C4- and D4-Modules via Perspective Submodules

Meltem Altun-Özarslan

Hacettepe University, Ankara, Turkey

Joint work with Y. Ibrahim, A. Ç. Özcan, and M. Yousif

Noncommutative Rings and their Applications
12 - 15 June 2017

Outline

Definition

Let R be a ring and M a right R-module.

M is called a *Ci-module* if it has the following Ci properties for $i = 1, 2, 3$.

$C1$: Every submodule of M is essential in a direct summand of M.

$C2$: Whenever A and B are submodules of M with $A \cong B$ and $B \subseteq^{\oplus} M$, then $A \subseteq^{\oplus} M$.

$C3$: Whenever A and B are direct summands of M with $A \cap B = 0$, then $A + B \subseteq^{\oplus} M$.

Definition

Let R be a ring and M a right R-module.

M is called a *Ci-module* if it has the following Ci properties for $i = 1, 2, 3$.

$C1$: Every submodule of M is essential in a direct summand of M.

$C2$: Whenever A and B are submodules of M with $A \cong B$ and $B \subseteq^{\oplus} M$, then $A \subseteq^{\oplus} M$.

$C3$: Whenever A and B are direct summands of M with $A \cap B = 0$, then $A + B \subseteq^{\oplus} M$.

Definition

Let R be a ring and M a right R-module.

M is called a *Ci-module* if it has the following Ci properties for $i = 1, 2, 3$.

$C1$: Every submodule of M is essential in a direct summand of M.

$C2$: Whenever A and B are submodules of M with $A \cong B$ and $B \subseteq^{\oplus} M$, then $A \subseteq^{\oplus} M$.

$C3$: Whenever A and B are direct summands of M with $A \cap B = 0$, then $A + B \subseteq^{\oplus} M$.

Definition

Let R be a ring and M a right R-module.

M is called a *Ci-module* if it has the following Ci properties for $i = 1, 2, 3$.

$C1$: Every submodule of M is essential in a direct summand of M.

$C2$: Whenever A and B are submodules of M with $A \cong B$ and $B \subseteq^{\oplus} M$, then $A \subseteq^{\oplus} M$.

$C3$: Whenever A and B are direct summands of M with $A \cap B = 0$, then $A + B \subseteq^{\oplus} M$.

Definition

A module M is called a *Di-module* if it satisfies the following *Di*-conditions.

$D1$: For every submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $A \cap M_2 \ll M_2$.

$D2$: Whenever A and B are submodules of M with $M/A \cong B$ and $B \subseteq^{\oplus} M$, then $A \subseteq^{\oplus} M$.

$D3$: Whenever A and B are direct summands of M with $A + B = M$, then $A \cap B \subseteq^{\oplus} M$.

Definition

M is called *continuous* if it is $C1$ and $C2$, and *quasi-continuous* if it is $C1$ and $C3$.

$$C2 \Rightarrow C3$$

quasi-injective $\Rightarrow$ continuous

Definition

A module M is called *discrete* if it is both a $D1$- and a $D2$ -module, *quasi-discrete* if it is both a $D1$- and a $D3$-module.

quasi-projective $\Rightarrow D2 \Rightarrow D3$

Definition

M is called *continuous* if it is $C1$ and $C2$, and *quasi-continuous* if it is $C1$ and $C3$.

$$C2 \Rightarrow C3$$

quasi-injective $\Rightarrow$ continuous

Definition

A module M is called *discrete* if it is both a $D1$- and a $D2$ -module, *quasi-discrete* if it is both a $D1$- and a $D3$-module.

quasi-projective $\Rightarrow D2 \Rightarrow D3$

Proposition [Amin et al., 2015]

If M is a $C3$-module, then for every decomposition $M = A \oplus B$ and every homomorphism $f : A \to B$ with $\ker f \subseteq^{\oplus} A$, then $\mathrm{Im} f \subseteq^{\oplus} B$.

[Amin et al., 2015]

The following are equivalent for a module M:

(1) If $M = A \oplus B$ and $f : A \to B$ is a monomorphism, then $\mathrm{Im} f \subseteq^{\oplus} B$.

