Nondistributive rings

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Definition. By *a ring* we mean a set R of no fewer than two elements, together with two binary operations called the addition and multiplication, in which

(1) R is an abelian group with respect to the addition.

(2) R is a semigroup with unit with respect to the multiplication.

(3) $(x+y)z = xz + yz$ and $x(y+z) = xy + xz$ for all $x, y, z \in R$. $\square$

From the last postulate

$$0x = (0+0)x = 0x + 0x \text{ and } x0 = x(0+0) = x0 + x0,$$

it follows that

$$0x = x0 = 0$$

for every $x \in R$.

Example. For an abelian additive group G, we denote by $End(G)$ the set of group endomorphisms of G. With the addition defined by

$$(f+g)(x) = f(x) + g(x)$$

and the multiplication defined by

$$(fg)(x) = f(g(x))$$

for all $f, g \in End(G)$ and $x \in G$, the set $End(G)$ forms a ring.

The left distributiveness in $End(G)$ follows from the additivity of group endomorphisms

$$(f(g+h))(x) = f(g(x)+h(x)) = f(g(x)) + f(h(x)) = (fg+fh)(x)$$

for all $f, g, h \in End(G)$ and $x \in G$.

From the additivity of group endomorphisms

$$f(0) = f(0+0) = f(0) + f(0),$$

it follows that

$$f(0) = 0$$

for every $f \in End(G)$. $\square$




Example. For a (not necessarily abelian) additive group G, we denote by $M_0(G)$ the set of maps from G into itself preserving 0

$$M_0(G) = \left\{ f \colon G \to G \mid f(0) = 0 \right\}.$$

With the addition defined by

$$(f+g)(x) = f(x) + g(x)$$

for all $f, g \in M_0(G)$ and $x \in G$, the set $M_0(G)$ forms a (not necessarily abelian) group.

With the multiplication defined by

$$(fg)(x) = f(g(x))$$

for all $f, g \in M_0(G)$ and $x \in G$, the set $M_0(G)$ forms a semigroup with unit $id_G \colon G \to G$, $id_G(x) = x$ for every $x \in G$.

The right distributiveness in $M_0(G)$ follows from both definitions of the addition and multiplication in $M_0(G)$

$$((f+g)h)(x) = (f+g)(h(x)) = f(h(x)) + g(h(x)) = (fh+gh)(x)$$

for all $f, g, h \in M_0(G)$ and $x \in G$.

The left distributiveness in $M_0(G)$ does not hold

$$(f(g+h))(x) = f(g(x)+h(x)) \neq f(g(x)) + f(h(x)) = (fg+fh)(x)$$

where $f, g, h \in M_0(G)$ and $x \in G$, unless f is a group endomorphism of G.

For the zero map $0_G \colon G \to G$, $0_G(x) = 0$ where $x \in G$, from the definition of the set $M_0(G)$, it follows that

$$(f0_G)(x) = f(0) = 0 = 0_G(x)$$

for all $f \in M_0(G)$ and $x \in G$. $\square$




Definition. By *a near ring* we mean a set N of no fewer than two elements, together with two binary operations called the addition and multiplication, in which

(1) N is a (not necessarily abelian) group with respect to the addition.

(2) N is a semigroup with unit with respect to the multiplication.

(3) $(x+y)z = xz + yz$ for all $x, y, z \in N$.

(4) $x0 = 0$ for every $x \in N$. This postulate means that we require a near ring to be *zerosymmetric.* $\square$

From the third postulate $0x = (0+0)x = 0x + 0x$, it follows that

$$0x = 0$$

for every $x \in R$.

Definition. By *a nondistributive ring* we mean a set N of no fewer than two elements, together with two binary operations called the addition and multiplication, in which

(1) N is a (not necessarily abelian) group with respect to the addition, with the neutral element denoted by 0.

(2) N is a semigroup with unit with respect to the multiplication, with the neutral element denoted by 1.

