C4- and D4-Modules via Perspective Submodules

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Outline

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Let $R$ be a ring and $M$ a right $R$-module.

$M$ is called a $Ci$-module if it has the following $Ci$ properties for $i = 1, 2, 3$.

$C1$: Every submodule of $M$ is essential in a direct summand of $M$.

$C2$: Whenever $A$ and $B$ are submodules of $M$ with $A \cong B$ and $B \subseteq^{\oplus} M$, then $A \subseteq^{\oplus} M$.

$C3$: Whenever $A$ and $B$ are direct summands of $M$ with $A \cap B = 0$, then $A + B \subseteq^{\oplus} M$. 
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A module $M$ is called a *Di-module* if it satisfies the following *Di*-conditions.

**D1:** For every submodule $A$ of $M$, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $A \cap M_2 \ll M_2$.

**D2:** Whenever $A$ and $B$ are submodules of $M$ with $M/A \cong B$ and $B \subseteq \oplus M$, then $A \subseteq \oplus M$.

**D3:** Whenever $A$ and $B$ are direct summands of $M$ with $A + B = M$, then $A \cap B \subseteq \oplus M$. 
Definition

$M$ is called continuous if it is $C_1$ and $C_2$, and quasi-continuous if it is $C_1$ and $C_3$.

\[ C_2 \Rightarrow C_3 \]

quasi-injective $\Rightarrow$ continuous

Definition

A module $M$ is called discrete if it is both a $D_1$- and a $D_2$-module, quasi-discrete if it is both a $D_1$- and a $D_3$-module.

\[ \text{quasi-projective} \Rightarrow D_2 \Rightarrow D_3 \]
Definition

$M$ is called *continuous* if it is $C_1$ and $C_2$, and *quasi-continuous* if it is $C_1$ and $C_3$.

$$C_2 \Rightarrow C_3$$

quasi-injective $\Rightarrow$ continuous

Definition

A module $M$ is called *discrete* if it is both a $D_1$- and a $D_2$-module, *quasi-discrete* if it is both a $D_1$- and a $D_3$-module.

quasi-projective $\Rightarrow$ $D_2 \Rightarrow D_3$
Proposition [Amin et al., 2015]

If $M$ is a $C_3$-module, then for every decomposition $M = A \oplus B$ and every homomorphism $f : A \rightarrow B$ with $\ker f \subseteq \oplus A$, then $\text{Im} f \subseteq \oplus B$.

[Amin et al., 2015]

The following are equivalent for a module $M$:

(1) If $M = A \oplus B$ and $f : A \rightarrow B$ is a monomorphism, then $\text{Im} f \subseteq \oplus B$.

(2) If $M = A \oplus B$ and $f : A \rightarrow B$ is a homomorphism with $\ker f \subseteq \oplus A$, then $\text{Im} f \subseteq \oplus B$. 
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A module $M$ is called a C4-module if it satisfies any of the equivalent conditions in the above.

C3 $\Rightarrow$ C4

A module $M$ is called (summand-) square-free if whenever $N \subseteq M$ and $N = Y_1 \oplus Y_2$ with $Y_1 \cong Y_2$ (and $Y_1, Y_2 \subseteq \oplus M$), then $Y_1 = Y_2 = 0$.

summand-square-free $\Rightarrow$ C4
Definition [Ding et al., 2017]

A module $M$ is called a $C4$-module if it satisfies any of the equivalent conditions in the above.

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summand-square-free $\Rightarrow$ $C4$
Definition [Ding et al., 2017]

A module $M$ is called a \textit{C4-module} if it satisfies any of the equivalent conditions in the above.

$C3 \Rightarrow C4$

Definition

A module $M$ is called (\textit{summand-}) \textit{square-free} if whenever $N \subseteq M$ and $N = Y_1 \oplus Y_2$ with $Y_1 \cong Y_2$ (and $Y_1, Y_2 \subseteq \oplus M$), then $Y_1 = Y_2 = 0$.

