On Generalized Perfect Rings

Pınar AYDOĞDU

Hacettepe University / TURKEY

(Joint work with D. Herbera)

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Definitions [A. Amini, B. Amini, Ershad, Sharif-2007]

Let $R$ be an associative ring with 1. All modules are unital. Ring homomorphisms preserve 1.

- Let $F$ and $M$ be right $R$-modules such that $F_R$ is flat. A module epimorphism $f : F \to M$ is said to be a $G$-flat cover of $M$ if $\text{Ker}(f)$ is a small submodule of $F$. 

A ring $R$ is called right generalized perfect (right $G$-perfect, for short) if every right $R$-module has a $G$-flat cover.

A ring $R$ is called $G$-perfect if it is both left and right $G$-perfect.
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- A ring $R$ is called $\textit{right generalized perfect}$ (right $G$-perfect, for short) if every right $R$-module has a $G$-flat cover.
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- A ring $R$ is called right generalized perfect (right $G$-perfect, for short) if every right $R$-module has a $G$-flat cover.
- A ring $R$ is called $G$-perfect if it is both left and right $G$-perfect.
\begin{itemize}
  \item \{ perfect rings \} \subseteq \{ G\text{-perfect rings} \}
  \item \{ Von Neumann regular rings \} \subseteq \{ G\text{-perfect rings} \}
  \item \{ G\text{-perfect rings} \} is closed under finite products and quotients.
\end{itemize}
Definition (due to Auslander and Enochs)

Let $\mathcal{C}$ be a class of right $R$-modules, and let $M_R$ be a right $R$-module. A module homomorphism $f : C \to M$ is a $\mathcal{C}$-precover of $M$ if it satisfies that

(i) $C \in \mathcal{C}$;
(ii) any diagram with $C' \in \mathcal{C}$

\[
\begin{array}{ccc}
C' & \rightarrow & M \\
\downarrow & & \downarrow \\
C & \xrightarrow{f} & M \\
\end{array}
\]

can be completed to a commutative diagram. The homomorphism $f : C \to M$ is said to be right minimal if for any $g \in \text{End}_R(C)$, $f = fg$ implies $g$ bijective.
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Let $C$ be a class of right $R$-modules, and let $M_R$ be a right $R$-module.

A module homomorphism $f : C \rightarrow M$ is a $C$-precover of $M$ if it satisfies that

(i) $C \in C$;
(ii) any diagram with $C' \in C$

\[
\begin{array}{c}
C' \\
\downarrow \\
\vdots \\
C \\
\downarrow \\
f \\
\rightarrow \\
M \\
\rightarrow \\
0
\end{array}
\]

can be completed to a commutative diagram.

The homomorphism $f$ is a $C$-cover if, in addition, it is right minimal.

Recall that $f : C \rightarrow M$ is said to be right minimal if for any $g \in \text{End}_R(C)$, $f = fg$ implies $g$ bijective.
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In the case of perfect rings, projective covers, flat covers and $G$-flat covers coincide.

In the case of von Neumann regular rings, flat covers are $G$-flat covers.
\[ \mathcal{E} = \{ B \in \text{Mod-}R \mid \text{Ext}^1_R(L, B) = 0 \text{ for any flat } L_R \} \] is called the class of (Enochs) cotor\text{s}ion modules.
$\mathcal{C} = \{ B \in \text{Mod-}R | \text{Ext}^1_R(L, B) = 0 \text{ for any flat } L_R \}$ is called the class of (Enochs) cotorsion modules.

- Kernel of any flat cover is a cotorsion module.
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- Kernel of any flat cover is a cotorsion module.
- Any \( M_R \) fits into an exact sequence

\[ 0 \rightarrow B \rightarrow L \xrightarrow{g} M \]

where \( L \) is flat and \( B \) is cotorsion. \( g \) is a flat precover.
Example due to A. Amini, B. Amini, Ershad, Sharif-2007

Let $R$ be a regular ring which is not a right $V$-ring.
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**Case 1**

$\text{Soc}(E/M) = 0$. $\pi: E \to E/M$ and $i: E/M \to E$ are both $G$-flat covers of $E/M$. But $E \ncong E/M$.