(3) $0x = x0 = 0$ for every $x \in N$. This postulate is called *zerosymmetric.*

We say that a nondistributive ring is *abelian* (respectively, *commutative*) if the additive group mentioned above is abelian (respectively, the multiplicative semigroup mentioned above is commutative). $\square$

   

Example. For a nonempty set X with a fixed element 0, we denote by $Map_0(X)$ the set of maps from X into itself preserving 0

$$Map_0(X) = \left\{ f \colon X \to X \mid f(0) = 0 \right\}.$$

With the multiplication defined by

$$(fg)(x) = f(g(x))$$

for all $f, g \in Map_0(X)$ and $x \in X$, the set $Map_0(X)$ forms a semigroup with unit $id_X \colon X \to X$, $id_X(x) = x$ for every $x \in X$.

For the zero map $0_X \colon X \to X$, $0_X(x) = 0$ where $x \in X$, we have

$$(0_X f)(x) = 0 = 0_X(x)$$

and

$$(f 0_X)(x) = f(0) = 0 = 0_X(x)$$

where $f \in Map_0(X)$ and $x \in X$.

Assume that elements of the set $Map_0(X)$ are indexed by elements of an additive group G, with the zero map $0_X = f_0$. We can make the above assumption, since every nonempty set admits a group structure (the statement is equivalent to the Axiom of Choice). With the addition defined by

$$f_a + f_b = f_{a+b}$$

for all $a, b \in G$, the set $Map_0(X)$ forms a group with the neutral element $f_0 = 0_X$.

All of this means that the set $Map_0(X)$ together with both operations, the addition and multiplication, defined above is a nondistributive ring. $\square$




For a nondistributive ring N, we denote by N^+ the additive group of N.

A well known result in the ring theory asserts that

(1) every ring R is isomorphic to the ring $End(R_R)$ of endomorphisms of R viewed as a right module over itself.

(2) $End(R_R)$ is a subring of the ring $End(R^+)$ of group endomorphisms of R^+.

   

   

Example. For a nondistributive ring N, we denote by $r.Hom(N)$ the set of right homogeneous maps from N into itself

$$r.Hom(N) = \Big\{ f\colon N \to N \mid f(xn) = f(x)n \text{ for all } n, x \in N \Big\}.$$

With the multiplication defined by

$$(fg)(x) = f(g(x))$$

for all $f, g \in r.Hom(N)$ and $x \in N$, the set $r.Hom(N)$ forms a semigroup with unit $id_N\colon N \to N$, $id_N(x) = x$ and zero $0_N\colon N \to N$, $0_N(x) = 0$ for every $x \in N$.

We define a map $\lambda\colon N \to r.Hom(N)$ by sending $m \in N$ to the left multiplication $\lambda_m\colon N \to N$ on N defined by $\lambda_m(x) = mx$ for every $x \in N$. Since

$$\lambda_m(xn) = m(xn) = (mx)n = \lambda_m(x)n$$

for all $m, n, x \in N$, we have indeed $\lambda_m \in r.Hom(N)$. Since $\lambda_0 = 0_N$, $\lambda_1 = id_N$ and

$$\lambda_{mn}(x) = (mn)x = m(nx) = (\lambda_m \lambda_n)(x)$$

for all $m, n, x \in N$, it follows that the map λ is a semigroup homomorphism. It is also evident that for all $m, n \in N$ if $\lambda_m = \lambda_n$, then

$$m = \lambda_m(1) = \lambda_n(1) = n,$$

and that

$$f(x) = f(1)x = \lambda_{f(1)}(x)$$

for all $f \in r.Hom(N)$, $x \in N$.