(2) If $M = A \oplus B$ and $f : A \to B$ is a homomorphism with $\ker f \subseteq^{\oplus} A$, then $\mathrm{Im} f \subseteq^{\oplus} B$.

Proposition [Amin et al., 2015]

If M is a $C3$-module, then for every decomposition $M = A \oplus B$ and every homomorphism $f : A \to B$ with $\ker f \subseteq^{\oplus} A$, then $\mathrm{Im} f \subseteq^{\oplus} B$.

[Amin et al., 2015]

The following are equivalent for a module M:

(1) If $M = A \oplus B$ and $f : A \to B$ is a monomorphism, then $\mathrm{Im} f \subseteq^{\oplus} B$.

(2) If $M = A \oplus B$ and $f : A \to B$ is a homomorphism with $\ker f \subseteq^{\oplus} A$, then $\mathrm{Im} f \subseteq^{\oplus} B$.

Definition [Ding et al., 2017]

A module M is called *a C4-module* if it satisfies any of the equivalent conditions in the above.

$C3 \Rightarrow C4$

Definition

A module M is called *(summand-) square-free* if whenever $N \subseteq M$ and $N = Y_1 \oplus Y_2$ with $Y_1 \cong Y_2$ (and $Y_1, Y_2 \subseteq^{\oplus} M$), then $Y_1 = Y_2 = 0$.

summand-square-free $\Rightarrow$ $C4$

Definition [Ding et al., 2017]

A module M is called *a C4-module* if it satisfies any of the equivalent conditions in the above.

$$C3 \Rightarrow C4$$

Definition

A module M is called *(summand-) square-free* if whenever $N \subseteq M$ and $N = Y_1 \oplus Y_2$ with $Y_1 \cong Y_2$ (and $Y_1, Y_2 \subseteq^{\oplus} M$), then $Y_1 = Y_2 = 0$.

$$\text{summand-square-free} \Rightarrow C4$$

Definition [Ding et al., 2017]

A module M is called *a C4-module* if it satisfies any of the equivalent conditions in the above.

$$C3 \Rightarrow C4$$

Definition

A module M is called *(summand-) square-free* if whenever $N \subseteq M$ and $N = Y_1 \oplus Y_2$ with $Y_1 \cong Y_2$ (and $Y_1,\ Y_2 \subseteq^{\oplus} M$), then $Y_1 = Y_2 = 0$.

summand-square-free $\Rightarrow C4$

Definition [Ding et al., 2017]

A module M is called *a C4-module* if it satisfies any of the equivalent conditions in the above.

$$C3 \Rightarrow C4$$

Definition

A module M is called *(summand-) square-free* if whenever $N \subseteq M$ and $N = Y_1 \oplus Y_2$ with $Y_1 \cong Y_2$ (and $Y_1,\ Y_2 \subseteq^{\oplus} M$), then $Y_1 = Y_2 = 0$.

summand-square-free $\Rightarrow C4$

Definition

Two direct summands A and B of a module M are *perspective* exactly when there exists a common direct sum complement C, i.e.,

$$M = A \oplus C = B \oplus C.$$

Theorem 1.1

The following are equivalent for a module M:

(1) M is a $C4$-module.

(2) If A and B are perspective direct summands of M with $A \cap B = 0$, then $A \oplus B \subseteq^{\oplus} M$.

(3) If A and B are perspective direct summands of M with $A \cap B \subseteq^{\oplus} M$, then $A + B \subseteq^{\oplus} M$.

Definition

Two direct summands A and B of a module M are *perspective* exactly when there exists a common direct sum complement C, i.e.,

$$M = A \oplus C = B \oplus C.$$

Theorem 1.1

The following are equivalent for a module M:

(1) M is a $C4$-module.

(2) If A and B are perspective direct summands of M with $A \cap B = 0$, then $A \oplus B \subseteq^{\oplus} M$.

(3) If A and B are perspective direct summands of M with $A \cap B \subseteq^{\oplus} M$, then $A + B \subseteq^{\oplus} M$.

Theorem 1.2

Let $M = \oplus_{i \in I} M_i$ be a module, where M_i is fully invariant in M for every $i \in I$. Then M is a $C4$-module if and only if each M_i is a $C4$-module.