\textit{summand-square-free} $\Rightarrow$ \textit{C4}
Definition

Two direct summands $A$ and $B$ of a module $M$ are *perspective* exactly when there exists a common direct sum complement $C$, i.e.,

$$M = A \oplus C = B \oplus C.$$ 

Theorem 1.1

The following are equivalent for a module $M$:

1. $M$ is a $C_4$-module.
2. If $A$ and $B$ are perspective direct summands of $M$ with $A \cap B = 0$, then $A \oplus B \subseteq \langle M \rangle$.
3. If $A$ and $B$ are perspective direct summands of $M$ with $A \cap B \subseteq \langle M \rangle$, then $A + B \subseteq \langle M \rangle$. 
## Definition

Two direct summands $A$ and $B$ of a module $M$ are *perspective* exactly when there exists a common direct sum complement $C$, i.e.,

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Theorem 1.2

Let $M = \bigoplus_{i \in I} M_i$ be a module, where $M_i$ is fully invariant in $M$ for every $i \in I$. Then $M$ is a $C4$-module if and only if each $M_i$ is a $C4$-module.
Definition 1.3

A module $M$ is said to satisfy the restricted ACC on summands (r-ACC on summands, for short) if, $M$ has no strictly ascending chains of non-zero summands

\[
A_1 \subseteq A_2 \subseteq \cdots \quad B_1 \subseteq B_2 \subseteq \cdots
\]

with $A_i \cong B_i$ and $A_i \cap B_i = 0$ for all $i \geq 1$.

ACC on summands $\Rightarrow$ r-ACC on summands

summand-square-free $\Rightarrow$ r-ACC on summands
Definition 1.3

A module $M$ is said to satisfy the restricted ACC on summands ($r$-ACC on summands, for short) if, $M$ has no strictly ascending chains of non-zero summands

$$A_1 \subsetneq A_2 \subsetneq \cdots$$
$$B_1 \subsetneq B_2 \subsetneq \cdots$$

with $A_i \cong B_i$ and $A_i \cap B_i = 0$ for all $i \geq 1$.

$\text{ACC on summands} \Rightarrow r$ - ACC on summands

summand-square-free $\Rightarrow r$ - ACC on summands
Theorem 1.4

If $M$ is a $C_4$-module that satisfies the restricted ACC on summands, then $M = A \oplus B \oplus K$ where $A \cong B$ is a $C_2$-module and $K$ is a summand-square-free module.
Let $M$ be a right $R$-module and $S = \text{End}_R(M)$.

- If $S$ is a right $C_2$-module, then $M_R$ is $C_2$; the converse is true if $\ker(\alpha)$ is generated by $M$ whenever $\alpha$ is such that $r_S(\alpha)$ is a direct summand of $S$ [Nicholson and Yousif, 2003].

- If $S$ is a right $C_3$-module, then $M_R$ is $C_3$ [Amin et al., 2015]; the converse is true if for every pair of idempotents $e, f \in S$ with $eS \cap fS = 0$ we have $eM \cap fM = 0$ by [Mazurek et al., 2015].

- If $S$ is a right $C_4$-module, then $M_R$ is $C_4$; the converse is true if for every pair of idempotents $e, f \in S$ with $eS \cap fS = 0$ we have $eM \cap fM = 0$ [Ding et al., 2017].
Let $M$ be a right $R$-module and $S = \text{End}_R(M)$.

- If $S_S$ is a right $C_2$-module, then $M_R$ is $C_2$; the converse is true if $\ker(\alpha)$ is generated by $M$ whenever $\alpha$ is such that $r_S(\alpha)$ is a direct summand of $S_S$ [Nicholson and Yousif, 2003].

- If $S_S$ is a right $C_3$-module, then $M_R$ is $C_3$ [Amin et al., 2015]; the converse is true if for every pair of idempotents $e, f \in S$ with $eS \cap fS = 0$ we have $eM \cap fM = 0$ by [Mazurek et al., 2015].

- If $S_S$ is a right $C_4$-module, then $M_R$ is $C_4$; the converse is true if for every pair of idempotents $e, f \in S$ with $eS \cap fS = 0$ we have $eM \cap fM = 0$ [Ding et al., 2017].
Theorem 2.1

Let $M$ be a right $R$-module with $S = \text{End}_R(M)$. Then $S$ is a right $C4$-ring, if $M$ is a $C4$-module and one of the following is satisfied.