All of this means that $\lambda\colon N \to r.Hom(N)$ is a semigroup isomorphism, and, in consequence, elements of the set $r.Hom(N)$ are indexed by elements of the additive group N^+. With the addition defined by

$$\lambda_m + \lambda_n = \lambda_{m+n}$$

for all $m, n \in N^+$, the semigroup $r.Hom(N)$ forms a nondistributive ring isomorphic to N. $\square$




Theorem. Every nondistributive ring N embeds into the nondistributive ring $Map_0(N)$ of maps from N, viewed as a set, into itself preserving 0.

Proof. According to the previous example, the map $\lambda\colon N \to r.Hom(N)$ defined by $\lambda(m) = \lambda_m$ for every $m \in N$ is a nondistributive ring isomorphism.

It is also evident that $r.Hom(N)$ is a subsemigroup of the multiplicative semigroup $Map_0(N)$ with unit id_N and zero 0_N.

If $Map_0(N)$ is an infinite set and if $\mathcal{F}(Map_0(N))$ denotes the ring (without unit) of finite subsets of $Map_0(N)$, then $N^+ \times \mathcal{F}(Map_0(N))^+$ is a group of order

$$\left|N^+ \times \mathcal{F}(Map_0(N))^+\right| = \left|\mathcal{F}(Map_0(N))^+\right| = \left|Map_0(N)\right|,$$

which means that elements of the set $Map_0(N)$ can be indexed by elements of the additive group $N^+ \times \mathcal{F}(Map_0(N))^+$, with $f_m = \lambda_m$ for every $m \in N^+$. From this we conclude that $r.Hom(N) \subseteq Map_0(N)$ also viewed as nondistributive rings.

The same conclusion holds if $Map_0(N)$ is a finite set. $\square$

Example. For a nonempty and finite set $X = \left\{x_1, x_2, \ldots, x_n\right\}$, we denote by $\mathcal{P}(X)$ the family of subsets of X.

With the multiplication defined by

$$AB = A \cap B$$

for all $A, B \in \mathcal{P}(X)$, the set $\mathcal{P}(X)$ forms a commutative semigroup with unit X and zero $\emptyset$.

Assume that elements of the set $\mathcal{P}(X)$ are indexed by elements of $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\ldots\times\mathbb{Z}/2\mathbb{Z}$, the direct product of n-copies of the additive group $\mathbb{Z}/2\mathbb{Z}$ as follows: for all $i \in \{1, 2, \ldots, n\}$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \mathbb{Z}/2\mathbb{Z}$ we write $x_i \in A_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)}$ if and only if $\varepsilon_i = 1$. Then the addition defined by

$$A_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)} + A_{(\eta_1,\eta_2,\ldots,\eta_n)} = A_{(\varepsilon_1+\eta_1,\varepsilon_2+\eta_2,\ldots,\varepsilon_n+\eta_n)}$$

coincides with the symmetric difference

$$\begin{aligned} A_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)} \triangle A_{(\eta_1,\eta_2,\ldots,\eta_n)} &= \\ &= \Big(A_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)} \setminus A_{(\eta_1,\eta_2,\ldots,\eta_n)}\Big) \cup \Big(A_{(\eta_1,\eta_2,\ldots,\eta_n)} \setminus A_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)}\Big). \end{aligned}$$

All this means that the set $\mathcal{P}(X)$ together with both operations, the addition and multiplication, defined above is a commutative ring. $\square$

   

Example. Let $\mathcal{P}(X)$ be the same commutative semigroup with unit and zero as previously.

Assume that this time elements of the set $\mathcal{P}(X)$ are indexed by elements of the group $D_8 \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \ldots \times \mathbb{Z}/2\mathbb{Z}$, where

$$\begin{aligned} D_8 = \Big\{ & \sigma_0 = (1), \sigma_1 = (1,2,3,4), \sigma_2 = (1,3)(2,4), \sigma_3 = (1,4,3,2), \\ & \sigma_4 = (2,4), \sigma_5 = (1,2)(3,4), \sigma_6 = (1,3), \sigma_7 = (1,4)(2,3) \Big\} \end{aligned}$$