Definition 1.3

A module M is said to satisfy *the restricted ACC on summands* (r - ACC on summands, for short) if, M has no strictly ascending chains of non-zero summands

$$\begin{array}{ccc} A_1 & \subsetneqq & A_2 \subsetneqq \cdots \\ B_1 & \subsetneqq & B_2 \subsetneqq \cdots \end{array}$$

with $A_i \cong B_i$ and $A_i \cap B_i = 0$ for all $i \geqslant 1$.

ACC on summands $\Rightarrow$ r - ACC on summands

summand-square-free $\Rightarrow$ r - ACC on summands

Definition 1.3

A module M is said to satisfy *the restricted ACC on summands* (r - ACC on summands, for short) if, M has no strictly ascending chains of non-zero summands

$$\begin{array}{ccc} A_1 & \subsetneqq & A_2 \subsetneqq \cdots \\ B_1 & \subsetneqq & B_2 \subsetneqq \cdots \end{array}$$

with $A_i \cong B_i$ and $A_i \cap B_i = 0$ for all $i \geqslant 1$.

ACC on summands $\Rightarrow$ r - ACC on summands

summand-square-free $\Rightarrow$ r - ACC on summands

Theorem 1.4

If M is a $C4$-module that satisfies *the restricted ACC on summands, then* $M = A \oplus B \oplus K$ where $A \cong B$ is a $C2$-module and K is a summand-square-free module.

Let M be a right R-module and $S = \mathrm{End}_R(M)$.

- If S_S is a right $C2$-module, then M_R is $C2$; the converse is true if $ker(\alpha)$ is generated by M whenever α is such that $r_S(\alpha)$ is a direct summand of S_S [Nicholson and Yousif, 2003].

- If S_S is a right $C3$-module, then M_R is $C3$ [Amin et al., 2015]; the converse is true if for every pair of idempotents $e, f \in S$ with $eS \cap fS = 0$ we have $eM \cap fM = 0$ by [Mazurek et al., 2015].

- If S_S is a right $C4$-module, then M_R is $C4$; the converse is true if for every pair of idempotents $e,\ f \in S$ with $eS \cap fS = 0$ we have $eM \cap fM = 0$ [Ding et al., 2017].

Let M be a right R-module and $S = \mathrm{End}_R(M)$.

- If S_S is a right $C2$-module, then M_R is $C2$; the converse is true if $ker(\alpha)$ is generated by M whenever α is such that $r_S(\alpha)$ is a direct summand of S_S [Nicholson and Yousif, 2003].

- If S_S is a right $C3$-module, then M_R is $C3$ [Amin et al., 2015]; the converse is true if for every pair of idempotents $e, f \in S$ with $eS \cap fS = 0$ we have $eM \cap fM = 0$ by [Mazurek et al., 2015].

- If S_S is a right $C4$-module, then M_R is $C4$; the converse is true if for every pair of idempotents e, $f \in S$ with $eS \cap fS = 0$ we have $eM \cap fM = 0$ [Ding et al., 2017].

Theorem 2.1

Let M be a right R-module with $S = \mathrm{End}_R(M)$. Then S is a right $C4$-ring, if M is a $C4$-module and one of the following is satisfied.

(1) M is k-local-retractable.

(2) For any $\alpha \in S$, $ker(\alpha)$ is generated by M.

(3) For every pair of perspective idempotents $e, f \in S$ with $eS \cap fS = 0$, we have $eM \cap fM = 0$.

Proposition 2.2

Let M be a right R-module with $S = \mathrm{End}_R(M)$. Then the following are equivalent:

(1) M is a $C4$-module.

(2) For every pair of perspective idempotents $e, f \in S$ with $eM \cap fM = 0$, there exists an idempotent g of S such that $eM = gM$ and $fM \subseteq (1 - g)M$.

Proposition 2.3

A right R-module M is $C4$ if and only if for any idempotents $e, f \in End_R(M)$, if $kere = kerf = ker(e - f)$, then $(1 - e)fM \subseteq^{\oplus} M$.