1. $M$ is k-local-retractable.
2. For any $\alpha \in S$, $\text{ker}(\alpha)$ is generated by $M$.
3. For every pair of perspective idempotents $e, f \in S$ with $eS \cap fS = 0$, we have $eM \cap fM = 0$. 
Proposition 2.2

Let $M$ be a right $R$-module with $S = \text{End}_R(M)$. Then the following are equivalent:

(1) $M$ is a $C4$-module.

(2) For every pair of perspective idempotents $e, f \in S$ with $eM \cap fM = 0$, there exists an idempotent $g$ of $S$ such that $eM = gM$ and $fM \subseteq (1 - g)M$.

Proposition 2.3

A right $R$-module $M$ is $C4$ if and only if for any idempotents $e, f \in \text{End}_R(M)$, if $\text{kere} = \text{kerf} = \text{ker}(e - f)$, then $(1 - e)fM \subseteq \oplus M.$
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Proposition 3.1

Let \( R_i \ (i \in I) \) be any collection of rings, and let \( R \) be the direct product \( \prod_{i \in I} R_i \). Then \( R \) is a right \( C4 \)-ring if and only if every \( R_i \) is a right \( C4 \)-ring.

Proposition 3.2

If \( R \) is a right \( C4 \)-ring, then so is \( eRe \) for any idempotent \( e \in R \) such that \( ReR = R \).
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Let $R_i (i \in I)$ be any collection of rings, and let $R$ be the direct product $\prod_{i \in I} R_i$. Then $R$ is a right C4-ring if and only if every $R_i$ is a right C4-ring.

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If $R$ is a right C4-ring, then so is $eRe$ for any idempotent $e \in R$ such that $ReR = R$. 
Example 3.3

The condition $ReR = R$ is not superfluous in Proposition 3.2: Let $R$ be the algebra of matrices, over a field $F$, of the form

$$
\begin{pmatrix}
a & x & 0 & 0 & 0 & 0 \\
0 & b & 0 & 0 & 0 & 0 \\
0 & 0 & c & y & 0 & 0 \\
0 & 0 & 0 & a & 0 & 0 \\
0 & 0 & 0 & 0 & b & z \\
0 & 0 & 0 & 0 & 0 & c \\
\end{pmatrix}
$$

$\Rightarrow e := e_{11} + e_{22} + e_{33} + e_{44} + e_{55}$, where $e_{ij}$ are the matrices with $(i,j)$-entry 1 and all other entries zero.

$\Rightarrow e$ is an idempotent of $R$ such that $ReR \neq R$.

$\Rightarrow R$ is a quasi-Frobenius ring by [Koike, 1995] $\Rightarrow R$ is right $C4$.

$\Rightarrow eRe \cong \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ is not a right $C4$-ring.
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\( R \) is a quasi-Frobenius ring by [Koike, 1995] \( \Rightarrow \) \( R \) is right C4.

\( eRe \cong \begin{pmatrix} F & F \\ 0 & F \end{pmatrix} \) is not a right C4-ring.
Right C4 rings

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\begin{pmatrix}
a & x & 0 & 0 & 0 & 0 \\
0 & b & 0 & 0 & 0 & 0 \\
0 & 0 & c & y & 0 & 0 \\
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0 & 0 & 0 & 0 & b & z \\
0 & 0 & 0 & 0 & 0 & c
\end{pmatrix}
$$

- $e := e_{11} + e_{22} + e_{33} + e_{44} + e_{55}$, where $e_{ij}$ are the matrices with $(i,j)$-entry 1 and all other entries zero.
- $e$ is an idempotent of $R$ such that $ReR \not= R$.
- $R$ is a quasi-Frobenius ring by [Koike, 1995] $\Rightarrow R$ is right C4.
- $eRe \cong \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ is not a right C4-ring.
Let $R$ be a ring and $M$ an $R$-$R$-bimodule. Then the trivial extension $R \rtimes M$ is a ring whose underlying group is $R \times M$ with the multiplication defined by

$$(r, m)(s, n) = (rs, rn + ms)$$

where $r, s \in R$ and $m, n \in M$.

