Nondistributive rings

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Definition. By a ring we mean a set $R$ of no fewer than two elements, together with two binary operations called the addition and multiplication, in which

(1) $R$ is an abelian group with respect to the addition.

(2) $R$ is a semigroup with unit with respect to the multiplication.

(3) $(x + y)z = xz + yz$ and $x(y + z) = xy + xz$ for all $x, y, z \in R$. □

From the last postulate

$$0x = (0 + 0)x = 0x + 0x \quad \text{and} \quad x0 = x(0 + 0) = x0 + x0,$$

it follows that

$$0x = x0 = 0$$

for every $x \in R$. 

Example. For an abelian additive group $G$, we denote by $End(G)$ the set of group endomorphisms of $G$. With the addition defined by

$$(f + g)(x) = f(x) + g(x)$$

and the multiplication defined by

$$(fg)(x) = f(g(x))$$

for all $f, g \in End(G)$ and $x \in G$, the set $End(G)$ forms a ring.

The left distributiveness in $End(G)$ follows from the additivity of group endomorphisms

$$(f(g + h))(x) = f(g(x) + h(x)) = f(g(x)) + f(h(x)) = (fg + fh)(x)$$

for all $f, g, h \in End(G)$ and $x \in G$.

From the additivity of group endomorphisms

$$f(0) = f(0 + 0) = f(0) + f(0),$$

it follows that

$$f(0) = 0$$

for every $f \in End(G)$. 

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Example. For a (not necessarily abelian) additive group $G$, we denote by $M_0(G)$ the set of maps from $G$ into itself preserving 0

$$M_0(G) = \left\{ f: G \to G \mid f(0) = 0 \right\}.$$ 

With the addition defined by

$$(f + g)(x) = f(x) + g(x)$$

for all $f, g \in M_0(G)$ and $x \in G$, the set $M_0(G)$ forms a (not necessarily abelian) group.

With the multiplication defined by

$$(fg)(x) = f(g(x))$$

for all $f, g \in M_0(G)$ and $x \in G$, the set $M_0(G)$ forms a semigroup with unit $id_G: G \to G$, $id_G(x) = x$ for every $x \in G$. 
The right distributiveness in $M_0(G)$ follows from both definitions of the addition and multiplication in $M_0(G)$

$((f + g)h)(x) = (f + g)(h(x)) = f(h(x)) + g(h(x)) = (fh + gh)(x)$

for all $f, g, h \in M_0(G)$ and $x \in G$.

The left distributiveness in $M_0(G)$ does not hold

$(f(g + h))(x) = f(g(x) + h(x)) \neq f(g(x)) + f(h(x)) = (fg + fh)(x)$

where $f, g, h \in M_0(G)$ and $x \in G$, unless $f$ is a group endomorphism of $G$.

For the zero map $0_G: G \to G$, $0_G(x) = 0$ where $x \in G$, from the definition of the set $M_0(G)$, it follows that

$\quad (f0_G)(x) = f(0) = 0 = 0_G(x)$

for all $f \in M_0(G)$ and $x \in G$. 

\qed
Definition. By a near ring we mean a set $N$ of no fewer than two elements, together with two binary operations called the addition and multiplication, in which

(1) $N$ is a (not necessarily abelian) group with respect to the addition.

(2) $N$ is a semigroup with unit with respect to the multiplication.

(3) $(x + y)z = xz + yz$ for all $x, y, z \in N$.

(4) $x0 = 0$ for every $x \in N$. This postulate means that we require a near ring to be zerosymmetric.

From the third postulate $0x = (0 + 0)x = 0x + 0x$, it follows that $0x = 0$ for every $x \in R$. 
Definition. By *a nondistributive ring* we mean a set \( N \) of no fewer than two elements, together with two binary operations called the addition and multiplication, in which

1. \( N \) is a (not necessarily abelian) group with respect to the addition, with the neutral element denoted by 0.

