Commutators in Division Algebras

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A Question!

Consider a special property P in a ring:

- Commutativity
- Algebraicity

Whether one can specify a set or a substructure S, such that the property P for S implies the property P for the whole ring.
\textbf{Set of Generators}

\begin{itemize}
  \item Question 1:
  \begin{align*}
    \text{Let } A \text{ be an algebraic structure generated by set } S. \text{ Whether property } P \text{ for } S \text{ implies property } P \text{ for } A?
  \end{align*}
  \item Question 2:
  \begin{align*}
    \text{Let } A \text{ be an algebraic structure generated by set } S. \text{ Whether property } P \text{ for } S \text{ implies property } Q \text{ for } A?
  \end{align*}
\end{itemize}
Candidates for $S$

- **General Division Algebras**
  - Multiplicative and additive commutators
  - Subgroups $D'$ and $[D, D]$.
  - Normal subgroups of $D^*$.

- **Division Algebras with Involutions**
  - Symmetric elements
  - Skew-symmetric elements
  - Unitary elements.
Some Basic Definitions

Let \( D \) be a division algebra over its center \( F \).

Definition

- Denote by \( D' \) the multiplicative subgroup of \( D^* \) generated by the all multiplicative commutators of \( D \).

Definition

- Denote by \([D,D]\) the additive subgroup of commutators generated by the all additive commutators of \( D \).

Definition

- Denote by \( T(D) \) the vector space generated by the all multiplicative commutators over \( F \).
Some Definitions

Let $D$ be division ring with center $F$.

**Definition**
- We say $A$ is **radical** over $B$ if for every element $a \in A$ there exists integer $n = n(a)$ such that $a^n \in B$.

**Definition**
- Element $a \in A$ is called **periodic** if there exists integer $n$ such that $a^n = 1$.

**Definition**
- Element $a \in A$ is called **algebraic** of degree $n$ over center if satisfies a polynomial $f(x) \in F[x]$ of degree $n$. 
Commutators in Division Rings

and

Their Generating Role
**Commutators as Generators**

"first course in non-commutative rings" due to T.Y. LAM

**Theorem (Corollary 13.19, p. 211)**

Let $D$ be a non-commutative division ring with center $F$. Then $D$ is generated as an $F$-algebra by all **additive commutators** of $D$.

**Theorem (Corollary 13.9, p. 207)**

A non-commutative division ring $D$ is generated as a **division ring** by all of its **multiplicative commutators**.

**Conjecture (M.A, Akbari-Arianejad-Madadi)**

A division ring $D$ with center $F$ is generated as a **vector space** over $F$ by all of its **multiplicative commutators**.
Theorem (M.A., Akbari-Ariannejad-Madadi)

If $D$ is algebraic with characteristic zero, then $T(D) = D$. 
Let $K/F$ be a field extension with $\text{dim}_F K = n$. For $a \in K$, define

$$L_a : K \to K,$$

where $L_a(b) = ab$.

**Definition**

The Trace function is defined for all $a \in K$ by

$$T_{K/F}(a) = \text{Tr}(L_a).$$
Theorem

Let $K/F$ be a field extension with $\dim_F K = n$ and

$$f(x) = x^m + b_{m-1}x^{m-1} + \cdots + b_1x + b_0$$

be the minimal polynomial of $a \in K$. Then

$$T_{K/F}(a) = -\frac{n}{m}b_{m-1}.$$
Wedderburn’s Theorem

Theorem (Wedderburn)

Let $D$ be a division ring with center $F$.

$a \in D^\ast$ be algebraic with minimal polynomial $f(x) \in F[x]$ of degree $n$.

Then

$$f(x) = (x - a_1) \ldots (x - a_n) \in D[x].$$

Remark

Note that linear factors are not unique!
Trace Formula

Let \( a \in D^* \) be algebraic with minimal polynomial

\[
f(x) = (x - a) \ldots (x - a_{n-1}) \in D[x].
\]

Then

\[
T_{F(a)/F(a)} = a + a_1 + \ldots + a_{n-1} = a + d_1 a d_1^{-1} + \ldots d_{n-1} a d_{n-1}^{-1}
\]

\[
= a(1 + a^{-1} d_1 a d_1^{-1} + a^{-1} d_2 a d_2^{-1} + \ldots + a^{-1} d_{n-1} a d_{n-1}^{-1})
\]

\[
= ad,
\]

where \( d \in F(a) \cap T(D) \).

