On radicals of differential polynomial rings

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(Based on joint works with Agata Smoktunowicz and Pace P. Nielsen)

Lens
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2. Radicals of polynomial rings

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- R - **with bounded index of nilpotence** - there is n such that $a^n = 0$ for any $a \in R$

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- (Puczylowski, Smoktunowicz) If R is nil then $R[x]$ is Brown-McCoy radical

References

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- Let R be a ring. A homomorphism of the additive group of R, $\delta : R \to R$, which satisfies $\delta(ab) = \delta(a)b + a\delta(b)$ for any $a, b \in R$, is called a **derivation** on R.

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- Let R be a ring and let δ be a derivation on R. Consider the set $R[x;\delta]$ of all polynomials $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$. With natural addition and multiplications defined using the rule

$$xa = ax + \delta(a)$$

  $R[x;\delta]$ has a structure of a ring and is called *the differential polynomial ring (Ore extension)*.

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- (Nielsen, Z.): If R is a ring with bounded index of nilpotence, then $R[x,\delta]$ is locally nilpotent
- (Nielsen, Z.): There exists a commutative ring R with bounded index of nilpotence and a derivation δ of R such that $R[x;\delta]$ is not prime radical

- **Question:** Let R be a prime radical ring with a derivation δ. Is then $R[x; \delta]$ Jacobson radical?

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- Is I always nil?
- YES! If R is commutative (Ferrero et al.), right Noetherian (D. Jordan), ...
- Answer is "NO" in general!
- (Smoktunowicz, 2015) There exists a ring R and a derivation δ of R such that $I = J(R[x;\delta]) \cap R$ is not nil.

THANK YOU FOR YOUR ATTENTION.