Proposition 3.4

Let $R$ be a ring and $M$ an $R$-$R$-bimodule.

1) If $R \rtimes M$ is a right C4-ring, and for any idempotents $e, f \in R$, $eR \cap fR = 0$ implies $eM \cap fM = 0$, then $R$ is a right C4-ring.

2) If $R$ is a right C4-ring, and $eM(1 - e) = 0$ for any idempotent $e \in R$, then $R \rtimes M$ is a right C4-ring.
Let $R$ be a ring and $M$ an $R$-$R$-bimodule. Then the trivial extension $R \triangleleft M$ is a ring whose underlying group is $R \times M$ with the multiplication defined by

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where $r, s \in R$ and $m, n \in M$.

**Proposition 3.4**

Let $R$ be a ring and $M$ an $R$-$R$-bimodule.

1) If $R \triangleleft M$ is a right $C4$-ring, and for any idempotents $e, f \in R$, $eR \cap fR = 0$ implies $eM \cap fM = 0$, then $R$ is a right $C4$-ring.

2) If $R$ is a right $C4$-ring, and $eM(1 - e) = 0$ for any idempotent $e \in R$, then $R \triangleleft M$ is a right $C4$-ring.
**Proposition [Yousif et al., 2014]**

If $M$ is a $D3$-module, then for every decomposition $M = A \oplus B$ and every homomorphism $f : A \rightarrow B$ with $\text{Im} f \subseteq \oplus B$, then $\ker f \subseteq \oplus A$.

**[Yousif et al., 2014]**

The following are equivalent for a module $M$:

1. If $M = A \oplus B$ with $A, B \subseteq M$ and $f : A \rightarrow B$ is an epimorphism, then $\ker f \subseteq \oplus A$.

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Definition [Ding et al., 2017]
A module $M$ is called a $D4$-module if it satisfies any of the equivalent conditions in the above theorem.

$D3 \Rightarrow D4$

Definition [Ding et al., 2017]
A module $M$ is called summand-dual-square-free if $M$ has no proper direct summands $A$ and $B$ with $M = A + B$ and $M/A \cong M/B$.

summand-dual-square-free $\Rightarrow$ $D4$
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summand-dual-square-free $\Rightarrow D4$
Theorem 4.1

The following conditions on a module $M$ are equivalent:

(1) $M$ is a $D4$-module.

(2) If $A$ and $B$ are perspective direct summands of $M$ with $A + B = M$, then $A \cap B \subseteq \bigoplus M$.

(3) If $A$ and $B$ are perspective direct summands of $M$ with $A + B \subseteq \bigoplus M$, then $A \cap B \subseteq \bigoplus M$. 
Proposition 4.2

The following conditions on a module $M$ are equivalent:

1. $M$ is a $D4$- and summand-square-free module.
2. $M$ is a $C4$- and summand-dual-square-free module.

Corollary 4.3

$R_R$ is summand-square-free if and only if $R_R$ is $C4$ and summand-dual-square-free.
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$R_R$ is summand-square-free if and only if $R_R$ is $C4$ and summand-dual-square-free.
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A module $M$ is said to satisfy the restricted DCC on summands if, $M$ has no strictly descending chains of non-zero summands

$$
\begin{align*}
A_1 & \supsetneq A_2 \supsetneq \cdots \\
B_1 & \supsetneq B_2 \supsetneq \cdots
\end{align*}
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with $M/A_i \cong M/B_i$ and $A_i + B_i \subseteq^\oplus M$ for all $i \geq 1$. 

Theorem 4.5

If $M$ is a $D4$-module that satisfies the restricted $DCC$ on summands, then $M = A \oplus B \oplus K$ where $A \cong B$, $A$ and $B$ are $D2$-modules, and $K$ is a summand-dual-square-free module.

Corollary 4.6

If $R$ is $I$-finite, then $R_R = A \oplus B \oplus K$ with $A \cong B$ and $K$ a summand-dual-square-free module. Moreover, if $R$ is also a right $C4$-ring, then $R_R = A \oplus B \oplus K$ where $A \cong B$ are $C2$-modules and $K$ is both a summand-dual-square-free as well as a summand-square-free module.
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References

thank you