**Case 2**

$\text{Soc}(E/M) \neq 0$. There is $K_R \subseteq E$ such that $K/M$ is a simple $R$-module. $\pi: K \to K/M$ and $i: K/M \to K/M$ are both $G$-flat covers of $K/M$. But $K \ncong K/M$. 

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1. **Case 1** $\text{Soc}(E/M) = 0$. $\pi : E \to E/M$ and $i : E/M \to E/M$ are both $G$-flat covers of $E/M$. But $E \ncong E/M$. 

2. **Case 2** $\text{Soc}(E/M) \neq 0$. There is $K_R \subseteq E_R$ such that $K/M$ is a simple $R$-module. $\pi : K \to K/M$ and $i : K/M \to K/M$ are both $G$-flat covers of $K/M$. But $K \ncong K/M$. 

Example due to A. Amini, B. Amini, Ershad, Sharif-2007
Let $R$ be a regular ring which is not a right $V$-ring. Then there exist a right $R$-module $M$ such that $M \not\subseteq E = E(M)$.

- **Case 1** $\text{Soc}(E/M) = 0$. $\pi : E \rightarrow E/M$ and $i : E/M \rightarrow E/M$ are both $G$-flat covers of $E/M$. But $E \not\cong E/M$.

- **Case 2** $\text{Soc}(E/M) \neq 0$. There is $K_R \subseteq E_R$ such that $K/M$ is a simple $R$-module. $\pi : K \rightarrow K/M$ and $i : K/M \rightarrow K/M$ are both $G$-flat covers of $K/M$. But $K \not\cong K/M$. 
Some results from A. Amini, B. Amini, Ershad, Sharif-2007

- $R$ is right $G$-perfect $\implies J(R)$ is right $T$-nilpotent.
- $R$ is right duo and right $G$-perfect $\implies R/J(R)$ is von Neumann regular.

Conjecture: $R$ is right $G$-perfect $\implies$ semiregular ???

Our Answer: No!!
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**Conjecture:** $R$ is right $G$-perfect $\implies$ semiregular ???

**Our Answer:** No!!!
A pair $(\mathcal{X}, \mathcal{Y})$ of subclasses of $\text{Mod-} R$ is said to be a torsion pair if

(i) $\text{Hom}_R(X, Y) = \{0\}$ for any $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

(ii) If $X_R$ is a right $R$-module such that $\text{Hom}_R(X, Y) = \{0\}$ for any $Y \in \mathcal{Y}$ then $X \in \mathcal{X}$.

(iii) If $Y_R$ is a right $R$-module such that $\text{Hom}_R(X, Y) = \{0\}$ for any $X \in \mathcal{X}$ then $Y \in \mathcal{Y}$.

In this case, $\mathcal{X}$ is said to be a torsion class and $\mathcal{Y}$ is a torsion-free class. The objects of $\mathcal{X}$ are called torsion modules and the objects in $\mathcal{Y}$ are called torsion-free modules.
Basic Definitions

Let \((\mathcal{X}, \mathcal{Y})\) be a torsion pair. If \(M_R\) is a right \(R\)-module, the largest submodule of \(M_R\) that is an object of \(\mathcal{X}\) called the torsion submodule of \(M\) and is denoted by \(t(M)\). \(t\) is indeed a functor and a radical. So that, there is an exact sequence

\[
0 \to t(M) \to M \to M/t(M) \to 0
\]

where \(M/t(M) \in \mathcal{Y}\).
Basic Definitions

- A class of modules $\mathcal{X}$ is torsion if and only if it is closed under isomorphisms, extensions, coproducts and quotients.
- Dually, a class of modules $\mathcal{Y}$ is a torsion-free class if it is closed under isomorphism, extensions, submodules and products.

