of $\lambda_M$.
(d) $M$ is a generator for $R\text{-Mod}$.
(e) $M$ is a Kasch module, $t_{\xi(S)}(M) = \text{soc}_S(M)$ for every $S \in R\text{-simp}$, $\text{soc}(M)$ is the least essential element of $S_{fi}(M)$, and $e_M(\tau_D) = \text{soc}(M)$.
(f) For every $\tau \in R\text{-tors}$, $t_\tau = \alpha^M_{t_\tau(M)}$.
(g) For every $I \in S_{fi}(R)$, $\alpha^R_I = \alpha^M_{IM}$.
(h) If $M$ is projective, then, for every $\tau \in R\text{-tors}$, $t_\tau = \alpha^R_{t_\tau(R)}$. 


Mappings between $S_{fi}(M)$ and $R$-ler

**Theorem**

For a ring $R$, the following statements are equivalent.

(a) $R$ is semisimple.

(b) $R$-trad = $R$-ler.

(c) For every projective generator $P$, $\lambda_P : S_{fi}(P) \rightarrow R$-ler is well-defined and a lattice isomorphism.

(d) $\lambda_R : S_{fi}(R) \rightarrow R$-ler is well-defined and a lattice isomorphism.

(e) There is some projective module $P$ such that $\lambda_P : S_{fi}(P) \rightarrow R$-ler is well-defined and a lattice isomorphism.
Thank you!
References


