Classical left regular left quotient ring of a ring and its semisimplicity criteria

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*V. V. Bavula, The classical left regular left quotient ring of a ring and its semisimplicity criteria, Arxiv:math.RA:1504.07257.

talk-ClasLeftRegQuotRing.tex

R is a ring with 1, R^* is its group of units,

$\mathcal{C} = \mathcal{C}_R$ is the set of regular elements of R,

$Q = Q_{l,cl}(R) := \mathcal{C}^{-1}R$ is the **left quotient ring** (the **classical left ring of fractions**) of R (if it exists),

$\mathrm{Ore}_l(R)$ is the set of **left Ore sets** S (i.e. for all $s \in S$ and $r \in R$: $Sr \cap Rs \neq \emptyset$),

$\mathrm{ass}(S) := \{r \in R \,|\, sr = 0$ for some $s \in S\}$, an ideal of R,

$\mathrm{Den}_l(R, \mathfrak{a})$ is the set of **left denominator sets** S of R with $\mathrm{ass}(S) = \mathfrak{a}$ where $\mathfrak{a}$ is an ideal of R (i.e. $S \in \mathrm{Ore}_l(R)$, and $rs = 0$ implies $s'r = 0$ for some $s' \in S$),

Given $S \in \mathrm{Den}_l(R)$, $S^{-1}R := \{s^{-1}r \,|\, s \in S, r \in R\}$, the **localization** of R at S, Let $X \subseteq R$, a subset,

$$\text{ann}_l(X) = \{r \in R \,|\, rX = 0\}$$

is the **left annihilator** of X, it is a left ideal of R,

A ring R is a **left Goldie ring** if

(i) R satisfies ACC for left annihilators,

(ii) R contains no infinite direct sums of left ideals.

Thm (Goldie, 1958, 1960). *A ring R has a semisimple left quotient ring iff R is a semiprime left Goldie ring.*

Lessieur and Croisot (1959): prime case.

$\cdot r : R \to R$, $x \mapsto xr$,

${}'\mathcal{C}_R := \{r \in R \,|\, \ker(\cdot r) = 0\}$ is the **set of left regular elements** of R

$\mathcal{C}_R \subseteq {}'\mathcal{C}_R$

The ring ${}'Q_{l,cl}(R) := {}'\mathcal{C}_R^{-1} R$ is called the **classical left regular left quotient ring** of R

Goal: To give *semisimplicity criteria* for ${}'Q_{l,cl}(R)$

A subset S of R is called a **multiplicative set** if $1 \in S$, $SS \subseteq S$ and $0 \notin S$

Let $S \subseteq T$ be multiplicative sets in R such that $S \subseteq T$

S is called **dense** (or **left dense**) in T if for each element $t \in T$ there exists an element $r \in R$ such that $rt \in S$

For a left ideal I of R, let ${}'\mathcal{C}_I := \{i \in I \,|\, \cdot i : I \to I$, $x \mapsto xi$, is an injection$\}$

Thm 1 (First Criterion) *Let R be a ring, $'\mathcal{C} = {}'\mathcal{C}_R$ and $\mathfrak{a} := \mathrm{ass}_R('\mathcal{C})$. The following statements are equivalent.*

1. $'Q := {}'Q_{l,cl}(R)$ is a semisimple Artinian ring.

2. (a) $\mathfrak{a}$ is a semiprime ideal of R,

(b) the set $'\overline{\mathcal{C}} := \pi('\mathcal{C})$ is a dense subset of $'\mathcal{C}_{\overline{R}}$ where $\pi : R \to \overline{R} := R/\mathfrak{a}$, $r \mapsto \overline{r} := r + \mathfrak{a}$,

(c) $\mathrm{udim}({}_{\overline{R}}\overline{R}) < \infty$, and

(d) $'\mathcal{C}_V \neq \emptyset$ for all uniform left ideals V of $\overline{R}$.

3. $\mathfrak{a}$ is a semiprime ideal of R, $'\overline{\mathcal{C}}$ is a dense subset of $\mathcal{C}_{\overline{R}}$ and $Q_{l,cl}(\overline{R})$ is a semisimple Artinian ring.