2. \( N \) is a semigroup with unit with respect to the multiplication, with the neutral element denoted by 1.

3. \( 0x = x0 = 0 \) for every \( x \in N \). This postulate is called *zerosymmetric*.

We say that a nondistributive ring is *abelian* (respectively, *commutative*) if the additive group mentioned above is abelian (respectively, the multiplicative semigroup mentioned above is commutative).
Example. For a nonempty set $X$ with a fixed element $0$, we denote by $Map_0(X)$ the set of maps from $X$ into itself preserving $0$

$$Map_0(X) = \{ f : X \to X \mid f(0) = 0 \}.$$ 

With the multiplication defined by

$$(fg)(x) = f(g(x))$$

for all $f, g \in Map_0(X)$ and $x \in X$, the set $Map_0(X)$ forms a semigroup with unit $id_X : X \to X$, $id_X(x) = x$ for every $x \in X$.

For the zero map $0_X : X \to X$, $0_X(x) = 0$ where $x \in X$, we have

$$(0_Xf)(x) = 0 = 0_X(x)$$

and

$$(f0_X)(x) = f(0) = 0 = 0_X(x)$$

where $f \in Map_0(X)$ and $x \in X$. 
Assume that elements of the set $Map_0(X)$ are indexed by elements of an additive group $G$, with the zero map $0_X = f_0$. We can make the above assumption, since every nonempty set admits a group structure (the statement is equivalent to the Axiom of Choice). With the addition defined by

$$f_a + f_b = f_{a+b}$$

for all $a, b \in G$, the set $Map_0(X)$ forms a group with the neutral element $f_0 = 0_X$.

All of this means that the set $Map_0(X)$ together with both operations, the addition and multiplication, defined above is a nondistributive ring. \qed
For a nondistributive ring $N$, we denote by $N^+$ the additive group of $N$.

A well known result in the ring theory asserts that

(1) every ring $R$ is isomorphic to the ring $\text{End}(R_R)$ of endomorphisms of $R$ viewed as a right module over itself.

(2) $\text{End}(R_R)$ is a subring of the ring $\text{End}(R^+)$ of group endomorphisms of $R^+$. 
Example. For a nondistributive ring $N$, we denote by $r.Hom(N)$ the set of right homogeneous maps from $N$ into itself

$$r.Hom(N) = \{ f: N \to N \mid f(xn) = f(x)n \text{ for all } n, x \in N \}.$$ 

With the multiplication defined by

$$(fg)(x) = f(g(x))$$

for all $f, g \in r.Hom(N)$ and $x \in N$, the set $r.Hom(N)$ forms a semigroup with unit $id_N: N \to N$, $id_N(x) = x$ and zero $0_N: N \to N$, $0_N(x) = 0$ for every $x \in N$. 
We define a map $\lambda: N \to r.Hom(N)$ by sending $m \in N$ to the left multiplication $\lambda_m: N \to N$ on $N$ defined by $\lambda_m(x) = mx$ for every $x \in N$. Since

$$\lambda_m(xn) = m(xn) = (mx)n = \lambda_m(x)n$$

for all $m, n, x \in N$, we have indeed $\lambda_m \in r.Hom(N)$. Since $\lambda_0 = 0_N$, $\lambda_1 = id_N$ and $\lambda_{mn}(x) = (mn)x = m(nx) = (\lambda_m\lambda_n)(x)$ for all $m, n, x \in N$, it follows that the map $\lambda$ is a semigroup homomorphism. It is also evident that for all $m, n \in N$ if $\lambda_m = \lambda_n$, then

$$m = \lambda_m(1) = \lambda_n(1) = n,$$

and that

$$f(x) = f(1)x = \lambda_{f(1)}(x)$$

for all $f \in r.Hom(N)$, $x \in N$. 
All of this means that $\lambda : N \to r.\text{Hom}(N)$ is a semigroup isomorphism, and, in consequence, elements of the set $r.\text{Hom}(N)$ are indexed by elements of the additive group $N^+$. With the addition defined by

$$\lambda_m + \lambda_n = \lambda_{m+n}$$

for all $m, n \in N^+$, the semigroup $r.\text{Hom}(N)$ forms a nondistributive ring isomorphic to $N$. □
Theorem. Every nondistributive ring $N$ embeds into the nondistributive ring $Map_0(N)$ of maps from $N$, viewed as a set, into itself preserving $0$.