Theorem (M.A., Akbari-Arianejad-Madadi)

Let \( a \in D \) be algebraic and \( T_{F(a)/F(a)} \neq 0 \), then \( a^{-1} \in T(D) \).
\textbf{T(D) as Lie Ideal}

\textbf{Theorem (M. Aaghabali)}

- Let $D$ be an \textbf{algebraic} non-commutative division ring with center $F$. Then $T(D)$ is a \textbf{non-central Lie ideal} of $D$.

\textbf{Theorem (M. Aaghabali)}

- Let $D$ be a \textbf{centrally finite} division ring over $F$. Then $T(D) = D$. 

Theorem (M.A., Akbari-Ariannejad-Madadi)

Let $D$ be a division ring with center $F$. If $\dim_F T(D) = n < \infty$, then $\dim_F D < \infty$. 
Commutators in Division Rings

Commutators in Division Rings

and

Commutativity Conditions
Two Important Commutativity Conditions

Theorem (Wedderburn’s Little Theorem)

- Every finite division ring is commutative.

Theorem (Kaplansky)

- If $D$ is a division ring radical over its center, then $D$ is commutative.
Finiteness Conditions

Theorem (Herstein-Procesi-Schacher)

If $D$ is a division ring with center $F$ whose all additive commutators are radical over the center, then

$$\dim_F D \leq 4$$
Conjecture (Herstein)

- Every division ring whose all multiplicative commutators are radical over its center must be commutative.

- **General case is still open!**

- **Herstein (1978):** Statement holds when commutators are periodic.

- **Herstein (1978):** Statement holds for centrally finite division rings.

\[ \dim_{F} D = n^2 \leq \infty. \]
Commutativity Conditions

- **Herstein (1980):** Statement holds for division rings with uncountable centers.

- **Putcha-Yaqub (1974):** The conjecture is true if the radical degree is a power of 2.

- **Mahdavi-Akbari (1996):** The conjecture is true if the radical degree is a power of 6.
Herstein Conjecture (Special Case)

**Theorem (Mahdavi (1995))**

Let $D$ be an algebraic division algebra over its center $F$. If $D'$ is radical over the center, then $D$ is commutative.

**Theorem (Mahdavi (1995))**

Let $D$ be a division algebra over its center $F$. If $D'$ is radical over the center, then $D$ is commutative.
Theorem (M.A., Akbari-Ariannejad-Madadi)

Let $D$ be a division algebra over its center $F$. If $T(D)$ is radical over the center, then $D$ is commutative.
Jacobson Theorem

Theorem (Jacobson)

- Every division algebra algebraic over a finite field is commutative.

Theorem (Mahdavi (1996))

- Every division algebra whose multiplicative group of commutators is algebraic over a finite field is commutative.
Noether-Jacobson Theorem

Theorem (Noether-Jacobson)
- Every non-commutative algebraic division ring over its center contains a non-central separable element.

Theorem (Mahdavi (1995))
- Every non-commutative algebraic division ring over its center contains a non-central separable element in its multiplicative subgroup of commutators.
Commutators in Division Rings

and

Algebraicity Conditions
Let $D$ be a division ring:

- **Multiplicative Commutators**
- **Additive Commutators**
- **Subgroups** $D'$ and $[D, D]$

Whether one can deduce algebraicity of $D$ over center if mentioned sets and structures are algebraic over the center.
Algebraicity of $D'$ and $[D, D]$

**Theorem (Akbari-Mahdavi (1996))**

- Let $D'$ be algebraic over the center, then $D$ is algebraic over the center.

**Theorem (Akbari-Ariannejad-Mehraabaadi (1998))**

- Let $[D, D]$ be algebraic over the center, then $D$ is algebraic over the center, provided $\text{char}(D) = 0$. 
Algebraic commutators

**Theorem (M.A., Akbari-Ariannejad-Madadi)**

- Let \( D \) be a division algebra over its center \( F \). If all multiplicative commutators are algebraic over \( F \), then \( D \) is algebraic provided that \( F \) is UNCOUNTABLE.