Proposition 2.2

Let M be a right R-module with $S = \mathrm{End}_R(M)$. Then the following are equivalent:

(1) M is a $C4$-module.

(2) For every pair of perspective idempotents $e, f \in S$ with $eM \cap fM = 0$, there exists an idempotent g of S such that $eM = gM$ and $fM \subseteq (1 - g)M$.

Proposition 2.3

A right R-module M is $C4$ if and only if for any idempotents $e, f \in End_R(M)$, if $kere = kerf = ker(e - f)$, then $(1 - e)fM \subseteq^{\oplus} M$.

Proposition 3.1

Let R_i $(i \in I)$ be any collection of rings, and let R be the direct product $\prod_{i \in I} R_i$. Then R is a right $C4$-ring if and only if every R_i is a right $C4$-ring.

Proposition 3.2

If R is a right $C4$-ring, then so is eRe for any idempotent $e \in R$ such that $ReR = R$.

Proposition 3.1

Let R_i $(i \in I)$ be any collection of rings, and let R be the direct product $\prod_{i \in I} R_i$. Then R is a right $C4$-ring if and only if every R_i is a right $C4$-ring.

Proposition 3.2

If R is a right $C4$-ring, then so is eRe for any idempotent $e \in R$ such that $ReR = R$.

Example 3.3

The condition $ReR = R$ is not superfluous in Proposition 3.2: Let R be the algebra of matrices, over a field F, of the form

$$\begin{pmatrix} a & x & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & y & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & b & z \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}$$

- $e := e_{11} + e_{22} + e_{33} + e_{44} + e_{55}$, where e_{ij} are the matrices with (i, j)-entry 1 and all other entries zero.
- e is an idempotent of R such that $ReR \neq R$.
- R is a quasi-Frobenius ring by [Koike, 1995] $\Rightarrow$ R is right $C4$.
- $eRe \cong \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$ is not a right $C4$-ring.

Example 3.3

The condition $ReR = R$ is not superfluous in Proposition 3.2: Let R be the algebra of matrices, over a field F, of the form

$$\begin{pmatrix} a & x & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & y & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & b & z \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}$$

- $e := e_{11} + e_{22} + e_{33} + e_{44} + e_{55}$, where e_{ij} are the matrices with (i, j)-entry 1 and all other entries zero.
- e is an idempotent of R such that $ReR \neq R$.
- R is a quasi-Frobenius ring by [Koike, 1995] $\Rightarrow$ R is right $C4$.
- $eRe \cong \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$ is not a right $C4$-ring.

Example 3.3

The condition $ReR = R$ is not superfluous in Proposition 3.2: Let R be the algebra of matrices, over a field F, of the form

$$\begin{pmatrix} a & x & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & y & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & b & z \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}$$

- $e := e_{11} + e_{22} + e_{33} + e_{44} + e_{55}$, where e_{ij} are the matrices with (i, j)-entry 1 and all other entries zero.
- e is an idempotent of R such that $ReR \neq R$.
- R is a quasi-Frobenius ring by [Koike, 1995] $\Rightarrow$ R is right $C4$.
- $eRe \cong \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$ is not a right $C4$-ring.

- Let R be a ring and M an R-R-bimodule.
  Then the trivial extension $R \propto M$ is a ring whose underlying group is $R \times M$ with the multiplication defined by

$$(r, m)(s, n) = (rs, rn + ms)$$

  where $r, s \in R$ and $m, n \in M$.

Proposition 3.4

Let R be a ring and M an R-R-bimodule.

1) If $R \propto M$ is a right $C4$-ring, and for any idempotents $e, f \in R$, $eR \cap fR = 0$ implies $eM \cap fM = 0$, then R is a right $C4$-ring.
2) If R is a right $C4$-ring, and $eM(1 - e) = 0$ for any idempotent $e \in R$, then $R \propto M$ is a right $C4$-ring.

- Let R be a ring and M an R-R-bimodule.
  Then the trivial extension $R \propto M$ is a ring whose underlying group is $R \times M$ with the multiplication defined by

$$(r, m)(s, n) = (rs, rn + ms)$$

  where $r, s \in R$ and $m, n \in M$.

Proposition 3.4

Let R be a ring and M an R-R-bimodule.