If one of the equivalent conditions holds then $'\overline{\mathcal{C}} \in \mathrm{Den}_l(\overline{R}, 0)$, $'\overline{\mathcal{C}}$ is a dense subset of $\mathcal{C}_{\overline{R}}$ and $'Q \simeq {}'\overline{\mathcal{C}}^{-1}\overline{R} \simeq Q_{l,cl}(\overline{R})$. Furthermore, the ring $'Q$ is a simple ring iff the ideal $\mathfrak{a}$ is a prime ideal.

The set max.Den$_l(R)$ of **maximal left denominator sets** is a non-empty set

The next semisimplicity criterion for the ring $'Q_{l,cl}(R)$ is given via the set $'\mathcal{M}$ of maximal denominator sets of R that contain $'\mathcal{C}_R$

Thm 2 (Second Criterion) *Let R be a ring, $\mathfrak{a} = \text{ass}_R('\mathcal{C}_R)$. The following statements are equivalent.*

1. *$'Q_{l,cl}(R)$ is a semisimple Artinian ring.*

2. *$'\mathcal{M}$ is a finite nonempty set, $\bigcap_{S\in '\mathcal{M}} \text{ass}_R(S) = \mathfrak{a}$, for each $S \in {'\mathcal{M}}$, the ring $S^{-1}R$ is a simple Artinian ring and the set $'\overline{\mathcal{C}} := \{c+\mathfrak{a} \,|\, c \in {'\mathcal{C}_R}\}$ is a dense subset of $\mathcal{C}_{R/\mathfrak{a}}$ in $R/\mathfrak{a}$.*

Theorem below is a semisimplicity criterion for $'Q_{l,cl}(R)$ given via the **set** Min$_R(\mathfrak{a})$ **of minimal primes** of the ideal $\mathfrak{a}$, it describes explicitly the set $'\mathcal{M}$ in Theorem 2

Thm 3 (Third Criterion) *Let R be a ring, $'\mathcal{C} = {}'\mathcal{C}_R$ and $\mathfrak{a} = \text{ass}_R('\mathcal{C})$. The following statements are equivalent.*

1. $'Q_{l,cl}(R)$ *is a semisimple Artinian ring.*

2.(a) $\mathfrak{a}$ *is semiprime ideal of R and the set* $\text{Min}_R(\mathfrak{a})$ *is a finite set.*

(b) *For each $\mathfrak{p} \in \text{Min}_R(\mathfrak{a})$, the set $S_\mathfrak{p} := \{c \in R \mid c + \mathfrak{p} \in \mathcal{C}_{R/\mathfrak{p}}\}$ is a left denominator set of the ring R with $\text{ass}_R(S_\mathfrak{p}) = \mathfrak{p}$.*

(c) *For each $\mathfrak{p} \in \text{Min}_R(\mathfrak{a})$, the ring $S_\mathfrak{p}^{-1}R$ is a simple Artinian ring.*

(d) *The set $'\overline{\mathcal{C}} = \{c + \mathfrak{a} \mid c \in {}'\mathcal{C}\}$ is a dense subset of $\mathcal{C}_{R/\mathfrak{a}}$.*

Theorem below is a semisimplicity criterion for the ring ${}'Q_{l,cl}(R)$ in terms of left Goldie rings.

Thm 4 (Fourth Criterion) *The following statements are equivalent.*

1. *${}'Q_{l,cl}(R)$ is a semisimple Artinian ring.*

2.(a) *$\mathfrak{a}$ is a semiprime ideal of R and the set $\mathrm{Min}_R(\mathfrak{a})$ is finite.*

(b) *For each $\mathfrak{p} \in \mathrm{Min}_R(\mathfrak{a})$, the ring $R/\mathfrak{p}$ is a left Goldie ring.*

(c) *The set ${}'\overline{\mathcal{C}}$ is a dense subset of $\mathcal{C}_{\overline{R}}$.*

Theorem below is a useful semisimplicity criterion for the ring $'Q_{l,cl}(R)$ as often we have plenty of simple Artinian localizations of a ring.