Proof. According to the previous example, the map $\lambda \colon N \to r.Hom(N)$ defined by $\lambda(m) = \lambda_m$ for every $m \in N$ is a nondistributive ring isomorphism.

It is also evident that $r.Hom(N)$ is a subsemigroup of the multiplicative semi-group $Map_0(N)$ with unit $id_N$ and zero $0_N$.

If $Map_0(N)$ is an infinite set and if $\mathcal{F}(Map_0(N))$ denotes the ring (without unit) of finite subsets of $Map_0(N)$, then $N^+ \times \mathcal{F}(Map_0(N))^+$ is a group of order

$$|N^+ \times \mathcal{F}(Map_0(N))^+| = |\mathcal{F}(Map_0(N))^+| = |Map_0(N)|,$$

which means that elements of the set $Map_0(N)$ can be indexed by elements of the additive group $N^+ \times \mathcal{F}(Map_0(N))^+$, with $f_m = \lambda_m$ for every $m \in N^+$. From this we conclude that $r.Hom(N) \subseteq Map_0(N)$ also viewed as nondistributive rings.

The same conclusion holds if $Map_0(N)$ is a finite set. □
Example. For a nonempty and finite set \( X = \{x_1, x_2, \ldots, x_n\} \), we denote by \( \mathcal{P}(X) \) the family of subsets of \( X \).

With the multiplication defined by

\[
AB = A \cap B
\]

for all \( A, B \in \mathcal{P}(X) \), the set \( \mathcal{P}(X) \) forms a commutative semigroup with unit \( X \) and zero \( \emptyset \).

Assume that elements of the set \( \mathcal{P}(X) \) are indexed by elements of \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z} \), the direct product of \( n \)-copies of the additive group \( \mathbb{Z}/2\mathbb{Z} \) as follows: for all \( i \in \{1, 2, \ldots, n\} \) and \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \mathbb{Z}/2\mathbb{Z} \) we write \( x_i \in A(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \) if and only if \( \varepsilon_i = 1 \). Then the addition defined by

\[
A(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) + A(\eta_1, \eta_2, \ldots, \eta_n) = A(\varepsilon_1 + \eta_1, \varepsilon_2 + \eta_2, \ldots, \varepsilon_n + \eta_n)
\]

coincides with the symmetric difference

\[
A(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \triangle A(\eta_1, \eta_2, \ldots, \eta_n) = \left( A(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \setminus A(\eta_1, \eta_2, \ldots, \eta_n) \right) \cup \left( A(\eta_1, \eta_2, \ldots, \eta_n) \setminus A(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \right).
\]

All this means that the set \( \mathcal{P}(X) \) together with both operations, the addition and multiplication, defined above is a commutative ring.
Example. Let $\mathcal{P}(X)$ be the same commutative semigroup with unit and zero as previously.

Assume that this time elements of the set $\mathcal{P}(X)$ are indexed by elements of the group $D_8 \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \ldots \times \mathbb{Z}/2\mathbb{Z}$, where

$$D_8 = \{ \sigma_0 = (1), \sigma_1 = (1, 2, 3, 4), \sigma_2 = (1, 3)(2, 4), \sigma_3 = (1, 4, 3, 2),$$

$$\sigma_4 = (2, 4), \sigma_5 = (1, 2)(3, 4), \sigma_6 = (1, 3), \sigma_7 = (1, 4)(2, 3) \}$$

