Generalized Hereditary Noetherian Prime Rings

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Introduction

Let R be a hereditary Noetherian prime ring, σ be an automorphism of R and δ be a derivation on R. The skew polynomial ring $R[x;\sigma]$ and differential polynomial ring $R[x;\delta]$ have the following properties:

- Global dimensions are two and
- Any ideal which is a left v-ideal or right v-ideal is left and right projective.

Definition: A prime Goldie ring R is a generalized hereditary Noetherian prime ring (a G-HNP ring for short) if

- Any ideal A of R with $A = A_v$ or $A = {}_vA$ is a projective ideal, that is, left and right projective.
- R is τ-Noetherian.

A G-HNP ring is called a strongly G-HNP ring if any one-sided v-ideal is projective.

From the view-point of global dimensions HNP rings have global dimension one. However, the examples of G-HNP rings show that the class of G-HNP rings ranges from rings with global dimension two to rings with infinite global dimension.

Preliminaries

Let R be a prime Goldie ring with its quotient ring Q, I be a (fractional) right R-ideal and J be a left R-ideal. We use the notation:

- $I^* = \{q \in Q |\ qI \subseteq R\}$, a left R-ideal and
- $J^+ = \{q \in Q |\ Jq \subseteq R\}$, a right R-ideal
- We define $I_v = I^{*+}$, which contains I
- We call I a right v-ideal if $I_v = I$
- Similarly, we define ${}_vJ = J^{+*} \supseteq J$ and J is called a left v-ideal if ${}_vJ = J$
- An R-ideal A is said to be a v-ideal if $A_v = A =_v A$.

Preliminaries

A right R-ideal I is right projective if and only if $II^* = \mathcal{O}_l(I) = \{q \in Q \mid qI \subseteq I\}$, which is equivalent to $II^* \ni 1$ since $II^* \subseteq \mathcal{O}_l(I)$ and I is a left $\mathcal{O}_l(I)$-module. Similarly a left R-ideal J is left projective if and only if $J^+J = \mathcal{O}_r(J) = \{q \in Q \mid Jq \subseteq J\}$.

Examples

Let R be an HNP ring. Then the skew polynomial ring $R[x;\sigma]$ and the differential polynomial ring $R[x;\delta]$ are strongly G-HNP rings with global dimension 2, where σ is an automorphism of R and δ is a derivation of R.

Examples

Let U be a commutative uniqe factorization domain (a UFD for short) and let $S = \begin{pmatrix} U[x] & U[x] \\ U[x] & U[x] \end{pmatrix}$ be the 2×2 matrix ring over the polynomial ring $U[x]$ with indeterminate x.
Then $S = M_2(U)[x]$ is a non-commutative unique factorization ring (a UFR for short).
Let $I = \begin{pmatrix} xU[x] & xU[x] \\ U[x] & U[x] \end{pmatrix}$ be a right ideal of S and consider the idealizer R of I in S:

$$R = \mathbf{I}_S(I) = \{s \in S \mid sI \subseteq I\}.$$

Then $R = \begin{pmatrix} U[x] & xU[x] \\ U[x] & U[x] \end{pmatrix}$ and is a G-HNP ring such that $n \leq \operatorname{gl.dim} R \leq 2n$ if $\operatorname{gl\,dim} U[x] = n$. This contains the examples of G-HNP rings with infinite global dimension.

Lemmas

- A maximal projective ideal of R is either invertible or idempotent.
- A finite set of distinct idempotent maximal projective ideals $M_1, \ldots, M_n$ (ideals maximal amongst the projective ideals) such that $\mathcal{O}_r(M_1) = \mathcal{O}_l(M_2)$,...,$\mathcal{O}_r(M_n) = \mathcal{O}_l(M_1)$ is called a cycle. Let $M_1, \ldots, M_n$ be a cycle. Then $P = M_1 \cap \ldots \cap M_n$ is an invertible ideal.
- Let I be an essential right ideal of R with $I = I_v$. Then any descending chain of right v-ideals $R \supseteq I_1 \supseteq I_2 \supseteq \ldots \supseteq I$ must stabilize.
- An ideal is a maximal invertible ideal (ideal maximal amongst the invertible ideals) if and only if it is the intersection of a cycle.

Theorem : Let R be a G-HNP ring. Then the invertible ideals of R generate an Abelian group whose generators are maximal invertible ideals.

- An ideal I is called eventually idempotent if I^n is idempotent for $n \geq 1$ and we write $evI = inf\{n > 0 \mid I^n = I^{n+1}\}$. Clearly I is eventually idempotent if and only if $evI < \infty$. In this case evI is called the degree of eventuality of I.

Proposition : Let R be a prime Goldie ring satisfying the ascending chain conditions on one-sided v-ideals of R and let I be an ideal of R which is right projective. Then exactly one of the followings holds:
(1) I is eventually idempotent, that is, $I^k = I^{k+1}$ for some k.
(2) $\cap_i I^i = (0)$.

Lemmas

- Any prime projective ideal which is eventually idempotent is idempotent.
- Let A be a projective ideal of R. Then there are prime projective ideals $P_1, ..., P_m$ (it may happen that $P_i = P_j$ for $i \neq j$) such that $P_1 ... P_m \subseteq A \subseteq \cap P_i$.
- For an ideal A we denote by $N(A)$ the prime radical of A, that is, the intersection of all prime ideals containing A.
- Let A be a projective ideal.
  (1) Any minimal prime ideal over A is projective.
  (2) $N(A) = P_1 \cap ... \cap P_n$, where $P_1, ..., P_n$ are all projective ideals which are minimal over A and $N(A)$ is a projective ideal.
  (3) A is eventually idempotent if and only if so is $N(A)$ and $evN(A) \geq evA$.

- We denote by $P(R)$ the set of all projective ideals of R and $P_m(R) = \{A \in P(R) \mid A \subseteq P : \text{prime projective ideal} \Rightarrow P : \text{maximal projective ideal}\}$.
- If R has enough invertible ideals, that is, any projective ideal contains an invertible ideal, then $P(R) = P_m(R)$. If R is a PI ring, then R has enough invertible ideals by Posner's theorem.
- Examples given above have the property $P(R) = P_m(R)$.

Theorem : Let $I \in P_m(R)$. Then $I = XA$ where X is an invertible ideal and A is an eventually idempotent ideal. Let $N(A) = P_1 \cap \ldots \cap P_n$ where P_i are minimal primes over A. Then $evA \leq n$.

Remark : Let $I \in P(R) \setminus P_m(R)$. Then $I = XA$, where X is an invertible ideal and A is a projective ideal which is not contained in any invertible ideal and either eventually idempotent or $\cap_n A^n = (0)$. However, we do not have an example of G-HNP ring R with $P(R) \supset P_m(R)$. That is an open question.

Structure theorem for G-HNP rings

Let R be a G-HNP ring. Then

- $R = \cap R_P \cap S$, where P runs over all maximal invertible ideals, R_P is a semi-local HNP ring and S is a G-HNP ring with no invertible ideals. If R is a strongly G-HNP ring, then so is S.
- There is a one-to-one correspondence between $\mathrm{Spec}^*(R)$ and $\mathrm{Spec}^{\mathfrak{p}}(S)$ given by $P \longrightarrow P' = PS$ and $P' \longrightarrow P' \cap R$, where $P \in \mathrm{Spec}^*(R)$.
- Each regular element of R is a unit of R_P for almost all P.

Thanks

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