Notice that if a class of modules $\mathcal{Y}$ is closed by products, coproducts, subobjects, quotients and extensions then $\mathcal{Y}$ is a torsion class and a torsion free class at the same time. Therefore, one has a triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ such that $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are torsion pairs. Such a triple is called a TTF-triple.
Basic Definitions

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- Notice that if a class of modules $\mathcal{Y}$ is closed by products, coproducts, subobjects, quotients and extensions then $\mathcal{Y}$ is a torsion class and a torsion free class at the same time. Therefore, one has a triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ such that $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are torsion pairs. Such a triple is called a TTF-triple.
Let $0 \to M \xrightarrow{h} N \xrightarrow{f} K \to 0$ be an exact sequence of right $R$-modules and let $L \xrightarrow{g} K \to 0$ be an onto homomorphism. We consider the pullback of $f$ and $g$ to obtain a commutative diagram with exact rows and columns:

$$
\begin{array}{c}
0 & 0 \\
\downarrow & \downarrow \\
X = X = \text{Ker}g \\
\downarrow^{\varepsilon_2} & \downarrow \\
0 & 0
\end{array}
$$

$$
\begin{array}{c}
0 \to M \xrightarrow{\varepsilon_1} L' \xrightarrow{\pi_2} L \to 0 \\
\downarrow^{\varepsilon_2} & \downarrow^{\pi_1} & \downarrow^g \\
0 \to M \xrightarrow{h} N \xrightarrow{f} K \to 0 \\
\downarrow & \downarrow \\
0 & 0
\end{array}
$$

(1)
In (1),

- $L' = \{(x, y) \in N \oplus L | f(x) = g(y)\}$.
- The maps $\pi_1 : L' \to N$ and $\pi_2 : L' \to L$ are restrictions of the canonical projections $\pi_1 : N \oplus L \to N$ and $\pi_2 : N \oplus L \to L$, respectively.
- The homomorphism $\varepsilon_1 : M \to L'$ is defined by $\varepsilon_1(x) = (h(x), 0)$ for each $x \in M$, and $\varepsilon_2 : X \to L'$ is defined by $\varepsilon_2(y) = (0, y)$ for each $y \in X$. 
Let \((\mathcal{X}, \mathcal{Y})\) be a torsion pair in \(\text{Mod-}R\) such that the associated torsion radical \(t\) is exact. Assume that in diagram (1), \(M \in \mathcal{X}\) and \(K, L \in \mathcal{Y}\).

- If \(X\) is small in \(L\), then \(\varepsilon_2(X)\) is small in \(L'\).
- In particular, if \(L_R\) and \(M_R\) are flat, then \(\pi_1: L' \to N\) is a \(G\)-flat cover of \(N\).
- \(g\) is right minimal if and only if \(\pi_1\) is right minimal.
Useful facts on TTF-triples

Let $R$ and $S$ be rings such that there is an exact sequence

$$0 \rightarrow I \rightarrow R \xrightarrow{\varphi} S \rightarrow 0$$

where $\varphi$ is a ring morphism such that $RS$ becomes a flat module. Consider the following classes of modules

$$\mathcal{X} = \{X \in \text{Mod}-R \mid XI = X\}$$
$$\mathcal{Y} = \{Y \in \text{Mod}-R \mid YI = \{0\}\}$$
$$\mathcal{Z} = \{Z \in \text{Mod}-R \mid \text{ann}_Z(I) = \{0\}\}$$

then $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a TTF-triple such that the torsion pair $(\mathcal{X}, \mathcal{Y})$ is hereditary and $\text{Ext}^i_R(X, Y) = 0$ for any $i \geq 0$, $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Moreover, the torsion radical associated to the torsion class $\mathcal{X}$ is naturally equivalent to the exact functor $- \otimes R I$, and the torsion radical associated to the class $\mathcal{Y}$ is naturally equivalent to the functor $\text{Hom}_R(S, -)$. 
Useful facts on **TTF-triples**

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then \((\mathcal{X}, \mathcal{Y}, \mathcal{Z})\) is a TTF-triple such that the torsion pair \((\mathcal{X}, \mathcal{Y})\) is hereditary and \( \text{Ext}^i_R(X, Y) = 0 \) for any \( i \geq 0, X \in \mathcal{X} \) and \( Y \in \mathcal{Y} \). Moreover, the torsion radical associated to the torsion class \( \mathcal{X} \) is naturally equivalent to the exact functor \(- \otimes_R I\), and the torsion radical associated to the class \( \mathcal{Y} \) is naturally equivalent to the functor \( \text{Hom}_R(S, -) \).
Corollary

Let $R$ and $S$ be rings such that there is an exact sequence

$$0 \to I \to R \xrightarrow{\varphi} S \to 0$$

where $\varphi$ is a ring morphism such that $S$ becomes a flat $R$-module on the right and on the left. Then:

(i) $M_R$ is flat if and only if $M \otimes_R S$ is a flat right $S$-module and $MI$ is a flat right $R$-module.