Thm 5 *The following statements are equivalent.*

1. *$'Q_{l,cl}(R)$ is a semisimple Artinian ring.*

2. *There are left denominator sets $S_1, \ldots, S_n$ of the ring R such that*

   (a) *the rings $S_i^{-1}R$ are simple Artinian rings,*

   (b) *$\mathfrak{a} = \bigcap_{i=1}^{n} \mathrm{ass}_R(S_i)$, and*

   (c) *$'\overline{\mathcal{C}}$ is a dense subset of $\mathcal{C}_{\overline{R}}$.*

Semisimplicity criteria for the ring $Q_{l,cl}(R)$. The next theorem shows that the a.c.c. condition on *left* annihilators in Goldie's Theorem can be replaced by the a.c.c. condition on *right* annihilators (or even by a weaker condition) and adding some extra condition.

Thm 6 *Let R be a ring, $\mathcal{C} = \mathcal{C}_R$ and $'\mathcal{C} = {}'\mathcal{C}_R$. The following statements are equivalent.*

1. *$Q_{l,cl}(R)$ is a semisimple Artinian ring.*

2. *R is a semiprime ring,* udim$({}_RR) < \infty$*, the ring R satisfies the a.c.c. on right annihilators and $'\mathcal{C}_U \neq \emptyset$ for all uniform left ideals U of R.*

3. *The ring R is a semiprime ring,* udim$({}_RR) <$ *∞, the set $\{\mathrm{ker}(c_R\cdot) \,|\, c \in {}'\mathcal{C}\}$ satisfies the*

a.c.c. and ${}'\mathcal{C}_U \neq \emptyset$ *for all uniform left ideals* U *of* R.

4. *The ring* R *is a semiprime ring,* $\mathrm{udim}({}_RR) < \infty$*, the set* $\{\mathrm{ker}(r_R\cdot) \,|\, r \in R\}$ *satisfies the a.c.c. and* ${}'\mathcal{C}_U \neq \emptyset$ *for all uniform left ideals* U *of* R.

Below is another semisimplicity criterion for the ring $Q_{l,cl}(R)$ via ${}'\mathcal{C}_R$.

Thm 7 *Let R be a ring. The following statements are equivalent.*

1. *$Q_{l,cl}(R)$ is a semisimple Artinian ring.*

2. *R is a semiprime ring,* $\mathrm{udim}({}_RR) < \infty$, *${}'\mathcal{C}_R = \mathcal{C}_R$ and ${}'\mathcal{C}_U \neq \emptyset$ for all uniform left ideals U of R.*

The rings $'Q_{l,cl}(\mathbb{I}_1)$ and $Q'_{r,cl}(\mathbb{I}_1)$. Let K be a field of characteristic zero,

$A_n = K\langle x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\rangle$ be the **Weyl algebra**

$\mathbb{I}_n = K\langle x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \int_1, \dots, \int_n\rangle$ be the **algebra of polynomial integro-differential operators**.

The ring $Q(A_n) := Q_{l,cl}(A_n)$ is a division ring and $Q(A_n) = {}'Q_{l,cl}(A_n)$.

Lemma 8 *For all K-algebras A and $n \geq 1$, the rings $Q_{l,cl}(\mathbb{I}_n \otimes A)$ and $Q_{r,cl}(\mathbb{I}_n \otimes A)$ do not exist.*

Thm 9 *$'Q_{l,cl}(\mathbb{I}_1) \simeq Q(A_1)$ and $Q'_{r,cl}(\mathbb{I}_1) \simeq Q(A_1)$ are division rings.*

Explicit descriptions of the sets ${}'\mathcal{C}_{\mathbb{I}_1}$ and $\mathcal{C}'_{\mathbb{I}_1}$ are given

This and some other results demonstrate that on many occasions the ring ${}'Q_{l,cl}(R)$ has 'somewhat better properties' than $Q_{l,cl}(R)$ which for $R = \mathbb{I}_n$ even **does not exist**

Conjecture. *${}'Q_{l,cl}(\mathbb{I}_n) \simeq Q_{l,cl}(A_n)$ is a division ring.*