is the dihedral group of order eight, provided that $\emptyset = A(\sigma_0, 0, 0, \ldots, 0)$. With the addition defined by

$$A(\sigma_i, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-3}) + A(\sigma_j, \eta_1, \eta_2, \ldots, \eta_{n-3}) = A(\sigma_i \sigma_j, \varepsilon_1 + \eta_1, \varepsilon_2 + \eta_2, \ldots, \varepsilon_{n-3} + \eta_{n-3}),$$

for all $A(\sigma_i, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-3}), A(\sigma_j, \eta_1, \eta_2, \ldots, \eta_{n-3}) \in \mathcal{P}(X)$, the set $\mathcal{P}(X)$ forms a non-abelian group with the neutral element $A(\sigma_0, 0, 0, \ldots, 0) = \emptyset$. 
All of this means that the set $\mathcal{P}(X)$ together with both operations, the addition and multiplication, defined above is a nonabelian and commutative nondistributive ring.

If the distributiveness held in $\mathcal{P}(X)$, writing $A_{\sigma_i}$ instead of $A_{(\sigma_i,0,0,\ldots,0)}$ for every $i \in \{0, 1, 2, \ldots, 7\}$, we would obtain

$$A_{\sigma_1}A_{\sigma_2} = A_{\sigma_1}(A_{\sigma_1} + A_{\sigma_1}) = A_{\sigma_1}A_{\sigma_1} + A_{\sigma_1}A_{\sigma_1} = A_{\sigma_1} \cap A_{\sigma_1} + A_{\sigma_1} \cap A_{\sigma_1} = A_{\sigma_1} + A_{\sigma_1} = A_{\sigma_2}$$

and thus

$$A_{\sigma_2} = A_{\sigma_2} \cap A_{\sigma_2} = (A_{\sigma_1} + A_{\sigma_1})A_{\sigma_2} = A_{\sigma_1}A_{\sigma_2} + A_{\sigma_1}A_{\sigma_2} = A_{\sigma_2} + A_{\sigma_2} = A_{\sigma_0},$$

a contradiction. □
Example. Let $Q_8 \cup \{0\}$ be the noncommutative semigroup with unit and zero, obtained from the quaternion group

$$Q_8 = \{ \pm 1, \pm i, \pm j, \pm k \}$$

of order eight by adjoining the zero element.

Assume that elements of the set $Q_8 \cup \{0\}$ are indexed by elements of the group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ as follows:

$$
\begin{align*}
x(0,0) &= 0, & x(1,0) &= 1, & x(2,0) &= -1 \\
x(0,1) &= -i, & x(0,2) &= i, & x(1,1) &= -j \\
x(2,2) &= j, & x(2,1) &= -k, & x(1,2) &= k.
\end{align*}
$$
With the addition defined by
\[ x_{(a,b)} + x_{(c,d)} = x_{(a+c, b+d)} \]
for all \( a, b, c, d \in \mathbb{Z}/3\mathbb{Z} \), the set \( Q_8 \cup \{0\} \) forms an abelian group.

All of this means that the set \( Q_8 \cup \{0\} \) together with both operations, the addition and multiplication, defined above is an abelian and noncommutative near field.

The left distributiveness does not hold since
\[ i(1 + i) = ik = -j \]
but
\[ i + i^2 = i - 1 = j. \]
Definition. Let $N$ be a nondistributive ring, and let $S \subseteq N$ be a multiplicatively closed set. We call a nondistributive ring $S^{-1}N$ a nondistributive ring of left quotients of $N$ with respect to $S$ if there exists a homomorphism $\eta: N \to S^{-1}N$ of nondistributive rings, for which

1. $\eta(s)$ is invertible in $S^{-1}N$ for every $s \in S$.

2. $\eta(s)$ is left distributive in $S^{-1}N$ for every $s \in S$.

3. every element of $S^{-1}N$ is of the form $\eta(s)^{-1}\eta(n)$ where $n \in N$ and $s \in S$.