1) If $R \propto M$ is a right $C4$-ring, and for any idempotents $e, f \in R$, $eR \cap fR = 0$ implies $eM \cap fM = 0$, then R is a right $C4$-ring.
2) If R is a right $C4$-ring, and $eM(1 - e) = 0$ for any idempotent $e \in R$, then $R \propto M$ is a right $C4$-ring.

Proposition [Yousif et al., 2014]

If M is a $D3$-module, then for every decomposition $M = A \oplus B$ and every homomorphism $f : A \to B$ with $\mathrm{Im} f \subseteq^{\oplus} B$, then $\ker f \subseteq^{\oplus} A$.

[Yousif et al., 2014]

The following are equivalent for a module M:

(1) If $M = A \oplus B$ with $A, B \subseteq M$ and $f : A \to B$ is an epimorphism, then $\ker f \subseteq^{\oplus} A$.

(2) If $M = A \oplus B$ with $A, B \subseteq M$ and $f : A \to B$ is a homomorphism with $\mathrm{Im} f \subseteq^{\oplus} B$, then $\ker f \subseteq^{\oplus} A$.

Proposition [Yousif et al., 2014]

If M is a $D3$-module, then for every decomposition $M = A \oplus B$ and every homomorphism $f : A \to B$ with $\mathrm{Im} f \subseteq^{\oplus} B$, then $\ker f \subseteq^{\oplus} A$.

[Yousif et al., 2014]

The following are equivalent for a module M:

(1) If $M = A \oplus B$ with $A, B \subseteq M$ and $f : A \to B$ is an epimorphism, then $\ker f \subseteq^{\oplus} A$.

(2) If $M = A \oplus B$ with $A, B \subseteq M$ and $f : A \to B$ is a homomorphism with $\mathrm{Im} f \subseteq^{\oplus} B$, then $\ker f \subseteq^{\oplus} A$.

Definition [Ding et al., 2017]

A module M is called a *D4-module* if it satisfies any of the equivalent conditions in the above theorem.

$$D3 \Rightarrow D4$$

Definition [Ding et al., 2017]

A module M is called *summand-dual-square-free* if M has no proper direct summands A and B with $M = A + B$ and $M/A \cong M/B$.

$$\text{summand-dual-square-free} \Rightarrow D4$$

Definition [Ding et al., 2017]

A module M is called a *D4-module* if it satisfies any of the equivalent conditions in the above theorem.

$$D3 \Rightarrow D4$$

Definition [Ding et al., 2017]

A module M is called *summand-dual-square-free* if M has no proper direct summands A and B with $M = A + B$ and $M/A \cong M/B$.

summand-dual-square-free $\Rightarrow D4$

Theorem 4.1

The following conditions on a module M are equivalent:

(1) M is a $D4$-module.

(2) If A and B are perspective direct summands of M with $A + B = M$, then $A \cap B \subseteq^{\oplus} M$.

(3) If A and B are perspective direct summands of M with $A + B \subseteq^{\oplus} M$, then $A \cap B \subseteq^{\oplus} M$.

Proposition 4.2

The following conditions on a module M are equivalent:

(1) M is a $D4$- and summand-square-free module.

(2) M is a $C4$- and summand-dual-square-free module.

Corollary 4.3

R_R is summand-square-free if and only if R_R is $C4$ and summand-dual-square-free .

Proposition 4.2

The following conditions on a module M are equivalent:

(1) M is a $D4$- and summand-square-free module.

(2) M is a $C4$- and summand-dual-square-free module.

Corollary 4.3

R_R is summand-square-free if and only if R_R is $C4$ and summand-dual-square-free .

Definition 4.4

A module M is said to satisfy *the restricted DCC on summands* if, M has no strictly descending chains of non-zero summands

$$\begin{array}{ccc} A_1 & \supsetneq & A_2 \supsetneq \cdots \\ B_1 & \supsetneq & B_2 \supsetneq \cdots \end{array}$$

with $M/A_i \cong M/B_i$ and $A_i + B_i \subseteq^{\oplus} M$ for all $i \geqslant 1$.