(ii) Let $M$ be a right $S$-module, then $M$ is cotorsion as a right $R$-module if and only if it is cotorsion as an $S$-module.
Proposition[A, Herbera-2016]

Let $S \subseteq T$ be an extension of rings. Let

$$R = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) | n \in \mathbb{N}, x_i \in T, x \in S\}.$$ 

Then, the following statements hold.
Proposition[A, Herbera-2016]

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$$R = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) | n \in \mathbb{N}, x_i \in T, x \in S\}.$$ 

Then, the following statements hold.

(i) The map $\varphi : R \to S$ defined by $\varphi(x_1, x_2, \ldots, x_n, x, x, \ldots) = x$ is a ring homomorphism with kernel

$$I = \bigoplus_{\mathbb{N}} T = \bigoplus_{i \in \mathbb{N}} e_i R,$$

where $e_i = (0, \ldots, 0, 1^{(i)}, 0, 0, \ldots)$ for any $i \in \mathbb{N}$. 

(ii) $I$ is a two-sided, countably generated idempotent ideal of $R$ which is pure and projective on both sides. Therefore, $S$ is flat as a right and as a left $R$-module.

(iii) For any $i \in \mathbb{N}$, the canonical projection into the $i$-th component $\pi_i : R \to T$ has kernel $(1 - e_i)R$ so that $T$ is projective as a right and as a left $R$-module via the $R$-module structure induced by $\pi_i$. 

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Then, the following statements hold.

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(iii) For any $i \in \mathbb{N}$, the canonical projection into the $i$-th component $\pi_i : R \rightarrow T$ has kernel $(1-e_i)R$ so that $T$ is projective as a right and as a left $R$-module via the $R$-module structure induced by $\pi_i$. 
Remark

Let $R$ be a ring as in the Proposition. Then there is a TTF-triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ associated to the pure exact sequence

$$0 \rightarrow I \rightarrow R \xrightarrow{\varphi} S \rightarrow 0$$

where $\mathcal{X} = \{ X \in \text{Mod} - R \mid X = \bigoplus_{i \in \mathbb{N}} Xe_i \}$,
$\mathcal{Y} = \{ Y \in \text{Mod} - R \mid YI = \{0\} \}$
$\mathcal{Z} = \{ Z \in \text{Mod} - R \mid \text{ann}_Z(I) = \{0\} \}$. Also, for any $i \in \mathbb{N}$, the split sequence

$$0 \rightarrow R(1 - e_i) \rightarrow R \xrightarrow{\pi_i} T \rightarrow 0$$

yields a corresponding (split) TTF-triple $(\mathcal{X}_i, \mathcal{Y}_i, \mathcal{Z}_i)$. 
Proposition[A, Herbera-2016]

(i) $J(R)$ contains $J = \bigoplus_{\mathbb{N}} J(T)$. Moreover, $J$ is essential on both sides into $J(R)$. In particular, $J(R) = 0$ if and only if $J(T) = 0$.

(ii) $R$ is von Neumann regular if and only if $S$ and $T$ are von Neumann regular.

(iii) Let $M_R$ be a right $R$-module. Then $M_R$ is flat if and only if $M \otimes_R S$ is a flat right $S$-module and, for any $i \in \mathbb{N}$, $Me_i$ is a flat right $T$-module.
Main Theorem [A, Herbera-2016]

Let $S \subseteq T$ be an extension of rings. Assume $T$ is von Neumann regular and that $S$ is right $G$-perfect. Then

$$R = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) \mid n \in \mathbb{N}, x_i \in T, x \in S\}$$

is a right $G$-perfect ring such that $J(R) = 0$. Moreover, if $S$ is a ring such that flat covers are $G$-flat covers, then also $R$ satisfies this property.
Proof

- By the properties of $R$, it readily follows that $J(R) = 0$. 

- Let $N$ be any right $R$-module. There is a pure exact sequence

$$
0 \rightarrow NI \xrightarrow{\sim} \bigoplus_{i \in N} Ne_i \rightarrow Nf \rightarrow N/NI \rightarrow 0
$$

- Since $T$ is von Neumann regular, for any $i \in N$, $Ne_i$ is a flat $T$-module.