is the dihedral group of order eight, provided that $\emptyset = A_{(\sigma_0,0,0,\ldots,0)}$. With the addition defined by

$$A_{(\sigma_i,\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{n-3})} + A_{(\sigma_j,\eta_1,\eta_2,\ldots,\eta_{n-3})} = A_{(\sigma_i\sigma_j,\varepsilon_1+\eta_1,\varepsilon_2+\eta_2,\ldots,\varepsilon_{n-3}+\eta_{n-3})},$$

for all $A_{(\sigma_i,\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{n-3})}, A_{(\sigma_j,\eta_1,\eta_2,\ldots,\eta_{n-3})} \in \mathcal{P}(X)$, the set $\mathcal{P}(X)$ forms a non-abelian group with the neutral element $A_{(\sigma_0,0,0,\ldots,0)} = \emptyset$.

All of this means that the set $\mathcal{P}(X)$ together with both operations, the addition and multiplication, deifned above is a nonabelian and commutative nondistributive ring.

If the distibutiveness held in $\mathcal{P}(X)$, writing A_{σ_i} instead of $A_{(\sigma_i,0,0,\ldots,0)}$ for every $i \in \{0, 1, 2, \ldots, 7\}$, we would obtain

$$\begin{aligned} A_{\sigma_1}A_{\sigma_2} = A_{\sigma_1}(A_{\sigma_1} + A_{\sigma_1}) = A_{\sigma_1}A_{\sigma_1} + A_{\sigma_1}A_{\sigma_1} = A_{\sigma_1} \cap A_{\sigma_1} + A_{\sigma_1} \cap A_{\sigma_1} = \\ = A_{\sigma_1} + A_{\sigma_1} = A_{\sigma_2} \end{aligned}$$

and thus

$$A_{\sigma_2} = A_{\sigma_2} \cap A_{\sigma_2} = (A_{\sigma_1} + A_{\sigma_1})A_{\sigma_2} = A_{\sigma_1}A_{\sigma_2} + A_{\sigma_1}A_{\sigma_2} = A_{\sigma_2} + A_{\sigma_2} = A_{\sigma_0},$$

a contradiction. $\square$

   




Example. Let $Q_8 \cup \{0\}$ be the noncommutative semigroup with unit and zero, obtained from the quaternion group

$$Q_8 = \Big\{ \pm 1, \pm i, \pm j, \pm k \Big\}$$

of order eight by adjoining the zero element.

Assume that elements of the set $Q_8 \cup \{0\}$ are indexed by elements of the group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ as follows:

$$\begin{array}{lll} x_{(0,0)} = 0, & x_{(1,0)} = 1, & x_{(2,0)} = -1 \\ x_{(0,1)} = -i, & x_{(0,2)} = i, & x_{(1,1)} = -j \\ x_{(2,2)} = j, & x_{(2,1)} = -k, & x_{(1,2)} = k. \end{array}$$

With the addition defined by

$$x_{(a,b)} + x_{(c,d)} = x_{(a+c,b+d)}$$

for all $a, b, c, d \in \mathbb{Z}/3\mathbb{Z}$, the set $Q_8 \cup \{0\}$ forms an abelian group.

All of this means that the set $Q_8 \cup \{0\}$ together with both operations, the addition and multiplication, defined above is an abelian and noncommutative near field.

The left distributiveness does not hold since

$$i(1 + i) = ik = -j$$

but

$$i + i^2 = i - 1 = j.$$

$\square$

   

Definition. Let N be a nondistributive ring, and let $S \subseteq N$ be a multiplicatively closed set. We call a nondistributive ring $S^{-1}N$ *a nondistributive ring of left quotients of N with respect to S* if there exists a homomorphism $\eta\colon N \to S^{-1}N$ of nondistributive rings, for which

(1) $\eta(s)$ is invertible in $S^{-1}N$ for every $s \in S$.

(2) $\eta(s)$ is left distributive in $S^{-1}N$ for every $s \in S$.