Theorem 4.5

If M is a $D4$-module that satisfies the restricted DCC on summands, then $M = A \oplus B \oplus K$ where $A \cong B$, A and B are $D2$-modules, and K is a summand-dual-square-free module.

Corollary 4.6

If R is I-finite, then $R_R = A \oplus B \oplus K$ with $A \cong B$ and K a summand-dual-square-free module. Moreover, if R is also a right $C4$-ring, then $R_R = A \oplus B \oplus K$ where $A \cong B$ are $C2$-modules and K is both a summand-dual-square-free as well as a summand-square-free module.

Theorem 4.5

If M is a $D4$-module that satisfies the restricted DCC on summands, then $M = A \oplus B \oplus K$ where $A \cong B$, A and B are $D2$-modules, and K is a summand-dual-square-free module.

Corollary 4.6

If R is I-finite, then $R_R = A \oplus B \oplus K$ with $A \cong B$ and K a summand-dual-square-free module. Moreover, if R is also a right $C4$-ring, then $R_R = A \oplus B \oplus K$ where $A \cong B$ are $C2$-modules and K is both a summand-dual-square-free as well as a summand-square-free module.

Theorem 4.5

If M is a $D4$-module that satisfies the restricted DCC on summands, then $M = A \oplus B \oplus K$ where $A \cong B$, A and B are $D2$-modules, and K is a summand-dual-square-free module.

Corollary 4.6

If R is I-finite, then $R_R = A \oplus B \oplus K$ with $A \cong B$ and K a summand-dual-square-free module. Moreover, if R is also a right $C4$-ring, then $R_R = A \oplus B \oplus K$ where $A \cong B$ are $C2$-modules and K is both a summand-dual-square-free as well as a summand-square-free module.

References

- I. Amin, Y. Ibrahim and M. Yousif, $C3$-Modules, Algebra Colloq., 4, 655-670, 2015.
- N. Ding, Y. Ibrahim, M. Yousif and Y. Zhou , $C4$-Modules, Comm. Algebra, http://dx.doi.org/10.1080/00927872.2016.1222412.
- W. K. Nicholson and M. Yousif, Quasi-Frobenius Rings, Cambridge Tracts in Mathematics 158, Cambridge University Press, 2003.
- R. Mazurek, P. P. Nielsen and M. Ziembowski, Commuting idempotents, square-free modules, and the exchange property. J. Algebra, 444, 52-80, 2015.

References

- K. Koike, Dual rings and cogenerator rings, Math. J. Okayama Univ, 37(1), 99-103, 1995.
- M. Yousif, I. Amin and Y. Ibrahim, D3-Modules, Comm. Algebra, 42:2, 578-592, 2014.
- N. Ding, Y. Ibrahim, M. Yousif and Y. Zhou, $D4$-Modules, J. Algebra Appl., Vol. 16, No. 5, 1750166 (25 pages), 2017.
- S. H. Mohamed and B. J. Müller, Continuous and Discrete Modules, London Math. Soc. Lecture Note Series, 147, Cambridge Univ. Press., Cambridge, 1990.


thank you
danke
謝謝
ngiyabonga
teşekkür ederim
gracias
merci
obrigado
dziękuję
bedankt
спасибо
Баярлалаа
рахмат
mersi
barka
welalin
tack
dank je
misaotra
matondo
paldies
grazzi
mahalo
tapadh leat
хвала
asante
manana
obrigada
tenki
murakoze
mochchakkeram
djiere dieuf
go raibh maith agat
mamnun
chokrane
faafetai lava
vinaka
спасибі
blagodaram
kiitos
dankie
nanni
nandri
dhanyavad
hvala
mauruuru
köszönöm
gràcie
bayarlalaa
enkosi
sobodi
dėkuji
chnorakaloutioun
grazias ago
gràcies
sulpáy
taiku
sukriya
kop khun krap
arigatô
takk
dakujem
trugarez
mèsi
sagolun
ありがとう
tanemirt
grazie
rahmet
najis tuke
terima kasih
dhanyavadagalu
shukriya
mercé
мерси
kam sah hamnida
rahmat
diolch
xièxie
감사합니다
ευχαριστώ
দোমাকে ধন্যবাদ