- Hence $NI$ is flat as a right $R$-module.
Proof

- By the properties of $R$, it readily follows that $J(R) = 0$.
- Let $N$ be any right $R$-module. There is a pure exact sequence

$$0 \longrightarrow NI \cong \bigoplus_{i \in \mathbb{N}} Ne_i \longrightarrow N \xrightarrow{f} N/NI \longrightarrow 0.$$
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- By the properties of $R$, it readily follows that $J(R) = 0$.
- Let $N$ be any right $R$-module. There is a pure exact sequence

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- Since $T$ is von Neumann regular, for any $i \in \mathbb{N}$, $Ne_i$ is a flat $T$-module.
- Hence $NI$ is flat as a right $R$-module.
...Proof...

Let $0 \to X \to L \xrightarrow{h} N/NI \to 0$ be a $G$-flat cover of the right $S$-module $N/NI$. Considering the pullback of $h$ and $f$ yields the following diagram with exact rows and columns:

$$
\begin{array}{cccccc}
0 & \to & NI & \to & L' & \xrightarrow{\pi_2} & L & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & h \\
0 & \to & NI & \to & N & \xrightarrow{f} & N/NI & \to & 0 \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
$$

$X = \text{Ker} h$
...Proof...

Since the radical associated to the torsion pair \((\mathcal{X}, \mathcal{Y})\) is exact and \(L \in \mathcal{Y}\), \(\pi_1\) is a \(G\)-flat cover of \(N\).
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Now assume, in addition, that \(0 \to X \to L \to^h N/NI \to 0\) is a flat cover of the right \(S\)-module \(N \otimes_R S\).
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In particular, \(X_S\) is cotorsion.
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In particular, \(X_S\) is cotorsion.

\(X_R\) is also a cotorsion module, hence \(0 \to X \to L' \xrightarrow{\pi_1} N \to 0\) is a flat precover of \(N\).
...Proof...

- Since the radical associated to the torsion pair $(\mathcal{X}, \mathcal{Y})$ is exact and $L \in \mathcal{Y}$, $\pi_1$ is a $G$-flat cover of $N$.
- Now assume, in addition, that $0 \to X \to L \xrightarrow{h} N/NI \to 0$ is a flat cover of the right $S$-module $N \otimes_R S$.
- In particular, $X_S$ is cotorsion.
- $X_R$ is also a cotorsion module, hence $0 \to X \to L' \xrightarrow{\pi_1} N \to 0$ is a flat precover of $N$.
- It follows that $\pi_1$ is also a flat cover.
Example 1 [A, Herbera-2016]

Let $F$ be a field, and let $S$ be any finite dimensional $F$-algebra such that $J(S) \neq 0$. 
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Let $F$ be a field, and let $S$ be any finite dimensional $F$-algebra such that $J(S) \neq 0$.

- Since $S$ is artinian, it is $G$-perfect.
Example 1 [A, Herbera-2016]

Let $F$ be a field, and let $S$ be any finite dimensional $F$-algebra such that $J(S) \neq 0$.

- Since $S$ is artinian, it is $G$-perfect.
- If $\dim_F(S) = n$, then $S \hookrightarrow T = \mathbb{M}_n(F)$ which is von Neumann regular.
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$$R = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) | n \in \mathbb{N}, x_i \in T, x \in S\}$$

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For a particular realization of such a ring $R$ consider, for example,

$$S = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}.$$  In this case, $T$ can be taken to be $M_2(F)$. 
Example 2 [A, Herbera-2016]

Let $R$ be as in Example (1).

- Then, $R \subseteq \prod \mathbb{M}_n(F) = T'$ which is a von Neumann regular ring.

- $R' = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) | n \in \mathbb{N}, x_i \in T', x \in R\}$ is also a $G$-perfect ring.
In general, it is difficult to compute Enochs flat covers. If projective covers exist, then they coincide with Enochs flat covers. So the question is:

**Question 1:** What is the relation, if any, between $G$-flat covers and Enochs flat covers?

**Question 2:** Let $R$ be a semiregular ring with right $T$-nilpotent Jacobson radical, is it $G$-perfect?