- Assume \( a \in D \setminus F \) and consider \( y \in D^* \) arbitrarily.
- Either \( y \in C_D(a) \) or \( y \notin C_D(a) \).
- \( y \notin C_D(a) \), for every \( r \in F \) we have:

\[ 0 \neq b = ay - ya = a(y + r) - (y + r)a = (a(y + r)a^{-1}(y + r)^{-1} - 1)(y + r)a \]

- For every \( r \in F \), \((y + r)ab^{-1}\) is algebraic over \( F \).

\[ f(t) \in F[t]; \quad f((y + r)ab^{-1}) = 0 \]

- Put \( c = ab^{-1} \), then

\[ ((y + r)c)^n + \alpha_1((y + r)c)^{n-1} + \cdots + \alpha_{n-1}((y + r)c)^1 + \alpha_n = 0 \]
Algebraic commutators

- \((y + r)(c((y + r)c)^{n-1} + \alpha_1 c((y + r)c)^{n-2}) + \cdots + \alpha_{n-1} c) = -\alpha_n\)

- \(-\alpha_n(y + r)^{-1} = c((y + r)c)^{n-1} + \alpha_1 c((y + r)c)^{n-2}) + \cdots + \alpha_{n-1} c\)

- Assume the set of all words of finite length consisting of two letters \(y, c\).
- Consider vector space generated by the set of all such words.
- Clearly, for every \(r \in F\), we have \((y + r)^{-1} \in W\).
- \(\dim_F W\) is countable but \(F\) is uncountable.
Algebraic commutators

- hence we could find that \( (y + r_1)^{-1}, \ldots, (y + r_m)^{-1} \) are linearly dependent over \( F \).

**Theorem**

Let \( D \) be a division algebra and \( K \) be a subfield of \( D \). For \( a \in D \), if \( \text{dim}_K K[a] \geq n \), then for any distinct elements \( \alpha_1, \ldots, \alpha_n \in Z(D) \), \( (a - \alpha_1)^{-1}, \ldots, (a - \alpha_n)^{-1} \) are linearly independent.

- thus \( y \) is algebraic.
- now, assume \( y \in C_D(a) \) and \( z \notin C_D(a) \).
- \( (y + r)z \notin C_D(a) \), for every \( r \in F \) is algebraic.
- repeating argument for \( (y + r)z \) we find that \( y \) is algebraic over \( F \).
Algebraic commutators

Lemma (M.A., Akbari-Ariannejad-Madadi)

- Let $D$ be a division ring with center $F$, $T(D)$ be Algebraic over $F$ and $Char(D) = 0$. Then for any two Algebraic elements $a, b \in D$, the set $S = \{a + b, aba, a^2b\}$ is Algebraic over $F$.

Theorem (M.A., Akbari-Ariannejad-Madadi)

- Let $D$ be a division ring with center $F$ and $Char(D) = 0$. Then $T(D)$ is Algebraic over $F$ if and only if $D$ is Algebraic over $F$. 
Algebraic commutators

**Theorem (Jacobson)**

- Every division ring whose elements are algebraic of bounded degree over its center is centrally finite.

**Theorem (Bell-Drensky-Sharifi (2013))**

- Every division ring whose elements are left algebraic of bounded degree over a not necessarily central subfield is centrally finite.
Algebraic commutators

Theorem (M.A., Akbari-Bien)

Let $D$ be a division ring with infinite center. If $D$ contains element $a$ such that $xax^{-1}a^{-1}$, for every $x \in D^*$ are left algebraic of bounded degree over a not necessarily central subfield, then $D$ is centrally finite.
Theorem (M.A., Akbari-Bien)

Let $D$ be a division ring with infinite center $F$ and not necessarily central subfield $K$. If $D$ contains a non-central normal subgroup $N$ left algebraic of bounded degree $n$ over $K$, then $D$ is centrally finite.
Theorem (M.A., Akbari-Bien)

Let $D$ be a division ring with infinite center and not necessarily central subfield $K$. Assume that $K$ contains a non-central algebraic element $a$ over the center. If all additive commutators $ax - xa$, for every $x \in D$ are left algebraic of bounded degree over $K$, then $D$ is centrally finite.
Algebraic commutators

Theorem (M.A., Akbari-Bien)

Let $D$ be a division ring with center $F$ and not necessarily central subfield $K$. Assume that $D'$ is left algebraic of bounded degree over a $K$, then $D$ is centrally finite.
Thank you for your attention