(3) every element of $S^{-1}N$ is of the form $\eta(s)^{-1}\eta(n)$ where $n \in N$ and $s \in S$.

(4) $\ker \eta = \left\{ n \in N \mid r(s+n) = rs \text{ for some } r, s \in S \right\}$. $\square$

   

For a multiplicatively closed set S in a nondistributive ring N, we let

$$U = \left\{ n \in N \mid r_2(s_2 + nr_1 - s_1) = r_2 s_2 \text{ for some } r_2, s_2 \in S \right\} \supseteq S.$$

Theorem. A nondistributive ring N has a nondistributive ring of left quotients $S^{-1}N$ with respect to a multiplicatively closed set $S \subseteq N$ if and only if S satisfies the following conditions

(1) for all $n \in N$ and $s \in S$ there exist $n_1 \in N$ and $s_1, r_2, s_2 \in S$ such that $r_2(s_2 + n_1 s - s_1 n) = r_2 s_2$.

(2) for all $m, n \in N$ and $s \in U$ there exist $r_1, s_1 \in S$ such that $r_1(s_1 + s(m + n) - sn - sm) = r_1 s_1$.

(3) for all $m, n \in N$ if $r(s + tmu - tnu) = rs$ for some $r, s, t, u \in S$, then $r_1(s_1 + m - n) = r_1 s_1$ for some $r_1, s_1 \in S$.

(4) for all $m, n \in N$ if $r(s + m) = rs$ and $t(u + n) = tu$ for some $r, s, t, u \in S$, then $r_1(s_1 + m - n) = r_1 s_1$ for some $r_1, s_1 \in S$.

(5) for all $m, n \in N$ if $r(s+n) = rs$ for some $r, s \in S$, then $r_1(s_1+m+n-m) = r_1 s_1$ for some $r_1, s_1 \in S$.

(6) for all $k, l, m, n \in N$ if $r(s+m-n) = rs$ for some $r, s \in S$, then $r_1(s_1 + kml - knl) = r_1 s_1$ for some $r_1, s_1 \in S$.

The additional assumption that N is an abelian nondistributive ring (respectively, a commutative nondistributive ring, a left nearring, a right nearring) implies the same for $S^{-1}N$. $\square$




Corollary. If S is a multiplicatively closed set in a nondistributive ring N, and if every element of the set U defined above is left distributive in N, then the nondistributive ring of left quotients $S^{-1}N$ exists if and only if S satisfies the following conditions

(1) for all $n \in$ and $s \in S$ there exist $n_1 \in N$ and $s_1 \in S$ such that $n_1 s = s_1 n$. Analogously as in the ring theory, we call this postulate *the left Ore condition* with respect to S.

(2) for all $m, n \in N$ if $ms = ns$ for some $s \in S$, then $s_1 m = s_1 n$ for some $s_1 \in S$. □

Corollary. If S is a multiplicatively closed set of right cancellable elements in a nondistributive ring N, and if every element of the set U defined above is left distributive in N, then the nondistributive ring of left quotients $S^{-1}N$ exists if and only if N satisfies the left Ore condition with respect to S. Under the additional assumption that every element of S is also left cancellable, the nondistributive ring N embeds into the nondistributive ring of left quotients $S^{-1}N$. □

Corollary. If a nondistributive ring N satisfies both the right cancellation law and the left Ore condition with respect to a multiplicatively closed set $S \subseteq N$, and if every element of the set U defined above is left distributive in N, then

(1) every element of U is also left cancellable.

(2) the nondistributive ring N embeds into the nondistributive ring of left quotients $S^{-1}N$. $\square$

   

Example. Any (not necessarily unital) ring R, in which the circle operation

$$x \circ y = x + y - xy$$

where $x, y \in R$, substitutes for the multiplication, satisfies only the first two of postulates from the definition of a nondistributive ring.

The neutral element of the addition is also the neutral element of the circle operation. $\square$