4. $\ker \eta = \{ n \in N \mid r(s + n) = rs \text{ for some } r, s \in S \}$. 

$\blacksquare$
For a multiplicatively closed set $S$ in a nondistributive ring $N$, we let
\[
U = \{ n \in N \mid r_2(s_2 + nr_1 - s_1) = r_2s_2 \text{ for some } r_2, s_2 \in S \} \supseteq S.
\]

Theorem. A nondistributive ring $N$ has a nondistributive ring of left quotients $S^{-1}N$ with respect to a multiplicatively closed set $S \subseteq N$ if and only if $S$ satisfies the following conditions

(1) for all $n \in N$ and $s \in S$ there exist $n_1 \in N$ and $s_1, r_2, s_2 \in S$ such that $r_2(s_2 + n_1s - s_1n) = r_2s_2$.

(2) for all $m, n \in N$ and $s \in U$ there exist $r_1, s_1 \in S$ such that $r_1(s_1 + s(m + n) - sn - sm) = r_1s_1$.

(3) for all $m, n \in N$ if $r(s + tmu - tnu) = rs$ for some $r, s, t, u \in S$, then $r_1(s_1 + m - n) = r_1s_1$ for some $r_1, s_1 \in S$.

(4) for all $m, n \in N$ if $r(s + m) = rs$ and $t(u + n) = tu$ for some $r, s, t, u \in S$, then $r_1(s_1 + m - n) = r_1s_1$ for some $r_1, s_1 \in S$. 

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(5) for all $m, n \in \mathbb{N}$ if $r(s + n) = rs$ for some $r, s \in S$, then $r_1(s_1 + m + n - m) = r_1s_1$ for some $r_1, s_1 \in S$.

(6) for all $k, l, m, n \in \mathbb{N}$ if $r(s + m - n) = rs$ for some $r, s \in S$, then $r_1(s_1 + kml - knl) = r_1s_1$ for some $r_1, s_1 \in S$.

The additional assumption that $N$ is an abelian nondistributive ring (respectively, a commutative nondistributive ring, a left nearring, a right nearring) implies the same for $S^{-1}N$. 

\[\square\]
Corollary. If $S$ is a multiplicatively closed set in a nondistributive ring $N$, and if every element of the set $U$ defined above is left distributive in $N$, then the nondistributive ring of left quotients $S^{-1}N$ exists if and only if $S$ satisfies the following conditions

(1) for all $n \in N$ and $s \in S$ there exist $n_1 \in N$ and $s_1 \in S$ such that $n_1s = s_1n$. Analogously as in the ring theory, we call this postulate the left Ore condition with respect to $S$.

(2) for all $m,n \in N$ if $ms = ns$ for some $s \in S$, then $s_1m = s_1n$ for some $s_1 \in S$.

Corollary. If $S$ is a multiplicatively closed set of right cancellable elements in a nondistributive ring $N$, and if every element of the set $U$ defined above is left distributive in $N$, then the nondistributive ring of left quotients $S^{-1}N$ exists if and only if $N$ satisfies the left Ore condition with respect to $S$. Under the additional assumption that every element of $S$ is also left cancellable, the nondistributive ring $N$ embeds into the nondistributive ring of left quotients $S^{-1}N$. 

Corollary. If a nondistributive ring $N$ satisfies both the right cancellation law and the left Ore condition with respect to a multiplicatively closed set $S \subseteq N$, and if every element of the set $U$ defined above is left distributive in $N$, then

(1) every element of $U$ is also left cancellable.

(2) the nondistributive ring $N$ embeds into the nondistributive ring of left quotients $S^{-1}N$. 

$\square$
Example. Any (not necessarily unital) ring $R$, in which the circle operation

$$x \circ y = x + y - xy$$

where $x, y \in R$, substitutes for the multiplication, satisfies only the first two of postulates from the definition of a nondistributive ring.

The neutral element of the addition is also the neutral element of the circle operation.